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Hemirings characterized by the properties of their fuzzy ideals with thresholds

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Abstract. We define fuzzy h-subhemiring, fuzzy h-ideals and fuzzy generalized h-bi-ideals with thresholds, and characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their fuzzy h-ideals, fuzzy h-bi-ideals and fuzzy h-quasi-ideals with thresholds.

1. Introduction

Semirings are algebraic structures with two binary operations, introduced by Vandiver [23]. In more recent times semirings have been deeply studied, especially in relation with applications [10]. Semirings have also been used for studying optimization, graph theory, theory of discrete event, dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes [9, 24]. Hemirings, which are semirings with commutative addition and zero element, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in computer sciences [10, 19]

Ideals of hemirings and semirings play a central role in the structure theory and are useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings apparantly have no analogues in hemirings using only ideals. In [11] Henriksen defined a more restricted class of ideals in semirings, called k-ideals, with the property that if the semiring R is the ring, then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now h-ideals, has been given and investigated by

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Izuka [12] and La Torre [16].

The theory of fuzzy sets was first developed by Zadeh [26] in 1965, and has been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [22] and he introduced the notion of fuzzy subgroups. In [3] J. Ahsan initiated the study of fuzzy semirings(See also [2]), fuzzy k-ideals in semirings are studied in [8], and fuzzy h-ideals are studied in [13, 17, 27]. The fuzzy algebraic structures play an important role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [1, 10, 24].

The notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets proposed and discussed in [20, 21]. Many authors used these concepts to generalize some concepts of algebra, for example [4, 5, 6, 14]. In [7, 18] (α, β)-fuzzy ideals of hemirings are defined.

In this paper we define fuzzy h-subhemiring, fuzzy h-ideal and fuzzy generalized h-bi-ideals with thresholds, and characterize h-hemiregular and h-intra-hemiregular hemiring by the properties of their fuzzy h-ideals, fuzzy h-bi-ideals, fuzzy generalized h-bi-ideals, fuzzy h-quasi-ideals with thresholds.

2. Preliminaries

A *semiring* is a set $R \neq \emptyset$ together with two binary operations addition and multiplication such that $(R, +)$ and $(R, \cdot)$ are semigroups and both algebraic structures are connected by the distributive laws:

$$a\,(b + c) = ab + ac \qquad \text{and} \qquad (a + b)\,c = ac + bc$$

for all $a, b, c \in R$.

An element $0 \in R$ is called a *zero* of the semiring $(R, +, \cdot)$ if $0x = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$. An additively commutative semiring with zero is called a *hemiring*. An element 1 of a hemiring R is called the *identity* of R if $1x = x1 = x$ for all $x \in R$. A hemiring with commutative multiplication is called a *commutative hemiring*. A non-empty subset A of a hemiring R is called a *subhemiring* of R if it contains zero and is closed with respect to the addition and multiplication of R. A non-empty subset I of a hemiring R is called a *left* (*right*) *ideal* of R if I is closed under addition and $RI \subseteq I$ ($IR \subseteq I$). A non-empty subset I of a hemiring R is called an *ideal* of R if it is both a left ideal and a right ideal of R. A non-empty subset

Q of a hemiring R is called a *quasi-ideal* of R if Q is closed under addition and $RQ \cap QR \subseteq Q$. A subhemiring B of a hemiring R is called a *bi-ideal* of R if $BSB \subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal but the converse is not true.

A left (right) ideal I of a hemiring R is called a *left* (*right*) *h-ideal* if for all $x, z \in R$ and for any $a, b \in I$ from $x + a + z = b + z$ it follows $x \in I$. A bi-ideal B of a hemiring R is called an *h-bi-ideal* of R if for all $x, z \in R$ and $a, b \in B$ from $x + a + z = b + z$ it follows $x \in B$ [25].

The *h-closure* $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \text{ for some } a, b \in A, z \in R\}.$$

A quasi-ideal Q of a hemiring R is called an *h-quasi-ideal* of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x + a + z = b + z$ implies $x \in Q$, for all $x, z \in R$ and $a, b \in Q$ [25]. Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general.

