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Interval-valued $(\in, \in \vee q_{\widetilde{m}})-$ fuzzy subquasigroups

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Abstract. In this paper we introduce the notion of interval-valued $(\in, \in \vee q_{\widetilde{m}})-$ fuzzy subquasigroups and present some of their properties. We characterize interval-valued $(\in, \in \vee q_{\widetilde{m}})-$ fuzzy subquasigroups by their level subsets. The implication-based such new fuzzy subquasigroups are also established.

1. Introduction

The notion of interval-valued fuzzy sets was first introduced by Zadeh [21] as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Thus, interval-valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany on approximate reasoning [10, 11], Roy and Biswas on medical diagnosis [16] and Turksen on multivalued logic [17].

Murali [12] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [13] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroups, $(\in, \in \vee q)$-fuzzy subgroups, was introduced in earlier paper Bhakat and Das [5] by using the combined notions of *belongines* and *quasi-coincidence* of fuzzy point and fuzzy set. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. On the other hand, Akram and Dudek applied this concept to subquasigroup in [2] and studied some of its properties. Further, it was discussed by same authors in [3]. In this paper we introduce the notion of

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interval-valued $(\in, \in \vee q_m)-$ fuzzy subquasigroups and present some of their properties. We characterize interval-valued $(\in, \in \vee q_m)-$ fuzzy subquasigroups by their level subsets. The implication-based such fuzzy subquasigroups are also established. Some recent results obtained by Akram-Dudek [3] are extended and strengthened.

2. Preliminaries

A groupoid $(G, \cdot)$ is called a *quasigroup* if for any a , $b \in G$ each of the equations $a \cdot x = b$, $x \cdot a = b$ has a unique solution in G. A quasigroup may be also defined as an algebra $(G, \cdot, \backslash, /)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x \cdot y)/y = x, \qquad x \backslash (x \cdot y) = y,$$

$$(x/y) \cdot y = x, \qquad x \cdot (x \backslash y) = y.$$

Such defined quasigroup is called an *equasigroup*.

A nonempty subset S of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *subquasigroup* if it is closed with respect to these three operations.

In this paper $\mathcal{G}$ always denotes an equasigroup $(G, \cdot, \backslash, /)$; G always denotes a nonempty set.

Definition 2.1. An interval number D is an interval $[a^-, a^+]$ with $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Denote the set of all interval numbers by $D[0,1]$. Then the interval $[a, a]$ can be simply identified with the number $a \in [0,1]$. For any two given interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+] \in D[0,1]$, we define

$$\mathrm{rmin}\{D_1, D_2\} = \mathrm{rmin}\{[a_1^-, b_1^+], [a_2^-, b_2^+]\} = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}],$$

$$\mathrm{rmax}\{D_1, D_2\} = \mathrm{rmax}\{[a_1^-, b_1^+], [a_2^-, b_2^+]\} = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}],$$

and take

- $D_1 \leqslant D_2 \iff a_1^- \leqslant a_2^-$ and $b_1^+ \leqslant b_2^+$,
- $D_1 = D_2 \iff a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \iff D_1 \leqslant D_2$ and $D_1 \neq D_2$,
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+]$, where $0 \leqslant k \leqslant 1$.

Then, $(D[0,1], \leqslant, \vee, \wedge)$ forms a complete lattice under set inclusion with $[0,0]$ acts as its least element and $[1,1]$ acts as its greatest element. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+] \in D[0,1]$ we define

- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- a_2^-, b_1^+ + b_2^+ - b_1^+ b_2^+]$.

Definition 2.2. Let G be a given set. Then, the interval-valued fuzzy set (briefly, IF set) A in G is defined by

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in G\}$$

where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy sets of G such that $\mu_A^-(x) \leqslant \mu_A^+(x)$ for all $x \in G$. Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$. Then

$$A = \{(x, \widetilde{\mu}_A(x)) : x \in G\},$$

where $\widetilde{\mu}_A : G \to D[0,1]$.

Definition 2.3. An interval-valued fuzzy set $\widetilde{\mu}$ in a quasigroups $\mathcal{G}$ is called an *interval-valued fuzzy subquasigroup* of $\mathcal{G}$ if the following condition is satisfied:

$$\widetilde{\mu}(x * y) \geq \mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \quad \forall\ x,\ y\ \in G.$$

Definition 2.4. An *interval-valued fuzzy empty set* $\widetilde{0}$ and *interval-valued fuzzy whole set* $\widetilde{1}$ in a set G are defined by $\widetilde{0}(x) = [0,0]$ and $\widetilde{1}(x) = [1,1]$, for all $x \in G$. We write $\widetilde{t} = [t_1, t_2]$ and $\widetilde{s} = [s_1, s_2]$ in the interval $D[0,1]$.

