Quasigroups and Related Systems **17** (2009), 107 – 118

New fuzzy subquasigroups

Muhammad Akram and Wieslaw A. Dudek

Abstract. In this paper we introduce new generalized fuzzy subquasigroups and study some of their important properties. We characterize these generalized subquasigroups by their level subsets. Some characterization of the generalized fuzzy subquasigroups are also established.

1. Introduction

During the last decade, there have been many applications of quasigroups in different areas, such as cryptography, modern physics [9], coding theory, cryptology and geometry. In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then it has become a vigorous area of research in different mathematical domains. Rosenfeld inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. Das [4] characterized fuzzy subgroups by their level subgroups. Murali [8] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroups, $(\in, \in \vee q)$-fuzzy subgroups, was introduced in earlier paper Bhakat and Das [3] by using the combined notions of *belongines* and *quasi-coincidence* of fuzzy point and fuzzy set. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. On the other hand, Akram and Dudek applied this concept to subquasigroup in [2] and studied some of its properties. In this paper using general form of the concept of quasi-coincidence of a fuzzy point with a fuzzy subset, the notion of an $(\in, \in \vee\, q_m)-$ fuzzy subquasigroup is introduced and some of its important properties are investigated. We characterize these generalized subquasigroups by their level subsets.

2000 Mathematics Subject Classification: 20N15, 94D05
Keywords: Quasigroup, $(\in, \in \vee q_m)$-fuzzy subquasigroup.

Some characterization of the generalized fuzzy subquasigroups are established. Some recent results obtained by Akram-Dudek [2] are extended and strengthened.

2. Preliminaries

A groupoid $(G,\cdot)$ is called a *quasigroup* if for any a , $b\in G$ each of the equations $a\cdot x=b$, $x\cdot a=b$ has a unique solution in G. A quasigroup may be also defined as an algebra $(G,\cdot,\backslash,/)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x\cdot y)/y=x, \qquad x\setminus(x\cdot y)=y,$$

$$(x/y)\cdot y=x, \qquad x\cdot(x\setminus y)=y.$$

Such defined quasigroup is called an *equasigroup*.

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* if it is closed with respect to these three operations.

In this paper $\mathcal{G}$ always denotes an equasigroup $(G,\cdot,\backslash,/)$; G always denotes a nonempty set.

A mapping $\mu: G\to[0,1]$ is called a *fuzzy set* in G. For any fuzzy set μ in G and any $t\in[0,1]$, we define the set

$$U(\mu;t)=\{x\in G\mid \mu(x)\geqslant t\},$$

which is called the *upper t-level* of μ.

Definition 2.1. A fuzzy set μ in a set G of the form

$$\mu(y)=\begin{cases} t\in(0,1] & \text{for } y=x,\\ 0 & \text{for } y\neq x,\end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

We say that a fuzzy point x_t *belong to* a fuzzy set μ and write $x_t\in\mu$, if $\mu(x)\geqslant t$. A fuzzy point x_t is *quasicoincident* with a fuzzy set μ, if $\mu(x)+t>1$. In this case we write $x_tq\mu$.

- $x_t\in\vee q\mu$ means that $x_t\in\mu$ or $x_tq\mu$,
- $x_t\in\wedge q\mu$ means that $x_t\in\mu$ and $x_tq\mu$.

Definition 2.2. A fuzzy set μ in G is called an $(\in, \in\vee q)$-*fuzzy subquasigroup* of $\mathcal{G}$, if it satisfies the following condition:

$$x_{t_1}, y_{t_2} \in \mu \Longrightarrow (x * y)_{\min\{t_1,t_2\}} \in \vee q\mu$$

for all $x, y \in G$, $t_1, t_2 \in (0,1]$ and $* \in \{\cdot, \backslash, /\}$.

3. New fuzzy subquasigroups

Let m be an element of $[0,\ 1)$ unless otherwise specified. By $x_t q_m \mu$, we mean $\mu(x) + t + m > 1$, $t \in (0, \frac{1-m}{2}]$. The notation $x_t \in \vee q_m \mu$ means that $x_t \in \mu$ or $x_t q_m \mu$.

