Quasigroups and Related Systems **18** (2010), 187 – 194

Para-associative groupoids

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Abstract. We study properties of left (right) division (cancellative) groupoids with associative-like identities: $x \cdot yz = zx \cdot y$ and $x \cdot zy = xy \cdot z$.

1. Introduction

A quasigroup can be defined as an algebra $(Q, \cdot)$ with one binary operation in which some equations are uniquely solvable or as an algebra $(Q, \cdot, \backslash, /)$ with three binary operations satisfying some identities. The first definition is motivated by Latin squares, the second – by universal algebras. In the case of quasigroups various connections between these three operations are well described.

In this note we describe connections between these three operations in para-associative division groupoids, i.e., left (right) division groupoids satisfying some identities similar to the associativity.

By the proving of many results given in this paper we have used Prover9-Mace4 prepared by W. McCune [7].

2. Basic facts and definitions

By a *binary groupoid* $(Q, \cdot)$ we mean a non-empty set Q together with a binary operation denoted by juxtaposition. Dots will be only used to avoid repetition of brackets. For example, the formula $((xy)(zy))(xz) = (xz)z$ will be written in the abbreviated form as $(xy \cdot zy) \cdot xz = xz \cdot z$. In this notion the associative law has the form

$$x \cdot yz = xy \cdot z. \tag{1}$$

2010 Mathematics Subject Classification: 20N02, 20N05
Keywords: groupoid, right division groupoid, left division groupoid, right cancellative groupoid, left cancellative groupoid, quasigroup, abelian group.

If we permute the arguments in each side of (1) we can obtain 16 new equations. Hosszú observed (see [5]) that all these equations can be reduced to one of the following four cases: (1),

$$x \cdot yz = z \cdot yx, \tag{2}$$

$$x \cdot yz = y \cdot xz, \tag{3}$$

$$x \cdot yz = zx \cdot y. \tag{4}$$

Unfortunately Hosszú gives only two examples of such reductions.

Example 2.1. The equation $yz \cdot x = yx \cdot z$ is equivalent to $x * (z * y) = z * (x * y)$, where $t * s = st$. □

Example 2.2. If in the identity

$$x \cdot zy = xy \cdot z \tag{5}$$

(called by Hosszú – *Tarki's associative law*) we put $z = x$ and replace xy by t, we obtain $xt = tx$. Hence, in groupoids $(Q, \cdot)$ in which each element $t \in Q$ can be written in the form xy, $x, y \in Q$, (5) implies each of the equations (1) – (4). □

M. A. Kazim and M. Naseeruddin considered in [6] the following laws:

$$xy \cdot z = zy \cdot x \tag{6}$$

$$x \cdot yz = z \cdot yx. \tag{7}$$

Groupoids satisfying (6) are called *left almost semigroups* (*LA-semigroups*), groupoids satisfying (7) are called *right almost semigroups* (*RA-semigroups*).

All these identities are strongly connected with para-associative rings. Namely, a non-associative ring R is *para-associative* of type (i, j, k) (cf. [2] or [4]) or an(i, j, k)-*associative ring*, if $x_1x_2 \cdot x_3 = x_i \cdot x_jx_k$ is valid for all $x_1, x_2, x_3 \in R$, where (i, j, k) is a fixed permutation of the set $\{1, 2, 3\}$.

As usual, the map $L_a : Q \to Q$, $L_ax = ax$ for all $x \in Q$, is a *left translation*, the map $R_a : Q \to Q$, $R_ax = xa$, is a *right translation*.

A groupoid $(Q, \cdot)$ is a *left cancellation groupoid*, if $ax = ay$ implies $x = y$ for all $a, x, y \in Q$, i.e., if L_a is an injective map for every $a \in Q$. Similarly, $(Q, \cdot)$ is a *right cancellation groupoid*, if $xa = ya$ implies $x = y$ for all $a, x, y \in G$, i.e., if R_a is an injective map for every $a \in Q$. A *cancellation groupoid* is a groupoid which is both a left and right cancellation groupoid.