Based on Bhakat and Das [4], we can extend the concept of quasi-coincidence of fuzzy point within a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set as follows:

Definition 2.5. An interval valued fuzzy set $\widetilde{\mu}$ of a quasigroup $\mathcal{G}$ of the form

$$\widetilde{\mu}(y) = \begin{cases} \widetilde{t} \in (\widetilde{0}, \widetilde{1}], & \text{if } y = x \\ \widetilde{0}, & \text{if } y \neq x \end{cases}$$

is called *fuzzy interval value with support* x and interval value $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. A fuzzy interval value $x_{\widetilde{t}}$ is said to be belong to an interval valued fuzzy set $\widetilde{\mu}$ written as $x_{\widetilde{t}} \in \widetilde{\mu}$ if $\widetilde{\mu}(x) \geqslant \widetilde{t}$. A fuzzy interval value $x_{\widetilde{t}}$ is said to be quasi-coincident with an interval valued fuzzy set $\widetilde{\mu}$ written as $x_{\widetilde{t}} q \widetilde{\mu}$ if $\widetilde{\mu}(x) + \widetilde{t} > \widetilde{1}$.

Let m be an element of $[0,1)$ and let $\widetilde{m}$ be an element of $D[0,1)$ unless otherwise specified. By $x_{\widetilde{t}}q_{\widetilde{m}}\widetilde{\mu}$, we mean $\widetilde{\mu}(x)+\widetilde{t}+\widetilde{m}>\widetilde{1}$, $\widetilde{t}\in D(0,\frac{1-m}{2}]$. For brevity, we write the following notions:

- $x_{\widetilde{t}}\in\widetilde{\mu}$ or $x_{\widetilde{t}}q_{\widetilde{m}}\widetilde{\mu}$ will be denoted by $x_{\widetilde{t}}\in\vee q_{\widetilde{m}}\widetilde{\mu}$.
- $x_{\widetilde{t}}\in\widetilde{\mu}$ and $x_{\widetilde{t}}q_{\widetilde{m}}\widetilde{\mu}$ will be denoted by "$x_{\widetilde{t}}\in\wedge q_{\widetilde{m}}\widetilde{\mu}$."
- The symbol $\overline{\in\wedge q_{\widetilde{m}}}$ means neither $\in$ nor $q_{\widetilde{m}}$ hold.

3. Interval-valued $(\in,\in\vee q_m)-$fuzzy subquasigroups

Definition 3.1. An interval-valued fuzzy set $\widetilde{\mu}$ in G is called an interval-valued $(\in,\in\vee q_{\widetilde{m}})$-*fuzzy subquasigroup* of $\mathcal{G}$, if

$$x_{\widetilde{t}_1},y_{\widetilde{t}_2}\in\widetilde{\mu}\Longrightarrow(x*y)_{\mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\}}\in\vee q_{\widetilde{m}}\widetilde{\mu}$$

for all $x,y\in G$, $\widetilde{t}_1,\widetilde{t}_2\in D(0,1]$ and $*\in\{\cdot,\backslash,/\}$.

Note that an interval-valued $(\in,\in\vee q_{\widetilde{m}})$-*fuzzy subquasigroup* with $m=0$ is called *an interval-valued* $(\in,\in\vee q)$-*fuzzy subquasigroup*.

Example 3.2. Let $G=\{0,a,b,c\}$ be a quasigroup with the following multiplication table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(i) Consider an interval-valued fuzzy set

$$\widetilde{\mu}(x)=\begin{cases}[0.65,0.7], & \text{if } x=0,\\ [0.75,0.8] & \text{if } x=a,\\ [0.35,0.4] & \text{if } x=b,\\ [0.35,0.4] & \text{if } x=c.\end{cases}$$

If $m=0.15$, then $U(\widetilde{\mu};\widetilde{t})=G$ for all $\widetilde{t}\in D(0,0.4]$. Hence $\widetilde{\mu}$ is an interval-valued $(\in,\in\vee q_{[0.15,0.15]})$-fuzzy subquasigroup of $\mathcal{G}$.

(ii) Now consider an interval-valued fuzzy set

$$\widetilde{\mu}(x) = \begin{cases} [0.42, 0.45] & \text{if } \ x = 0, \\ [0.40, 0.41] & \text{if } \ x = a, \\ [0.40, 0.41] & \text{if } \ x = c, \\ [0.47, 0.49] & \text{if } \ x = b. \end{cases}$$

In this case for $m = 0.04$ we have

$$U(\widetilde{\mu}; t) = \begin{cases} G & \text{if } \ t \in D(0, 0.4], \\ \{0, b\} & \text{if } \ t \in D(0.4, 0.45], \\ \{b\} & \text{if } \ t \in D(0.45, 0.48]. \end{cases}$$

Since $\{b\}$ is not a subquasigroup of $\mathcal{G}$, so $U(\widetilde{\mu}; t)$ is not a subquasigroup for $t \in D(0.45, 0.48]$. Hence $\widetilde{\mu}$ is not an interval-valued $(\in, \in \vee q_{[0.04,0.04]})$-fuzzy subquasigroup of a quasigroup $\mathcal{G}$. □

We now formulate a technical characterization.