Definition 3.1. A fuzzy set μ in G is called an $(\in, \in\vee q_m)$-*fuzzy subquasigroup* of $\mathcal{G}$, if

$$x_{t_1}, y_{t_2} \in \mu \Longrightarrow (x * y)_{\min\{t_1,t_2\}} \in \vee q_m\mu$$

for all $x, y \in G$, $t_1, t_2 \in (0,1]$ and $* \in \{\cdot, \backslash, /\}$.

We note that different types of fuzzy subquasigroups can be constructed for different values of $m \in [0,1)$. Hence an $(\in, \in\vee q_m)$-fuzzy subquasigroup with $m = 0$ is called an $(\in, \in\vee q)$-fuzzy subquasigroup.

Example 3.2. Let $G = \{0, a, b, c\}$ be a quasigroup with the following multiplication table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(i) Consider a fuzzy set μ defined by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = 0, \\ 0.8 & \text{if } x = a, \\ 0.4 & \text{if } x = b, \\ 0.4 & \text{if } x = c. \end{cases}$$

If $m = 0.2$, then $U(\mu; t) = G$ for all $t \in (0, 0.4]$. Hence μ is an $(\in, \in \vee q_{0.2})$–fuzzy subquasigroup of $\mathcal{G}$.

(ii) Now consider a fuzzy set

$$\mu(x)=\begin{cases}0.45 & \text{if } x=0,\\ 0.41 & \text{if } x=a,\\ 0.41 & \text{if } x=c,\\ 0.49 & \text{if } x=b.\end{cases}$$

In this case for $m=0.04$ we have

$$U(\mu;t)=\begin{cases}G & \text{if } t\in(0,0.4],\\ \{0,b\} & \text{if } t\in(0.4,0.45],\\ \{b\} & \text{if } t\in(0.45,0.48].\end{cases}$$

Since $\{b\}$ is not a subquasigroup of $\mathcal{G}$, so $U(\mu;t)$ is not a subquasigroup for $t\in(0.45,0.48]$. Hence μ is not an $(\in,\in\vee q_{0.04})$-fuzzy subquasigroup of a quasigroup $\mathcal{G}$. □

Proposition 3.3. *Every $(\in,\in)$-fuzzy subquasigroup is an $(\in,\in\vee q_m)$-fuzzy subquasigroup.*

Proof. Straightforward. □

The converse statement may not be true.

Example 3.4. Consider the $(\in,\in\vee q_{0.2})$-fuzzy subquasigroup of $\mathcal{G}$ defined in Example 3.2. Then μ is not an $(\in,\in)$-fuzzy subquasigroup of $\mathcal{G}$ since $a_{0.71}\in\mu$ and $a_{0.75}\in\mu$, but $(a*a)_{\min\{0.71,0.75\}}=0_{0.71}\overline{\in}\mu$. □

Theorem 3.5. *A fuzzy set μ in $\mathcal{G}$ is an $(\in,\in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if*

$$\mu(x*y)\geqslant\min\Big\{\mu(x),\mu(y),\frac{1-m}{2}\Big\} \tag{1}$$

holds for all x, $y\in G$.

Proof. Let μ be an $(\in,\in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. Assume that (1) is not valid. Then there exist x', $y'\in G$ such that

$$\mu(x'*y')<\min\Big\{\mu(x'),\mu(y'),\frac{1-m}{2}\Big\}.$$

If $\min(\mu(x'), \mu(y')) < \frac{1-m}{2}$, then $\mu(x' * y') < \min(\mu(x'), \mu(y'))$. Thus

$$\mu(x' * y') < t \leqslant \min\{\mu(x'), \mu(y')\} \qquad \text{for some } t \in (0, 1].$$

It follows that $x'_t \in \mu$ and $y'_t \in \mu$, but $(x' * y')_t \overline{\in} \mu$, a contradiction. Moreover, $\mu(x' * y') + t < 2t < 1 - m$, and so $(x' * y')_t \overline{q_m} \mu$. Hence, consequently $(x' * y')_t \overline{\in \vee q_m} \mu$, a contradiction.

On the other hand, if $\min\{\mu(x'), \mu(y')\} \geqslant \frac{1-m}{2}$, then $\mu(x') \geqslant \frac{1-m}{2}$, $\mu(y') \geqslant \frac{1-m}{2}$ and $\mu(x' * y') < \frac{1-m}{2}$. Thus $x'_{\frac{1-m}{2}} \in \mu$ and $y'_{\frac{1-m}{2}} \in \mu$, but $(x' * y')_{\frac{1-m}{2}} \overline{\in} \mu$. Also

$$\mu(x' * y') + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $(x' * y')_{\frac{1-m}{2}} \overline{q_m} \mu$. Hence $(x' * y')_{\frac{1-m}{2}} \overline{\in \vee q_m} \mu$, a contradiction. So (1) is valid.