By a *left division groupoid* (shortly: *ld-groupoid*) we mean a groupoid in which all left translations L_x are surjective. A *right division groupoid* (shortly: *rd-groupoid*) is a groupoid in which all right translations R_x are surjective. If all L_x and all R_x are surjective, then we say that such groupoid is a *division groupoid*.

Example 2.3. Let $(\mathbb{Z},+,\cdot)$ be the ring of integers. Consider on $\mathbb{Z}$ two operations: $x\circ y = x+3y$ and $x*y=[x/2]+3y$. It is possible to check that $(\mathbb{Z},\circ)$ is a left cancellation groupoid, $(\mathbb{Z},*)$ is a left cancellation right division groupoid. □

Definition 2.4. A groupoid $(Q,\circ)$ is called a *right quasigroup* (a *left quasigroup*) if, for all $a,b\in Q$, there exists a unique solution $x\in Q$ of the equation $x\circ a=b$ (respectively: $a\circ x=b$), i.e., if all right (left) translations of $(Q,\circ)$ are bijective maps of Q.

A groupoid which is a left and right quasigroup is called a *quasigroup*. A quasigroup with the identity is called a *loop*.

T. Evans [3] proved that a quasigroup $(Q,\cdot)$ can be considered as an equationally defined algebra. Namely, he proved

Theorem 2.5. *A groupoid $(Q,\cdot)$ is a quasigroup if and only if $(Q,\cdot,\backslash,/)$ is an algebra with three binary operations $\cdot$, $\backslash$ and $/$ satisfying the following four identities:*

$$x\cdot(x\backslash y)=y, \tag{8}$$

$$(y/x)\cdot x=y, \tag{9}$$

$$x\backslash(x\cdot y)=y, \tag{10}$$

$$(y\cdot x)/x=y. \tag{11}$$

Another characterization of quasigroups was given by G. Birkhoff in [1].

Theorem 2.6. *A groupoid $(Q,\cdot)$ is a quasigroup if and only if $(Q,\cdot,\backslash,/)$ is an algebra with three binary operations $\cdot$, $\backslash$ and $/$ satisfying the identities* $(8)-(11)$ *and*

$$(x/y)\backslash x=y, \tag{12}$$

$$y/(x\backslash y)=x. \tag{13}$$

In the case of groupoids connections between these three operations are described in [8] and [9]. Namely, the following theorem is true.

Theorem 2.7. *Let $(Q, \cdot)$ be an arbitrary groupoid. Then*

1. *$(Q, \cdot)$ is a left division groupoid if and only if there exists a left cancellation groupoid $(Q, \backslash)$ such that an algebra $(Q, \cdot, \backslash)$ satisfies* (8),
2. *$(Q, \cdot)$ is a right division groupoid if and only if there exists a right cancellation groupoid $(Q, /)$ such that an algebra $(Q, \cdot, /)$ satisfies* (9),
3. *$(Q, \cdot)$ is a left cancellation groupoid if and only if there exists a left division groupoid $(Q, \backslash)$ such that an algebra $(Q, \cdot, \backslash)$ satisfies* (10),
4. *$(Q, \cdot)$ is a right cancellation groupoid if and only if there exists a right division groupoid $(Q, /)$ such that an algebra $(Q, \cdot, /)$ satisfies* (11).

3. Cyclic associative law

In this section we study various groupoids satisfying the cyclic associative law (4).