Theorem 3.3. *An interval-valued fuzzy set $\widetilde{\mu}$ in $\mathcal{G}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$ if and only if*

$$\widetilde{\mu}(x * y) \geqslant \text{rmin}\Big\{\widetilde{\mu}(x), \widetilde{\mu}(y), \Big[\frac{1-m}{2}, \frac{1-m}{2}\Big]\Big\} \tag{1}$$

holds for all x, $y \in G$.

Proof. Let $\widetilde{\mu}$ be an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. Assume that (1) is not valid. Then there exist x', $y' \in G$ such that

$$\widetilde{\mu}(x' * y') < \text{rmin}\Big\{\widetilde{\mu}(x'), \widetilde{\mu}(y'), \Big[\frac{1-m}{2}, \frac{1-m}{2}\Big]\Big\}.$$

If $\text{rmin}(\widetilde{\mu}(x'), \widetilde{\mu}(y')) < [\frac{1-m}{2}, \frac{1-m}{2}]$, then $\widetilde{\mu}(x' * y') < \text{rmin}(\widetilde{\mu}(x'), \widetilde{\mu}(y'))$. Thus

$$\widetilde{\mu}(x' * y') < \widetilde{t} \leqslant \text{rmin}\{\widetilde{\mu}(x'), \widetilde{\mu}(y')\} \qquad \text{for some } \widetilde{t} \in D(0, 1].$$

It follows that $x'_{\widetilde{t}} \in \widetilde{\mu}$ and $y'_{\widetilde{t}} \in \widetilde{\mu}$, but $(x' * y')_{\widetilde{t}} \overline{\in} \widetilde{\mu}$, a contradiction. Moreover, $\widetilde{\mu}(x' * y') + \widetilde{t} < 2\widetilde{t} < [1-m, 1-m]$, and so $(x' * y')_{\widetilde{t}} \overline{q_{\widetilde{m}}} \widetilde{\mu}$. Hence, consequently $(x' * y')_{\widetilde{t}} \overline{\in \vee q_{\widetilde{m}}} \widetilde{\mu}$, a contradiction.

On the other hand, if $\mathrm{rmin}\{\widetilde{\mu}(x'),\widetilde{\mu}(y')\} \geqslant [\frac{1-m}{2},\frac{1-m}{2}]$, then

$\widetilde{\mu}(x') \geqslant [\frac{1-m}{2},\frac{1-m}{2}]$, $\widetilde{\mu}(y') \geqslant [\frac{1-m}{2},\frac{1-m}{2}]$ and $\widetilde{\mu}(x' * y') < [\frac{1-m}{2},\frac{1-m}{2}]$.

Thus $x'_{[\frac{1-m}{2},\frac{1-m}{2}]} \in \widetilde{\mu}$ and $y'_{[\frac{1-m}{2},\frac{1-m}{2}]} \in \widetilde{\mu}$, but $(x' * y')_{[\frac{1-m}{2},\frac{1-m}{2}]} \overline{\in} \widetilde{\mu}$. Also

$$\widetilde{\mu}(x' * y') + \left[\frac{1-m}{2},\frac{1-m}{2}\right] < \left[\frac{1-m}{2},\frac{1-m}{2}\right] + \left[\frac{1-m}{2},\frac{1-m}{2}\right] = [1-m,1-m],$$

i.e., $(x' * y')_{[\frac{1-m}{2},\frac{1-m}{2}]} \overline{q}_{\widetilde{m}} \widetilde{\mu}$. Hence $(x' * y')_{[\frac{1-m}{2},\frac{1-m}{2}]} \overline{\in \vee q_{\widetilde{m}}} \widetilde{\mu}$, a contradiction. So (1) is valid.