Conversely, assume that μ satisfies (1). Let $x, y \in G$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \mu$ and $y_{t_2} \in \mu$. Then

$$\mu(x * y) \geqslant \min\left\{\mu(x), \mu(y), \frac{1-m}{2}\right\} \geqslant \min\left\{t_1, t_2, \frac{1-m}{2}\right\}.$$

Assume that $t_1 \leqslant \frac{1-m}{2}$ or $t_2 \leqslant \frac{1-m}{2}$. Then $\mu(x * y) \geqslant \min\{t_1, t_2\}$, which implies that $(x * y)_{\min\{t_1, t_2\}} \in \mu$. Now suppose that $t_1 > \frac{1-m}{2}$ and $t_2 > \frac{1-m}{2}$. Then $\mu(x * y) \geqslant \frac{1-m}{2}$, and thus

$$\mu(x * y) + \min\{t_1, t_2\} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $(x * y)_{\min\{t_1, t_2\}} q_m \mu$. Hence $(x * y)_{\min\{t_1, t_2\}} \in \vee q_m \mu$, and consequently, μ is an $(\in, \in\!\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. $\square$

Theorem 3.6. *A fuzzy set μ of G is an $(\in, \in\!\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if each nonempty level set $U(\mu; t)$, $t \in (0, \frac{1-m}{2}]$, is a subquasigroup of $\mathcal{G}$.*

Proof. Assume that a fuzzy set μ is an $(\in, \in \vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $t \in (0, \frac{1-m}{2}]$ and $x, y \in U(\mu; t)$. Then $\mu(x) \geqslant t$ and $\mu(y) \geqslant t$. It follows from (1) that

$$\mu(x * y) \geqslant \min\left\{\mu(x), \mu(y), \frac{1-m}{2}\right\} \geqslant \min\left\{t, \frac{1-m}{2}\right\} = t,$$

so that $x * y \in U(\mu;t)$. Hence $U(\mu;t)$ is an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.

Conversely, suppose that the nonempty set $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$ for all $t \in (0, \frac{1-m}{2}]$. If the condition (1) is not true, then there exists $a,\ b \in G$ such that $\mu(a*b) < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$. Hence we can take $t \in (0,1]$ such that $\mu(a*b) < t_1 < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$. Then $t \in (0, \frac{1-m}{2}]$ and $a, b \in U(\mu;t)$. Since $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$, it follows that $a * b \in U(\mu;t)$, so $\mu(a*b) \geqslant t$. This is a contradiction. Therefore the condition (1) is valid, and so μ is an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.7. *Let μ be a fuzzy set of a quasigroup $\mathcal{G}$. Then the nonempty level set $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$ for all $t \in (\frac{1-m}{2}, 1]$ if and only if*

$$\max\left\{\mu(x*y), \frac{1-m}{2}\right\} \geqslant \min\{\mu(x), \mu(y)\}$$

for all $x, y \in G$.

Proof. Suppose that $U(\mu;t) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$. Assume that $\max\{\mu(x*y), \frac{1-m}{2}\} < \min\{\mu(x), \mu(y)\} = t$ for some $x,\ y \in G$, then $t \in (\frac{1-m}{2}, 1]$, $\mu(x*y) < t$, $x \in U(\mu;t)$ and $y \in U(\mu;t)$. Since $x,\ y \in U(\mu;t)$, $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$, so $x * y \in U(\mu;t)$, a contradiction.