A fuzzy subset f of a universe X is a function from X into the unit closed interval $[0,1]$, that is $f : X \to [0,1]$. A fuzzy subset f in a universe X of the form

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f in a set X, Pu and Liu [21] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geqslant t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ (resp. $x_t \in f$ and $x_t q f$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. Let f be a fuzzy subset of R and $t \in (0,1]$ then the set $U(f;t) = \{x \in R : f(x) \geqslant t\}$ is called the *level subset* of R. For any two fuzzy subsets f and g of X, $f \leqslant g$ means that, for all $x \in X$, $f(x) \leqslant g(x)$. The symbols $f \wedge g$, and $f \vee g$ will mean the following fuzzy subsets of X

$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \qquad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for all $x \in X$. More generally, if $\{f_i : i \in \Lambda\}$ is a family of fuzzy subsets of X, then $\wedge_{i \in \Lambda} f_i$ and $\vee_{i \in \Lambda} f_i$ are defined by

$$(\bigwedge_{i \in \Lambda} f_i)(x) = \min_{i \in \Lambda}\{f_i(x)\}, \qquad (\bigvee_{i \in \Lambda} f_i)(x) = \max_{i \in \Lambda}\{f_i(x)\}$$

and are called the *intersection* and the *union* of the family $\{f_i : i \in \Lambda\}$ of fuzzy subsets of X, respectively.

Definition 2.1. Let f and g be two fuzzy subsets in a hemiring R. The *h-intrinsic product* of f and g is defined by

$$(f \odot g)(x) = \sup\left\{ \bigwedge_{i=1}^{m} \left(f(a_i) \wedge g(b_i)\right) \wedge \bigwedge_{j=1}^{n} \left(f(a'_j) \wedge g(b'_j)\right)\right\}$$

if $x \in R$ can be expressed as $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$, and 0 otherwise.

Proposition 2.2. [25] *Let R be a hemiring and f, g, h, k be any fuzzy subsets of R. If $f \leqslant g$ and $h \leqslant k$, then $f \odot h \leqslant g \odot k$.* □

Lemma 2.3. [25] *Let R be a hemiring and $A, B \subseteq R$. Then we have*

(i) $A \subseteq B \Leftrightarrow \chi_A \leqslant \chi_B$,
(ii) $\chi_A \wedge \chi_B = \chi_{A \cap B}$,
(iii) $\chi_A \odot \chi_B = \chi_{\overline{AB}}$. □

Definition 2.4. A fuzzy subset f in a hemiring R is called a *fuzzy h-sub-hemiring* of R if for all $x, y, z, a, b \in R$ we have

(i) $f(x + y) \geqslant \min\{f(x), f(y)\}$,
(ii) $f(xy) \geqslant \min\{f(x), f(y)\}$,
(iii) $x + a + z = b + z \Rightarrow f(x) \geqslant \min\{f(a), f(b)\}$.

Definition 2.5. A fuzzy subset f in a hemiring R is called a *fuzzy left* (*right*) *h-ideal* of R if for all $x, y, z, a, b \in R$ we have

(i) $f(x + y) \geqslant \min\{f(x), f(y)\}$,
(ii) $f(xy) \geqslant f(y) \quad (f(xy) \geqslant f(x))$,
(iii) $x + a + z = b + z \Rightarrow f(x) \geqslant \min\{f(a), f(b)\}$.

A fuzzy subset f of R is called a *fuzzy h-ideal* of R if it is both a fuzzy left and a fuzzy right h-ideal of R.

Definition 2.6. [25] A fuzzy subset f in a hemiring R is called a *fuzzy h-bi-ideal* of R if for all $x, y, z, a, b \in R$ we have

(i) $f(x + y) \geqslant \min\{f(x), f(y)\}$,
(ii) $f(xy) \geqslant \min\{f(x), f(y)\}$,
(iii) $f(xyz) \geqslant \min\{f(x), f(z)\}$,
(iv) $x + a + z = b + z \Rightarrow f(x) \geqslant \min\{f(a), f(b)\}$.

Definition 2.7. [25] A fuzzy subset f in a hemiring R is called a *fuzzy h-quasi-ideal* of R if for all $x, y, z, a, b \in R$ we have

(i) $f(x+y) \geqslant \min\{f(x), f(y)\}$,

(ii) $(f \odot \mathcal{R}) \wedge (\mathcal{R} \odot f) \leqslant f$,

(iii) $x+a+z=b+z \Rightarrow f(x) \geqslant \min\{f(a), f(b)\}$,

where $\mathcal{R}$ is the fuzzy subset of R mapping every element of R on 1.

Note that if f is a fuzzy left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal), then $f(0) \geqslant f(x)$ for all $x \in R$.

Definition 2.8. [25] A hemiring R is said to be *h-hemiregular* if for each $x \in R$, there exist $a, b, z \in R$ such that $x + xax + z = xbx + z$.

Lemma 2.9. [25] *A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.* □

Definition 2.10. [25] A hemiring R is said to be *h-intra-hemiregular* if for each $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=i}^{n} b_j x^2 b_j' + z$.