Conversely, assume that $\widetilde{\mu}$ satisfies (1). Let $x, y \in G$ and $\widetilde{t}_1, \widetilde{t}_2 \in D(0,1]$ be such that $x_{\widetilde{t}_1}\widetilde{\mu}$ and $y_{\widetilde{t}_2} \in \widetilde{\mu}$. Then

$$\widetilde{\mu}(x*y) \geqslant \mathrm{rmin}\Big\{\widetilde{\mu}(x),\widetilde{\mu}(y),\left[\frac{1-m}{2},\frac{1-m}{2}\right]\Big\} \geqslant \mathrm{rmin}\Big\{\widetilde{t}_1,\widetilde{t}_2,\left[\frac{1-m}{2},\frac{1-m}{2}\right]\Big\}.$$

Assume that $\widetilde{t}_1 \leqslant [\frac{1-m}{2},\frac{1-m}{2}]$ or $\widetilde{t}_2 \leqslant [\frac{1-m}{2},\frac{1-m}{2}]$. Then $\widetilde{\mu}(x * y) \geqslant \mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\}$, which implies that $(x * y)_{\mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\}} \in \widetilde{\mu}$. Now suppose that $\widetilde{t}_1 > [\frac{1-m}{2},\frac{1-m}{2}]$ and $\widetilde{t}_2 > [\frac{1-m}{2},\frac{1-m}{2}]$. Then $\widetilde{\mu}(x * y) \geqslant [\frac{1-m}{2},\frac{1-m}{2}]$, and thus

$$\widetilde{\mu}(x*y) + \mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\} > \left[\frac{1-m}{2},\frac{1-m}{2}\right] + \left[\frac{1-m}{2},\frac{1-m}{2}\right] = [1-m,1-m],$$

i.e., $(x * y)_{\mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\}} q_{\widetilde{m}} \widetilde{\mu}$. Hence $(x * y)_{\mathrm{rmin}\{\widetilde{t}_1,\widetilde{t}_2\}} \in \vee q_{\widetilde{m}} \widetilde{\mu}$, and consequently, $\widetilde{\mu}$ is an interval-valued $(\in, \in\vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. □

The following Corollary follows when $m = 0$.

Corollary 3.4. *An interval-valued fuzzy set $\widetilde{\mu}$ in $\mathcal{G}$ is an interval-valued $(\in, \in \vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if*

$$\widetilde{\mu}(x * y) \geqslant \mathrm{rmin}\Big\{\widetilde{\mu}(x),\widetilde{\mu}(y)\Big\}$$

holds for all x, $y \in G$. □

Theorem 3.5. *An interval-valued fuzzy set $\widetilde{\mu}$ of G is an interval-valued $(\in, \in\vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$ if and only if each nonempty level set $U(\widetilde{\mu};\widetilde{t})$, $\widetilde{t} \in D(0, \frac{1-m}{2}]$, is a subquasigroup of $\mathcal{G}$.*

Proof. Assume that $\widetilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. Let $\widetilde{t} \in D(0, \frac{1-m}{2}]$ and $x, y \in U(\widetilde{\mu}; \widetilde{t})$. Then $\widetilde{\mu}(x) \geqslant \widetilde{t}$ and $\widetilde{\mu}(y) \geqslant \widetilde{t}$. It follows from Condition (1) that

$$\widetilde{\mu}(x*y) \geqslant \mathrm{rmin}\Big\{\widetilde{\mu}(x), \widetilde{\mu}(y), \Big[\frac{1-m}{2}, \frac{1-m}{2}\Big]\Big\} \geqslant \mathrm{rmin}\Big\{\widetilde{t}, \Big[\frac{1-m}{2}, \frac{1-m}{2}\Big]\Big\} = \widetilde{t},$$

so that $x * y \in U(\widetilde{\mu}; \widetilde{t})$. Hence $U(\widetilde{\mu}; \widetilde{t})$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$.

Conversely, suppose that the nonempty set $U(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$ for all $\widetilde{t} \in D(0, \frac{1-m}{2}]$. If the condition(1) is not true, then there exists a, $b \in G$ such that $\widetilde{\mu}(a * b) < \mathrm{rmin}\{\widetilde{\mu}(a), \widetilde{\mu}(b), [\frac{1-m}{2}, \frac{1-m}{2}]\}$. Hence we can take $\widetilde{t} \in D(0, 1]$ such that $\widetilde{\mu}(a * b) < \widetilde{t}_1 < \mathrm{rmin}\{\widetilde{\mu}(a), \widetilde{\mu}(b), [\frac{1-m}{2}, \frac{1-m}{2}]\}$. Then $\widetilde{t} \in D(0, \frac{1-m}{2}]$ and $a, b \in U(\widetilde{\mu}; \widetilde{t})$. Since $U(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$, it follows that $a*b \in U(\widetilde{\mu}; \widetilde{t})$, so $\widetilde{\mu}(a*b) \geqslant \widetilde{t}$. This is a contradiction. Therefore the condition (1) is valid, and so $\widetilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. □

We induce the following Corollary by putting $m = 0$.