The proof of the second part of Theorem is straightforward. □

Theorem 3.8. *Let μ be an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. If it satisfies $\mu(x) < \frac{1-m}{2}$ for all $x \in G$, then it is a fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let $x, y \in G$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. It follows from Theorem 3.5 that

$$\mu(x*y) \geqslant \min\left\{\mu(x), \mu(y), \frac{1-m}{2}\right\} = \min\{\mu(x), \mu(y)\} = \min\{t_1, t_2\},$$

so $(x*y)_{\min\{t_1,t_2\}} \in \mu$. Hence μ is a fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.9. *If $0 \leqslant m < n < 1$, then each $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$ is an $(\in, \in\vee q_n)$-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let μ be an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$ and let x,$y \in G$. Then

$$\mu(x*y) \geqslant \min\left\{\mu(x), \mu(y), \frac{1-m}{2}\right\} \geqslant \min\left\{\mu(x), \mu(y), \frac{1-n}{2}\right\}.$$

Thus from Theorem 3.5, it follows that μ is an $(\in, \in\vee q_n)$-fuzzy subquasigroup of $\mathcal{G}$. $\square$

Note that an $(\in, \in\vee q_n)$-fuzzy subquasigroup may not be an $(\in, \in\vee q_m)$-fuzzy subquasigroup for $0 \leqslant m < n < 1$.

Example 3.10. Let $G = \{0, a, b, c\}$ be a quasigroup defined in Example 3.2. Consider a fuzzy set

$$\mu(x) = \begin{cases} 0.42 & \text{if } \ x = 0, \\ 0.4 & \text{if } \ x = a, \\ 0.4 & \text{if } \ x = c, \\ 0.48 & \text{if } \ x = b. \end{cases}$$

If $n = 0.16$, then

$$U(\mu; t) = \begin{cases} G & \text{for } \ t \in (0, 0.4], \\ \{0, b\} & \text{for } \ t \in (0.4, 0.42]. \end{cases}$$

Since G and $\{0, b\}$ are subquasigroups of $\mathcal{G}$, so $U(\mu; t)$ is a subquasigroup for $t \in (0.4, 0.42]$. Hence μ is an $(\in, \in\vee q_{0.16})$-fuzzy subquasigroup of $\mathcal{G}$.

If $m = 0.04$, then

$$U(\mu; t) = \begin{cases} G & \text{for } \ t \in (0, 0.4], \\ \{b\} & \text{for } \ t \in (0.4, 0.48]. \end{cases}$$

Since $\{b\}$ is not a subquasigroup of $\mathcal{G}$, so $U(\mu; t)$ is not a subquasigroup for $t \in (0.4, 0.48]$. Hence μ is not an $(\in, \in\vee q_{0.04})$-fuzzy subquasigroup of $\mathcal{G}$. $\square$

Theorem 3.11. *A nonempty subset M of G is a subquasigroup of $\mathcal{G}$ if and only if its characteristic function is an $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let M be a subquasigroup of $\mathcal{G}$. Then $\chi_{_M}(x) = 1$ for $x \in M$ and $\chi_{_M}(x) = 0$ for $x \notin M$. Thus $U(\mu_M; t) = M$ for all $t \in (0, \frac{1-m}{2}]$. Hence, by Theorem 3.6, $\chi_{_M}$ is an $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.

Conversely, suppose that μ_M is an $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. Then

$$\mu(x * y) \geqslant \min\Big\{\chi_{_M}(x), \chi_{_M}(y), \frac{1-m}{2}\Big\} = \min\Big\{1, \frac{1-m}{2}\Big\} = \frac{1-m}{2}$$

for $x, y \in G$. Since $m \in [0, 1)$, it follows that $\chi_{_M}(x * y) = 1$, so $x * y \in M$. Hence M is a subquasigroup of $\mathcal{G}$. $\square$

Corollary 3.12. *For every subquasigroup M of $\mathcal{G}$ and every $t \in (0, \frac{1-m}{2}]$ there exists an $(\in, \in\!\vee q_m)$-fuzzy subquasigroup μ of $\mathcal{G}$ such that $U(\mu; t) = M$.*

Proof. Indeed, χ_M is this $(\in, \in\!\vee q_m)$-fuzzy subquasigroup. □

Theorem 3.13. *The intersection of any family of $(\in, \in\vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$ is an $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let $\mu = \bigcap_{i\in\Lambda} \mu_i$ for some $(\in, \in\!\vee q_m)$-fuzzy subquasigroups μ_i of $\mathcal{G}$. Then

$$\begin{aligned}\mu(x*y) &= \sup_{i\in\Lambda}\mu_i(x*y) \geqslant \sup_{i\in\Lambda}\min\{\mu_i(x), \mu_i(y), \tfrac{1-m}{2}\}\\ &= \min\{\sup_{i\in\Lambda}\mu_i(x), \sup_{i\in\Lambda}\mu_i(y), \tfrac{1-m}{2}\}\\ &= \min\{\textstyle\bigcap_{i\in\Lambda}\mu_i(x), \bigcap_{i\in\Lambda}\mu_i(y), \tfrac{1-m}{2}\} = \min\{\mu(x), \mu(y), \tfrac{1-m}{2}\}.\end{aligned}$$

Hence, by Theorem 3.5, μ is an $(\in, \in\!\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. □

The union of two $(\in, \in\!\vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$ is not an $(\in, \in\vee q_m)$-fuzzy subquasigroup, in general.