Lemma 2.11. [25] *A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.* □

Lemma 2.12. [25] *The following conditions are equivalent.*

(i) *R is both h-hemiregular and h-intra-hemiregular hemiring,*

(ii) *$B = \overline{B^2}$ for every h-bi-ideal B of R,*

(iii) *$Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.* □

3. Fuzzy ideals with thresholds (α, β)

In this section we will discuss fuzzy h-subhemiring, fuzzy h-ideals, fuzzy h-bi-ideals, fuzzy generalized h-bi-ideals and fuzzy h-quasi-ideals with thresholds (α, β) of a hemiring R.

Definition 3.1. Let $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$. Then a fuzzy subset f of a hemiring R is called a *fuzzy h-subhemiring with thresholds* (α, β) of R if it satisfies

(1) $\max\{f(x+y), \alpha\} \geqslant \min\{f(x), f(y), \beta\}$,

(2) $\max\{f(xy), \alpha\} \geqslant \min\{f(x), f(y), \beta\}$,

(3) $x+a+z=b+z \Rightarrow \max\{f(x), \alpha\} \geqslant \min\{f(a), f(b), \beta\}$

for all $x, y, z, a, b \in R$.

Definition 3.2. Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy subset f of a hemiring R is called a *fuzzy left* (resp. *right*) *h-ideal with thresholds* (α, β) of R if it satisfies $(1), (3)$ and

(4) $\max\{f(xy), \alpha\} \geqslant \min\{f(y), \beta\}$

(resp. $\max\{f(xy), \alpha\} \geqslant \min\{f(x), \beta\}$)

for all $x, y \in R$.

A fuzzy subset f of a hemiring R is called a *fuzzy h-ideal with thresholds* (α, β) of R if it is both fuzzy left and fuzzy right h-ideal with thresholds (α, β) of R.

Definition 3.3. [18] Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy subset f of a hemiring R is called a *fuzzy h-bi-ideal with thresholds* (α, β) of R if it satisfies $(1), (2), (3)$ and

(5) $\max\{f(xzy), \alpha\} \geqslant \min\{f(x), f(y), \beta\}$

for all $x, y, z, \in R$.

Definition 3.4. Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy subset f of a hemiring R is called a *fuzzy generalized h-bi-ideal with thresholds* (α, β) of R if it satisfies (3) and (5).

Definition 3.5. [18] Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy subset f of a hemiring R is called fuzzy h-quasi-ideal with thresholds (α, β) of R if it satisfies (1), (3) and

(6) $\max\{f(x), \alpha\} \geqslant \min\{(f \odot \mathcal{R})(x), (\mathcal{R} \odot f)(x), \beta\}$

for all $x \in R$, where $\mathcal{R}$ is the fuzzy subset of R mapping every element of R into 1.

As a simple consequence of the Transfer Principle for fuzzy sets proved in [15] we obtain

Theorem 3.6. *A fuzzy subset f of a hemiring R is a fuzzy h-subhemiring with thresholds (α, β) of R if and only if $U(f;t) \neq \emptyset$ is h-subhemiring of R for all $t \in (\alpha, \beta)$.* $\square$

Theorem 3.7. *A fuzzy subset f of a hemiring R is a fuzzy left h-ideal (right h-ideal, h-ideal, generalized h-bi-ideal, h-bi-ideal, h-quasi-ideal) with thresholds (α, β) of R if and only if $U(f;t) \neq \emptyset$ is a left h-ideal (right h-ideal, h-ideal, generalized h-bi-ideal, h-bi-ideal, h-quasi-ideal) of R for all $t \in (\alpha, \beta)$.* $\square$

Theorem 3.8. *A non-empty subset A of a hemiring R is h-ideal (h-bi-ideal, generalized h-bi-ideal, h-quasi-ideal) of R if and only if the characteristic function χ_A is fuzzy h-ideal (h-bi-ideal, generalized h-bi-ideal, h-quasi-ideal) of R with thresholds (α, β) of R for all $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$.* □

Theorem 3.9. *Let f be a fuzzy h-bi-ideal with thresholds (α, β) of R , then $f \wedge \beta$ is fuzzy h-bi-ideal with thresholds (α, β) of R.*

Proof. Let $a, b, x, y, z \in R$. Then $(f \wedge \beta)(x) = f(x) \wedge \beta$ for all $x \in R$ and

$$\begin{aligned}\max\{(f \wedge \beta)(x+y), \alpha\} &= \max\{f(x+y) \wedge \beta, \alpha\} \\ &= \min\{\max\{f(x+y), \alpha\}, \beta\} \geqslant \min\{f(x), f(y), \beta\} \\ &= \min\{(f \wedge \beta)(x), (f \wedge \beta)(y), \beta\}.\end{aligned}$$