Corollary 3.6. *An interval-valued fuzzy set $\widetilde{\mu}$ of G is an interval-valued $(\in, \in \vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if each nonempty level set $U(\widetilde{\mu}; \widetilde{t})$, $\widetilde{t} \in D(0, 1]$, is a subquasigroup of $\mathcal{G}$.* □

Theorem 3.7. *Let $\widetilde{\mu}$ be an interval-valued fuzzy set of a quasigroup $\mathcal{G}$. Then the nonempty level set $U(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$ for all $\widetilde{t} \in D(\frac{1-m}{2}, 1]$ if and only if*

$$\mathrm{rmax}\Big\{\widetilde{\mu}(x * y), \Big[\frac{1-m}{2}, \frac{1-m}{2}\Big]\Big\} \geqslant \mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$$

for all $x, y \in G$.

Proof. Suppose that $U(\widetilde{\mu}; \widetilde{t}) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$. Assume that $\mathrm{rmax}\{\widetilde{\mu}(x*y), [\frac{1-m}{2}, \frac{1-m}{2}]\} < \mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} = \widetilde{t}$ for some x, $y \in G$, then $\widetilde{t} \in D(\frac{1-m}{2}, 1]$, $\widetilde{\mu}(x*y) < \widetilde{t}$, $x \in U(\widetilde{\mu}; \widetilde{t})$ and $y \in U(\widetilde{\mu}; \widetilde{t})$. Since x, $y \in U(\widetilde{\mu}; \widetilde{t})$, $U(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$, so $x * y \in U(\widetilde{\mu}; \widetilde{t})$, a contradiction.

The proof of the second part of Theorem is straightforward. □

The following Corollary follows when $m = 0$.

Corollary 3.8. *Let $\widetilde{\mu}$ be an interval-valued fuzzy set of a quasigroup $\mathcal{G}$. Then for every $\widetilde{t} \in D(0.5, 1]$ each nonempty level set $U(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$ if and only if*

$$\mathrm{rmax}\Big\{\widetilde{\mu}(x * y)\Big\} \geqslant \mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$$

for all $x, y \in G$. □

Theorem 3.9. *For any finite strictly increasing chain of subquasigroups of $\mathcal{G}$ there exists an interval-valued$(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup $\widetilde{\mu}$ of $\mathcal{G}$ whose level subquasigroups are precisely the members of the chain with $\widetilde{\mu}_{[\frac{1-m}{2}, \frac{1-m}{2}]} = G_0 \subset G_1 \subset \ldots \subset G_n = G$.*

Proof. Let $\{\widetilde{t}_i \,|\, \widetilde{t}_i \in D(0, \frac{1-m}{2}], i = 1, \ldots, n\}$ be such that $[\frac{1-m}{2}, \frac{1-m}{2}] > \widetilde{t}_1 > \widetilde{t}_2 > \widetilde{t}_3 > \ldots > \widetilde{t}_n$. Consider the interval-valued fuzzy set $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(x) = \begin{cases} [\frac{1-m}{2}, \frac{1-m}{2}] & \text{if } \; x \in G_0, \\ \widetilde{t}_k & \text{if } \; x \in G_k \setminus G_{k-1},\; k = 1, \ldots, n \end{cases}$$

Let $x, y \in G$ be such that $x \in G_i \setminus G_{i-1}$ and $y \in G_j \setminus G_{j-1}$, where $1 \leqslant i, j \leqslant n$. We consider the following cases:
Case I: when $i \geqslant j$, then $x \in G_i$, $y \in G_i$, so $x * y \in G_i$. Thus

$$\widetilde{\mu}(x * y) \geqslant \widetilde{t}_i = \mathrm{rmin}\{\widetilde{t}_i, \widetilde{t}_j\} = \mathrm{rmin}\Big\{\widetilde{\mu}(x), \widetilde{\mu}(y), \left[\frac{1-m}{2}, \frac{1-m}{2}\right]\Big\}.$$

Case II: when $i < j$, then $x \in G_j$, $y \in G_j$, so $x * y \in G_j$. Thus

$$\widetilde{\mu}(x * y) \geqslant t_j = \mathrm{rmin}\{t_i, t_j\} = \mathrm{rmin}\Big\{\widetilde{\mu}(x), \widetilde{\mu}(y), \left[\frac{1-m}{2}, \frac{1-m}{2}\right]\Big\}.$$

Hence $\widetilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. □

The following Corollary follows when $m = 0$.