Example 3.14. Let $\mathcal{G}$ be as in Example 3.2. Consider two fuzzy sets:

$$\mu(x) = \begin{cases} 0.6 & \text{if } x = 0,\\ 0.7 & \text{if } x = a,\\ 0.3 & \text{if } x = b,\\ 0.3 & \text{if } x = c,\end{cases} \qquad \text{and} \qquad \nu(x) = \begin{cases} 0.4 & \text{if } x = 0,\\ 0.5 & \text{if } x = b,\\ 0.3 & \text{if } x = a,\\ 0.3 & \text{if } x = c.\end{cases}$$

For $m = 0.2$ we have

$$U(\mu; t) = \begin{cases} G & \text{if } t \in (0, 0.3],\\ \{0, a\} & \text{if } t \in (0.4, 0.4],\end{cases} \qquad U(\nu; t) = \begin{cases} G & \text{if t } \in (0,\ 0.3],\\ \{0, b\} & \text{if t } \in (0.3,\ 0.4].\end{cases}$$

Since G, $\{0, a\}$ and $\{0, b\}$ are subquasigroups of $\mathcal{G}$, μ and ν are $(\in, \in\!\vee q_{0.2})$-fuzzy subquasigroups by Theorem 3.6.

The union $\mu \cup \nu$ has the form

$$(\mu \cup \nu)(x) = \begin{cases} 0.6 & \text{if } x = 0,\\ 0.7 & \text{if } x = a,\\ 0.5 & \text{if } x = b,\\ 0.3 & \text{if } x = c.\end{cases}$$

For $m = 0.2$ we have

$$U(\mu \cup \nu; t) = \begin{cases} G & \text{if } t \in (0, 0.3], \\ \{0, a, b\} & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since $\{0, a, b\}$ is not a subquasigroup, $\mu \cup \nu$ is not an $(\in, \in \vee q_{0.2})$-fuzzy subquasigroup of $\mathcal{G}$. $\square$

Theorem 3.15. *The union of ordered family of $(\in, \in\vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$ is an $(\in, \in\vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let $\{\mu_i \,|\, i \in \Lambda\}$ be an ordered family of $(\in, \in\vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$, i.e., $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$. Then for $\mu := \bigcup_{i\in\Lambda} \mu_i$ we have

$$\begin{aligned} \mu(x * y) &= \inf_{i\in\Lambda} \mu_i(x * y) \geqslant \inf_{i\in\Lambda} \min\{\mu_i(x)), \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min\{\inf_{i\in\Lambda} \mu_i(x), \inf_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min\{\textstyle\bigcup_{i\in\Lambda} \mu_i(x), \bigcup_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}\} = \min\{\mu(x), \mu(y), \tfrac{1-m}{2}\}. \end{aligned}$$

It is easy to see that

$$\inf_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\} \leqslant \bigcup_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\}.$$

Suppose that

$$\inf_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\} \neq \bigcup_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\}.$$

Then there exists s such that

$$\inf_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\} < s < \bigcup_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\}.$$

Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$, there exists $k \in \Lambda$ such that $s < \min\Big\{\mu_k(x), \mu_k(y), \frac{1-m}{2}\Big\}$. On the other hand, $\min\Big\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\Big\} > s$ for all $i \in \Lambda$, a contradiction. Hence

$$\begin{aligned} \inf_{i\in\Lambda} \min\Big\{\mu_i(x), \mu_i(y), \tfrac{1-m}{2}\Big\} &= \min\Big\{\textstyle\bigcup_{i\in\Lambda} \mu_i(x), \bigcup_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}\Big\} \\ &= \min\Big\{\mu(x), \mu(y), \tfrac{1-m}{2}\Big\}. \end{aligned}$$