Similarly we can show that

$$\max\{(f \wedge \beta)(xy), \alpha\} \geqslant \min\{(f \wedge \beta)(x), (f \wedge \beta)(y), \beta\} \quad \text{and}$$
$$\max\{(f \wedge \beta)(xzy), \alpha\} \geqslant \min\{(f \wedge \beta)(x), (f \wedge \beta)(y), \beta\}.$$

Now let $x + a + z = b + z$, then

$$\begin{aligned}\max\{(f \wedge \beta)(x), \alpha\} &= \max\{f(x) \wedge \beta, \alpha\} = \min\{\max(f(x), \alpha), \beta\} \\ &\geqslant \min\{f(a), f(b), \beta\} = \min\{(f \wedge \beta)(a), (f \wedge \beta)(b), \beta\}\end{aligned}$$

This shows that $f \wedge \beta$ is a fuzzy h-bi-ideal with thresholds (α, β) of R. □

Similarly we can show:

Theorem 3.10. *Let f be a fuzzy h-bi-ideal (h-subhemiring, generalized h-bi-ideal, h-ideal, h-quasi-ideal) with thresholds (α, β) of R, then $f \wedge \beta$ is a fuzzy h-bi-ideal (h-subhemiring, generalized h-bi-ideal, h-ideal, h-quasi-ideal) with thresholds (α, β) of R.* □

Definition 3.11. Let f, g be fuzzy subsets of a hemiring R. Then for all $\in R$ we define

$$(f \wedge_\alpha^\beta g)(x) = \{(f \wedge g)(x) \wedge \beta\} \vee \alpha,$$
$$(f \vee_\alpha^\beta g)(x) = \{(f \vee g)(x) \wedge \beta\} \vee \alpha,$$
$$(f \odot_\alpha^\beta g)(x) = \{(f \odot g)(x) \wedge \beta\} \vee \alpha,$$
$$(f +_\alpha^\beta g)(x) = (\sup\{f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2)\} \wedge \beta) \vee \alpha$$

for all possible expressions of x in the form $x + (a_1 + b_1) + z = (a_2 + b_2) + z$.

Lemma 3.12. *Let A, B be subsets of R, then*

$$(\chi_A +_\alpha^\beta \chi_B)(x) = (\chi_{\overline{A+B}}(x) \wedge \beta) \vee \alpha.$$

Proof. Let A, B be subsets of a hemiring R and $x \in R$. If $x \in \overline{A+B}$ then there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x+(a_1+b_1)+z = (a_2+b_2)+z$ for some $z \in R$. Thus

$$\begin{aligned}(\chi_A +_\alpha^\beta \chi_B)(x) &= (\sup\{\chi_A(a_1') \wedge \chi_A(a_2') \wedge \chi_B(b_1') \wedge \chi_B(b_2')\} \wedge \beta) \vee \alpha \\ &= (1 \wedge \beta) \vee \alpha = (\chi_{\overline{A+B}}(x) \wedge \beta) \vee \alpha.\end{aligned}$$

If $x \notin \overline{A+B}$ then there do not exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ for some $z \in R$. Thus $(\chi_A +_\alpha^\beta \chi_B)(x) = (0 \wedge \beta) \vee \alpha = (\chi_{\overline{A+B}}(x) \wedge \beta) \vee \alpha$. Hence $(\chi_A +_\alpha^\beta \chi_B)(x) = (\chi_{\overline{A+B}}(x) \wedge \beta) \vee \alpha$. □

Lemma 3.13. *A fuzzy subset f of a hemiring R satisfies* (1) *and* (3) *if and only if it satisfies*

(7) $f +_\alpha^\beta f \leqslant (f \wedge \beta) \vee \alpha$.

Proof. Let f satisfies (1), (3) and $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ for some $a_1, a_2, b_1, b_2, z \in R$. Then

$$\begin{aligned}(f +_\alpha^\beta f)(x) &= (\sup\{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \wedge \beta) \vee \alpha \\ &= (\sup\{(f(a_1) \wedge f(b_1) \wedge \beta) \wedge (f(a_2) \wedge f(b_2) \wedge \beta)\} \wedge \beta) \vee \alpha \\ &\leqslant (\sup\{(f(a_1 + b_1) \vee \alpha) \wedge (f(a_2 + b_2) \vee \alpha)\} \wedge \beta) \vee \alpha \\ &= (\sup\{\{f(a_1 + b_1) \wedge f(a_2 + b_2)\} \vee \alpha\} \wedge \beta) \vee \alpha \\ &= (\sup\{f(a_1 + b_1) \wedge f(a_2 + b_2) \wedge \beta\} \vee \alpha) \vee \alpha \\ &= (\sup\{f(a_1 + b_1) \wedge f(a_2 + b_2) \wedge \beta\} \wedge \beta) \vee \alpha \\ &\leqslant ((f(x) \vee \alpha) \wedge \beta) \vee \alpha \\ &= (f(x) \wedge \beta) \vee \alpha.\end{aligned}$$

Thus $f +_\alpha^\beta f \leqslant (f \wedge \beta) \vee \alpha$.