Theorem 3.1. *A right division groupoid $(Q, \cdot, /)$ satisfying* (4) *is an associative and commutative division groupoid.*

Proof. By Theorem 2.7 such groupoid satisfies (9). Hence

$$yz \cdot (x/y) \overset{(4)}{=} z \cdot (x/y)y \overset{(9)}{=} zx.$$

Using just proved identity, we obtain

$$xy \cdot z \overset{(4)}{=} y \cdot zx = y \cdot (yz \cdot (x/y)) \overset{(4)}{=} (x/y)y \cdot yz \overset{(9)}{=} x \cdot yz,$$

which proves the associativity. Moreover, for all $x, y \in Q$ we have

$$xy \overset{(9)}{=} x \cdot (y/z)z \overset{(4)}{=} zx \cdot (y/z) \overset{(1)}{=} z \cdot x(y/z) \overset{(4)}{=} (y/z)z \cdot x \overset{(9)}{=} yx.$$

So, $(Q, \cdot)$ is associative and commutative division groupoid. □

Corollary 3.2. *A right cancellation rd-groupoid $(Q, \cdot, /)$ satisfying* (4) *is a commutative group with respect to the operation $\cdot$ and satisfies the identities* (2) − (4).

Proof. By the previous theorem such groupoid is a commutative division groupoid. Since it also is a cancellation groupoid, it is a commutative group. Obviously it satisfies (2) − (4). □

Theorem 3.3. *A left cancellation rd-groupoid* $(Q,\cdot,\backslash,/)$ *satisfying* (4) *is a commutative group with respect to the operation* $\cdot$ *and satisfies the identities* (2) – (4).

Proof. By Theorem 2.7 such groupoid satisfies (9) and (10). Hence

$$xy \stackrel{(9)}{=} (x/x)x\cdot y \stackrel{(4)}{=} x\cdot y(x/x).$$

from this we obtain $x\backslash(xy) = y(x/x)$, which, in view of (9), gives

$$y = y(x/x). \tag{14}$$

So, for all $x, y \in Q$, we have

$$y\backslash y = x/x \tag{15}$$

Thus

$$y \stackrel{(9)}{=} (y/y)y \stackrel{(15)}{=} (x\backslash x)y \stackrel{(15)}{=} (x/x)y.$$

This, together with (14), shows that $e = x/x = x\backslash x$ is the identity of $(Q,\cdot)$.

Since

$$xy = xy\cdot e \stackrel{(4)}{=} y\cdot ex = yx.$$

$(Q,\cdot)$ is a commutative loop. Hence $xy\cdot z = yx\cdot z = x\cdot zy = x\cdot yz$, which means that it is a commutative group. Obviously it satisfies (2) – (4). □

Theorem 3.4. *A left division groupoid* $(Q,\cdot,\backslash)$ *satisfying* (4) *is a commutative division groupoid.*

Proof. By Theorem 2.7, such groupoid satisfies (8). Hence

$$zx \stackrel{(8)}{=} y(y\backslash z)\cdot x \stackrel{(4)}{=} (y\backslash z)\cdot xy.$$

Using just proved identity, we obtain

$$x\cdot yz \stackrel{(4)}{=} zx\cdot y = ((y\backslash z)\cdot xy)\cdot y \stackrel{(4)}{=} xy\cdot y(y\backslash z) \stackrel{(8)}{=} xy\cdot z,$$

which proves the associativity. Moreover, for all $x, y \in Q$ we have

$$xy \stackrel{(8)}{=} z(z\backslash x)\cdot y \stackrel{(4)}{=} (z\backslash x)\cdot yz \stackrel{(1)}{=} (z\backslash x)y\cdot z \stackrel{(4)}{=} y\cdot z(z\backslash x) \stackrel{(8)}{=} yx.$$

So, $(Q,\cdot)$ is associative and commutative division groupoid. □

Corollary 3.5. *A left cancellation ld-groupoid* $(Q,\cdot,\backslash)$ *satisfying* (4) *is a commutative group with respect to the operation* $\cdot$ *and satisfies the identities* (2) – (4).

Proof. By the previous theorem such groupoid is a commutative division groupoid. Since it also is a cancellation groupoid, it is a commutative group. Obviously it satisfies the identities (2) – (4). □

Theorem 3.6. *A right cancellation ld-groupoid* $(Q,\cdot,\backslash,/)$ *satisfying* (4) *is a commutative group with respect to the operation* $\cdot$ *and satisfies the identities* (2) – (4).