Theorem 3.5 completes the proof. $\square$

Theorem 3.16. *For any finite strictly increasing chain of subquasigroups of $\mathcal{G}$ there exists an $(\in, \in \vee q_m)$-fuzzy subquasigroup μ of $\mathcal{G}$ whose level subquasigroups are precisely the members of the chain with $\mu_{\frac{1-m}{2}} = G_0 \subset G_1 \subset \ldots \subset G_n = G$.*

Proof. Let $\{t_i \,|\, t_i \in (0, \frac{1-m}{2}], i = 1, \ldots, n\}$ be such that $\frac{1-m}{2} > t_1 > t_2 > t_3 > \ldots > t_n$. Consider the fuzzy set μ defined by

$$\mu(x) = \begin{cases} \frac{1-m}{2} & \text{if } \ x \in G_0, \\ t_k & \text{if } \ x \in G_k \setminus G_{k-1},\ k = 1, \ldots, n \end{cases}$$

Let $x, y \in G$ be such that $x \in G_i \setminus G_{i-1}$ and $y \in G_j \setminus G_{j-1}$, where $1 \leqslant i, j \leqslant n$. When $i \geqslant j$, then $x \in G_i$, $y \in G_i$, so $x * y \in G_i$. Thus

$$\mu(x * y) \geqslant t_i = \min\{t_i, t_j\} = \min\Big\{\mu(x), \mu(y), \frac{1-m}{2}\Big\}.$$

When $i < j$, then $x \in G_j$, $y \in G_j$, so $x * y \in G_j$. Thus

$$\mu(x * y) \geqslant t_j = \min\{t_i, t_j\} = \min\Big\{\mu(x), \mu(y), \frac{1-m}{2}\Big\}.$$

Hence μ is an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$. □

Definition 3.17. For any fuzzy set μ in $\mathcal{G}$ and $t \in (0,\, 1]$, we define two sets

$$[\mu]_t = \{x \in G \,|\, x_t \in \vee q_m\, \mu\}$$

and

$$Q(\mu; t) = \{x \in G \,|\, x_t q_m \mu\}.$$

It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$.

Example 3.18. Let $G = \{0, a, b, c\}$ be a quasigroup which is given in Example 3.2. Consider fuzzy sets

$$\mu(x) = \begin{cases} 0.67 & \text{if } \ x = 0, \\ 0.56 & \text{if } \ x = a, \\ 0.47 & \text{if } \ x = b, \\ 0.41 & \text{if } \ x = c, \end{cases} \qquad \nu(x) = \begin{cases} 0.60 & \text{if } \ x = 0, \\ 0.05 & \text{if } \ x = a, \\ 0.50 & \text{if } \ x = b, \\ 0.06 & \text{if } \ x = c. \end{cases}$$

(1) When $m = 0.6$, then $U(\mu; t) = G$ and $Q(\mu; t) = G$ for all $t \in (0, 0.2]$. Thus $[\mu]_t = G$ for all $t \in (0, 0.2]$. Hence $[\mu]_t$ is an $(\in, \in \vee q_{0.6})$-fuzzy subquasigroup of $\mathcal{G}$.

(2) When $m = 0.8$, then $U(\nu;t) = G$ and $Q(\nu;t) = \{0, b\}$ for all $t \in (0, 0.1]$. Thus $[\nu]_t = G$ for all $t \in (0, 0.1]$. Hence $[\nu]_t$ is an $(\in, \in \vee q_{0.8})$-fuzzy subquasigroup of $\mathcal{G}$. □

Problem 1. *Prove or disprove that each nonempty $[\mu]_t$ is an $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$.*

Problem 2. *Find a simple characterization of $[\mu]_t$.*

An $(\in, \in \vee q_m)$-fuzzy subquasigroup of a quasigroup $\mathcal{G}$ is *proper* if Imμ has at least two elements. Two $(\in, \in \vee q_m)$- fuzzy subquasigroups of $\mathcal{G}$ are *equivalent* if they have the same family of level subquasigroups. Otherwise, they are said to be non-equivalent.