Corollary 3.10. *For any finite strictly increasing chain of subquasigroups of $\mathcal{G}$ there exists an interval-valued $(\in, \in \vee q)$-fuzzy subquasigroup $\widetilde{\mu}$ of $\mathcal{G}$ whose level subquasigroups are precisely the members of the chain with $\widetilde{\mu}_{[0.5, 0.5]} = G_0 \subset G_1 \subset \ldots \subset G_n = G$.* □

Definition 3.11. For an interval-valued fuzzy set $\widetilde{\mu}$ in $\mathcal{G}$ and $\widetilde{t} \in D(0, 1]$, we define four sets:

(a) $Q(\widetilde{\mu};\widetilde{t}) = \{x \in G \mid x_{\widetilde{t}} q \widetilde{\mu}\}$,

(b) $Q^m(\widetilde{\mu};\widetilde{t}) = \{x \in G \mid x_{\widetilde{t}} q_{\widetilde{m}}\, \widetilde{\mu}\}$,

(c) $[\widetilde{\mu}]_{\widetilde{t}} = \{x \in G \mid x_{\widetilde{t}} \in \vee q\, \widetilde{\mu}\}$,

(d) $[\widetilde{\mu}]_t^m = \{x \in G \mid x_{\widetilde{t}} \in \vee q_{\widetilde{m}}\, \mu\}$.

It is clear that $[\widetilde{\mu}]_{\widetilde{t}}^m = U(\widetilde{\mu};\widetilde{t}) \cup Q^m(\widetilde{\mu};\widetilde{t})$.

Example 3.12. Let $G = \{0, a, b, c\}$ be a quasigroup which is given in Example 3.2. Consider interval-valued fuzzy sets

$$\widetilde{\mu}(x) = \begin{cases} [0.65,\ 0.67] & \text{if } x = 0, \\ [0.54,\ 0.56] & \text{if } x = a, \\ [0.45,\ 0.47] & \text{if } x = b, \\ [0.39,\ 0.41] & \text{if } x = c, \end{cases} \qquad \widetilde{\nu}(x) = \begin{cases} [0.58,\ 0.60] & \text{if } x = 0, \\ [0.03,\ 0.05] & \text{if } x = a, \\ [0.48,\ 0.50] & \text{if } x = b, \\ [0.04,\ 0.06] & \text{if } x = c. \end{cases}$$

(1) When $m = 0.6$, then $U(\widetilde{\mu};\widetilde{t}) = G$ and $Q(\widetilde{\mu};\widetilde{t}) = G$ for all $t \in D(0, 0.2]$. Thus $[\widetilde{\mu}]_{\widetilde{t}} = G$ for all $\widetilde{t} \in D(0, 0.2]$. Hence $[\widetilde{\mu}]_{\widetilde{t}}$ is an interval-valued $(\in, \in \vee q_{[0.6,0.6]})$-fuzzy subquasigroup of $\mathcal{G}$.

(2) When $m = 0.8$, then $U(\widetilde{\nu};\widetilde{t}) = G$ and $Q(\widetilde{\nu};\widetilde{t}) = \{0, b\}$ for all $\widetilde{t} \in D(0, 0.1]$. Thus $[\widetilde{\nu}]_{\widetilde{t}} = G$ for all $\widetilde{t} \in D(0, 0.1]$. Hence $[\widetilde{\nu}]_{\widetilde{t}}$ is an interval-valued $(\in, \in \vee q_{[0.8,0.8]})$-fuzzy subquasigroup of $\mathcal{G}$. □

We formulate a nice characterization.

Theorem 3.13. *An interval-valued fuzzy set $\widetilde{\mu}$ of $\mathcal{G}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$ if and only if for every $\widetilde{t} \in D(\frac{1-m}{2}, 1]$ each nonempty level $Q^m(\widetilde{\mu};\widetilde{t})$ is a subquasigroup of $\mathcal{G}$.*

Proof. Assume that $\widetilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$ and let $\widetilde{t} \in D(\frac{1-m}{2}, 1]$ be such that $Q^m(\widetilde{\mu};\widetilde{t}) \neq \emptyset$. Let $x,\ y \in Q^m(\widetilde{\mu};\widetilde{t})$. Then $x_{\widetilde{t}} q_{\widetilde{m}} \widetilde{\mu}$ and $y_{\widetilde{t}} q_{\widetilde{m}} \widetilde{\mu}$, i.e., $\widetilde{\mu}(x) + \widetilde{t} + \widetilde{m} > \widetilde{1}$ and $\widetilde{\mu}(y) + \widetilde{t} + \widetilde{m} > \widetilde{1}$. Using Theorem 3.3, we have

$$\widetilde{\mu}(x * y) \geqslant \text{rmin}\left\{\widetilde{\mu}(x), \widetilde{\mu}(y), \left[\frac{1-m}{2}, \frac{1-m}{2}\right]\right\}$$

$$\widetilde{\mu}(x * y) \geqslant \text{rmin}\,\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \quad \text{if } \text{rmin}\,\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \geqslant \left[\frac{1-m}{2}, \frac{1-m}{2}\right],$$

$$\widetilde{\mu}(x*y) \geqslant \left[\frac{1-m}{2}, \frac{1-m}{2}\right] \quad \text{if } \operatorname{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} < \left[\frac{1-m}{2}, \frac{1-m}{2}\right]$$

that is, $(x*y)_{\widetilde{t}}\, q_{\widetilde{m}}\, \widetilde{\mu}$. So $x*y \in Q^m(\widetilde{\mu}; \widetilde{t})$. Hence $Q^m(\widetilde{\mu}; \widetilde{t})$ is a subquasigroup of $\mathcal{G}$.