Proof. The proof is very similar to the proof of Theorem 3.3. □

4. Groupods in which x·zy = xy·z

Lemma 4.1. *A left division groupoid* $(Q,\cdot,\backslash)$ *satisfying* (5) *is commutative and associative.*

Proof. By Theorem 2.7 such groupoid satisfies (8). Hence

$$xy \stackrel{(8)}{=} y(y\backslash x)\cdot y \stackrel{(5)}{=} y\cdot y(y\backslash x) \stackrel{(8)}{=} yx$$

for all $x,y\in Q$. The associativity is obvious. □

Theorem 4.2. *A left cancellation ld-groupoid* $(Q,\cdot,\backslash)$ *satisfying* (5) *is a commutative group with the identity* $e = x\backslash x$ *and satisfies* (2) – (4).

Proof. Indeed, $xy \stackrel{(8)}{=} x(x\backslash x)\cdot y \stackrel{(5)}{=} x\cdot y(x\backslash x)$, which implies $y = y(x\backslash x)$. □

Corollary 4.3. *In a right cancellation ld-groupoid* $(Q,\cdot,\backslash,/)$ *satisfying* (5) *we have* $x\backslash y = y/x$ *for all* $x,y\in Q$.

Proof. By Lemma 4.1 such groupoid is commutative. Hence $y = xz = zx$ implies $x\backslash y = y/x$. □

Theorem 4.4. *A right cancellation ld-groupoid* $(Q,\cdot,\backslash,/)$ *satisfying* (5) *is a commutative group with respect to the operation* $\cdot$ *and satisfies the identities* (2) – (4).

Proof. By Lemma 4.1 such groupoid is associative and commutative. Hence it also is left cancellative. Theorem 4.2 completes the proof. □

Lemma 4.5. *A left cancellation groupoid* $(Q,\cdot,\backslash)$ *satisfying* (5) *is associative and commutative.*

Proof. In fact, using (5), we obtain

$$u(xy\cdot z)=uz\cdot xy=(uz\cdot y)x=(u\cdot yz)x=u(x\cdot yz).$$

This, by the left cancellativity, implies the associativity. Therefore,

$$x\cdot yz=xy\cdot z\overset{(5)}{=}x\cdot zy,$$

which shows that $(Q,\cdot)$ is also commutative. □

Theorem 4.6. *A left cancellation rd-groupoid* $(Q,\cdot,\backslash)$ *satisfying* (5) *is a commutative group with respect to the operation* $\cdot$ *and satisfies the identities* (2) − (4).

Proof. By Lemma 4.5 such groupoid is commutative. Hence it is a left division groupoid, too. Theorem 4.2 completes the proof. □

Theorem 4.7. *A right division groupoid* $(Q,\cdot,/)$ *satisfying* (5) *is associative and satisfies the identity* $x(y/y)=x$.

Proof. By Theorem 2.7 it satisfies (9). Hence

$$y\overset{(9)}{=}(x/y)y\overset{(9)}{=}(x/y)\cdot(y/y)y\overset{(5)}{=}(x/y)y\cdot(y/y)\overset{(9)}{=}x(y/y).$$

Let $e=y/y$. Then $xe=x$ for every $x\in Q$ and

$$xy\cdot z=(xy\cdot z)e\overset{(5)}{=}xy\cdot ez\overset{(5)}{=}x(ez\cdot y)\overset{(5)}{=}x(e\cdot yz)\overset{(5)}{=}(x\cdot yz)e=x\cdot yz,$$

which completes the proof. □

Note that a right cancellation rd-groupoid satisfying (5) may not be a group. A non-empty set Q with the multiplication defined by $xy=x$ is a simple example of a non-commutative right cancellation rd-groupoid without two-sided identity.

Acknowledgment: The author thanks to V. A. Shcherbacov for their useful comments and suggestions.

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Received July 22, 2010
Revised November 26, 2010

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