Theorem 3.19. *Let μ be a proper $(\in, \in \vee q_m)$-fuzzy subquasigroup μ of $\mathcal{G}$ having at least two values $t_1, t_2 < \frac{1-m}{2}$. If all $[\mu]_t$, $t \in (0, \frac{1-m}{2}]$, are subquasigroups, then μ can be decomposed into the union of two proper non-equivalent $(\in, \in \vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$.*

Proof. Let μ be a proper $(\in, \in \vee q_m)$-fuzzy subquasigroup of $\mathcal{G}$ with values $\frac{1-m}{2} > t_1 > t_2 > \ldots > t_n$, where $n \geqslant 2$. Let $G_0 = [\mu]_{\frac{1-m}{2}}$ and $G_k = [\mu]_{t_k}$ for $k = 1, 2, \ldots, n$. Then $\mu_{\frac{1-m}{2}} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ is the chain of $(\in, \in \vee q_m)$-subquasigroups.

Consider two fuzzy sets λ_1, $\lambda_2 \leqslant \mu$ defined by

$$\lambda_1(x) = \begin{cases} t_1 & \text{if } x \in G_1, \\ t_k, & \text{if } x \in G_k \setminus G_{k-1},\ k = 2, 3, \ldots, n, \end{cases}$$

$$\lambda_2(x) = \begin{cases} \mu(x) & \text{if } x \in G_0, \\ t_2, & \text{if } x \in G_2 \setminus G_0, \\ t_k, & \text{if } x \in G_k \setminus G_{k-1},\ k = 3, \ldots, n. \end{cases}$$

Then λ_1 and λ_2 are $(\in, \in \vee q_m)$-fuzzy subquasigroups of $\mathcal{G}$ with

$$G_1 \subset G_2 \subset \ldots \subset G_n$$

and

$$G_0 \subset G_2 \subset \ldots \subset G_n$$

being respectively chains of $(\in, \in \vee q_m)$-fuzzy subquasigroups. Obviously $\mu = \lambda_1 \cup \lambda_2$. Moreover, λ_1 and λ_2 are non-equivalent since $G_0 \neq G_1$. □

References

[1] **M. Akram**, *Fuzzy subquasigroups with respect to a s-norm*, Bul. Acad. Sci. Republ. Moldova, ser. Mathematica **2** (2008), 3 – 13.

[2] **M. Akram and W. A. Dudek**, *Generalized fuzzy subquasigroups*, Quasigroups and Related Systems **16** (2008), 133 – 146.

[3] **S. K. Bhakat and P. Das**, $(\in, \in \vee q)$-fuzzy subgroup, Fuzzy Sets and Systems **80** (1996), 359 – 368.

[4] **P. Das**, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **85** (1981), 264 – 269.

[5] **W. A. Dudek**, *Fuzzy subquasigroups*, Quasigroups and Related Systems **5** (1998), 81 – 98.

[6] **W. A. Dudek**, *On some old and new problems in n-ary groups*, Quasigroups and Related Systems **8** (2001), 15 – 36.

[7] **W. A. Dudek and Y. B. Jun**, *Fuzzy subquasigroups over a t-norm*, Quasigroups and Related Systems **6** (1999), 87 – 98.

[8] **V. Murali**, *Fuzzy points of equivalent fuzzy subsets.* Information Sciences **158** (2004), 277 – 288.

[9] **A. I. Nesterov**, *Some applications of quasigroups and loops in physics*, Inter. Conference Nonassociative Algebra and its Applications, Mexico, 2003.

[10] **P. M. Pu and Y. M. Liu**, *Fuzzy topology,I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571 – 599.

[11] **I. G. Rosenberg**, *Two properties of fuzzy subquasigroups of a quasigroup*, Fuzzy Sets and Systems **110** (2000), 447 – 450.

[12] **A. Rosenfeld**, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512 – 517.

[13] **L. A. Zadeh**, *Fuzzy sets*, Information Control. **8** (1965), 338 – 353.

[14] **H.-J. Zimmermann**, *Fuzzy set theory and applications*, Kluwer-Nijhoff Publishing, 1985.

Received May 19, 2009

M.Akram
Punjab University College of Information Technology, University of the Punjab, Old Campus, P. O. Box 54000, Lahore, Pakistan.
E-mail: m.akram@pucit.edu.pk

W.A.Dudek
Institute of Mathematics and Computer Science, Wroclaw University of Technology, Wyb. Wyspianskiego 27, 50-370 Wroclaw, Poland.
E-mail: dudek@im.pwr.wroc.pl