The proof of the sufficiency part is straightforward and is hence omitted. This completes the proof. □

Open problem. Prove or disprove that the following characterization is true.

An interval-valued fuzzy set $\widetilde{\mu}$ *of* $\mathcal{G}$ *is an interval-valued* $(\in, \in \vee q_m)$*-fuzzy subquasigroup of* $\mathcal{G}$ *if and only if for every* $\widetilde{t} \in D(\frac{1-m}{2}, 1]$ *each nonempty level* $[\widetilde{\mu}]^m_{\widetilde{t}}$ *is an subquasigroup of* $\mathcal{G}$.

4. Implication-based new fuzzy subquasigroups

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\vee$; $\wedge$; $\neg$; $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition p is denoted by $[p]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

1. $[x \in p] = p(x)$,
2. $[p \wedge q] = \min\{[p], [q]\}$,
3. $[p \rightarrow q] = \min\{1, 1 - [p] + [q]\}$,
4. $[\forall x p(x)] = \inf_{x \in U}\{p(x)\}$,
5. $\models p$ if and only if $[p] = 1$ for all valuations.

The truth valuation rules given in (4) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following:

A. Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(x, y) := \begin{cases} 1 & \text{if } x \leqslant y, \\ 0 & \text{otherwise} . \end{cases}$$

B. Gödel implication operator (I_G):

$$I_G(x, y) := \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise} . \end{cases}$$

C. The contraposition of Gödel implication operator $(\overline{I}_G)$:

$$\overline{I}_G(x, y) := \begin{cases} 1 & \text{if } x \leqslant y, \\ 1-x & \text{otherwise} . \end{cases}$$

Ying [19] introduced the concept of fuzzifying topology. We can extend this concept to a quasigroup, and we define an interval-valued fuzzfying subquasigroup as follows:

Definition 4.1. An interval-valued fuzzy set $\widetilde{\mu}$ in G is called an *interval-valued fuzzifying subquasigroup* of $\mathcal{G}$ if

$$\models \min\{[x \in \widetilde{\mu}], [y \in \widetilde{\mu}]\} \to [x * y \in \widetilde{u}]$$

for any x, $y \in G$.

Obviously, Definition 4.1 is equivalent to the Definition 2.3. Hence an interval-valued fuzzifying subquasigroup is a fuzzy subquasigroup. Ying [18] introduced the concept of t-topology, i.e., $\models_t p$ if and only if $[p] \geqslant t$ for all valuations. We give the definition of $\widetilde{t}$-implication-based subquasigroup.

Definition 4.2. Let $\widetilde{\mu}$ be an interval-valued fuzzy set of G and $\widetilde{t} \in D(0, 1]$. Then $\widetilde{\mu}$ is called a *$\widetilde{t}$-implication-based subquasigroup* of $\mathcal{G}$ if for any $x, y \in G$ $\models_{\widetilde{t}} \min\{[x \in \widetilde{\mu}], [y \in \widetilde{\mu}]\} \to [x * y \in \widetilde{\mu}]$.

The following proposition is obvious.

Proposition 4.3. *Let I be an implication operator. An interval-valued fuzzy set $\widetilde{\mu}$ of $\mathcal{G}$ is a $\widetilde{t}$-implication based interval-valued fuzzy subquasigroup of $\mathcal{G}$ if and only if $I(\mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}, \widetilde{\mu}(x * y)) \geqslant \widetilde{t}$ for all x, $y \in G$.* $\square$

We now formulate characterizations of implication-based interval-valued fuzzy subquasigroups.

Theorem 4.4. *Let $\widetilde{\mu}$ be an interval-valued fuzzy set in G. If $I = I_G$, then $\widetilde{\mu}$ is a $[\frac{1-m}{2}, \frac{1-m}{2}]$-implication- based interval-valued fuzzy subquasigroup of $\mathcal{G}$ if and only if $\widetilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Suppose that $\widetilde{\mu}$ is a $[\frac{1-m}{2},\frac{1-m}{2}]$-implication based subquasigroup of $\mathcal{G}$. Then

(*i*) $I_G(\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\},\widetilde{\mu}(x*y))\geqslant[\frac{1-m}{2},\frac{1-m}{2}]$ for all $x,\, y\in G$.

(*i*) implies that
$\widetilde{\mu}(x*y)\geqslant\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$ or $\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}\geqslant\widetilde{\mu}(x*y)\geqslant[\frac{1-m}{2},\frac{1-m}{2}]$.
It follows that

$$\widetilde{\mu}(x*y)\geqslant\min\left\{\widetilde{\mu}(x),\widetilde{\mu}(y),\left[\frac{1-m}{2},\frac{1-m}{2}\right]\right\}.$$

From Theorem 3.3, it follows that $\widetilde{\mu}$ is an interval-valued $(\in,\in\vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$.

Conversely, suppose that $\widetilde{\mu}$ is an interval-valued $(\in,\in\vee q_{\widetilde{m}})$-fuzzy subquasigroup of $\mathcal{G}$. From Theorem 3.3, if $\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y),[\frac{1-m}{2},\frac{1-m}{2}]\}=\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, then

$$I_G(\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y),\widetilde{\mu}(x*y)\}=\widetilde{1}\geqslant\left[\frac{1-m}{2},\frac{1-m}{2}\right].$$

Otherwise, $I_G(\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y),\widetilde{\mu}(x*y)\}\geqslant[\frac{1-m}{2},\frac{1-m}{2}]$. Hence $\widetilde{\mu}$ is a $[\frac{1-m}{2},\frac{1-m}{2}]$–implication based subquasigroup of $\mathcal{G}$. □

Theorem 4.5. *Let $\widetilde{\mu}$ be an interval-valued fuzzy set in G. If $I=\overline{I}_G$, then $\widetilde{\mu}$ is a $[\frac{1-m}{2},\frac{1-m}{2}]$-implication-based interval-valued fuzzy subquasigroup of $\mathcal{G}$ if and only if $\widetilde{\mu}$ satisfies the following assertion for all $x,\, y\in G$:*

(*ii*) $\mathrm{rmax}\{\widetilde{\mu}(x),[\frac{1-m}{2},\frac{1-m}{2}]\}\geqslant\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y),\widetilde{1}\}$.

Proof. Suppose that $\widetilde{\mu}$ is a $\left[\frac{1-m}{2},\frac{1-m}{2}\right]$–implication based interval-valued fuzzy subquasigroup of $\mathcal{G}$. Then

(*iii*) $\overline{I}_G(\min\{\widetilde{\mu}(x*y),\widetilde{\mu}(x),\widetilde{\mu}(y)\})\geqslant[\frac{1-m}{2},\frac{1-m}{2}]$ for all $x,y\in G$.

From (*iii*), it follows that $\overline{I}_G(\mathrm{rmin}\{\widetilde{\mu}(x*y),\widetilde{\mu}(x),\widetilde{\mu}(y)\})=\widetilde{1}$, that is, $\widetilde{\mu}(x*y)\geqslant\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$ or $\widetilde{1}-\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}\geqslant[\frac{1-m}{2},\frac{1-m}{2}]$, i.e., $\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}\leqslant[\frac{1-m}{2},\frac{1-m}{2}]$.
Thus

$$\mathrm{rmax}\left\{\widetilde{\mu}(x*y),\left[\frac{1-m}{2},\frac{1-m}{2}\right]\right\}\geqslant\mathrm{rmin}\{\widetilde{\mu}(x),\widetilde{\mu}(y),\widetilde{1}\}.$$

Hence $\widetilde{\mu}$ satisfies (*ii*).

The proof of converse part is obvious. □

Theorem 4.6. *Let $\widetilde{\mu}$ be an interval-valued fuzzy set in G. If $I = I_{GR}$, then $\widetilde{\mu}$ is a* $[0.5, 0.5]$*-implication- based interval-valued fuzzy subquasigroup of $\mathcal{G}$ if and only if $\widetilde{\mu}$ is an interval-valued fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Obvious. $\square$

Corollary 4.7. *Let $I = I_G$. Then $\widetilde{\mu}$ is a* $[0.5, 0.5]$*–implication-based interval-valued fuzzy subquasigroup of a quasigroup $\mathcal{G}$ if and only if $\widetilde{\mu}$ is an interval-valued* $(\in, \in \vee q_{\widetilde{m}})$*- fuzzy subquasigroup of $\mathcal{G}$.* $\square$

Corollary 4.8. *Let $I = \overline{I}_G$. Then $\widetilde{\mu}$ is a* $[0.5, 0.5]$*–implication-based interval-valued fuzzy subquasigroup of a quasigroup $\mathcal{G}$ if and only if $\widetilde{\mu}$ satisfies the following conditions:*

$$\mathrm{rmax}\{\widetilde{\mu}(x * y), [0.5, 0.5]\} \geqslant \mathrm{rmin}\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{1}\}$$

for all $x, y \in G$. $\square$

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