Conversely, assume that $(f +_\alpha^\beta f)(x) \leqslant (f \wedge \beta)(x) \vee \alpha$. Then for each $x, z \in R$ we have $0 + x + x + z = x + x + z$. Hence

$$\begin{aligned}f(0) \vee \alpha \geqslant (f \wedge \beta)(0) \vee \alpha &\geqslant (f +_\alpha^\beta f)(0) \\ &= (\sup\{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \wedge \beta) \vee \alpha \\ &\geqslant \sup\{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \wedge \beta \geqslant f(x) \wedge \beta.\end{aligned}$$

This means that for all $x \in R$ we have

$$f(0) \vee \alpha \geqslant f(x) \wedge \beta. \qquad (*)$$

Let $x, y \in R$. Then for all $a_1, a_2, b_1, b_2, z \in R$ such that $(x + y) + (a_1 + b_1) + z = (a_2 + b_2) + z$, we have

$$\begin{aligned}\max\{f(x+y), \alpha\} \geqslant \max\{(f \wedge \beta)(x+y), \alpha\} &\geqslant (f +_\alpha^\beta f)(x+y) \\ &= (\sup\{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \wedge \beta) \vee \alpha \\ &\geqslant (\{f(0) \wedge f(x) \wedge f(0) \wedge f(y)\} \wedge \beta) \vee \alpha,\end{aligned}$$

because $(x + y) + (0 + 0) + 0 = (x + y) + 0$.

From the above using$(*)$ we get $\max\{f(x+y),\alpha\} \geqslant \min\{f(x), f(y), \beta\}$, which proves (1).

Now let $a, b, x, z \in R$ be such that $x+a+z = b+z$. Then for all possible $a_1, a_2, b_1, b_2, z \in R$ satisfying the identity $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ we have

$$\begin{aligned}\max\{f(x),\alpha\} \geqslant \max\{(f\wedge\beta)(x),\alpha\} &\geqslant (f +_\alpha^\beta f)(x)\\ &= (\sup\{f(a_1)\wedge f(a_2)\wedge f(b_1)\wedge f(b_2)\}\wedge\beta)\vee\alpha\\ &\geqslant f(a)\wedge f(b)\wedge\beta \qquad\qquad \text{because } x+a+z=b+z\\ &= \min\{f(a), f(b), \beta\}.\end{aligned}$$

Thus f satisfies (3). □

Theorem 3.14. *A fuzzy subset f of a hemiring R is a fuzzy left (resp. right) h-ideal with thresholds (α,β) of R if and only if it satisfies (7) and*

(8) $\mathcal{R}\odot_\alpha^\beta f \leqslant (f\wedge\beta)\vee\alpha$ *(resp. $f\odot_\alpha^\beta \mathcal{R} \leqslant (f\wedge\beta)\vee\alpha$).*

Proof. Suppose f is a fuzzy left h-ideal with thresholds (α,β) of a hemiring R, then by Lemma 3.13, f satisfies (7). Now we show that f satisfies (8). Let $x \in R$. If $(\mathcal{R}\odot_\alpha^\beta f)(x) = 0$, then $\mathcal{R}\odot_\alpha^\beta f \leqslant (f\wedge\beta)\vee\alpha$. Otherwise, there exist elements $a_i, b_i, c_j, d_j, z \in R$ such that $x+\sum_{i=1}^m a_ib_i+z = \sum_{j=1}^n c_jd_j+z$. Thus

$$\begin{aligned}(\mathcal{R}\odot_\alpha^\beta f)(x) &= \Big(\sup\Big\{\bigwedge_{i=1}^m\Big(\mathcal{R}(a_i)\wedge f(b_i)\Big)\wedge\bigwedge_{j=1}^n\Big(\mathcal{R}(a_j')\wedge f(b_j')\Big)\Big\}\wedge\beta\Big)\vee\alpha\\ &= \Big(\sup\Big\{\bigwedge_{i=1}^m f(b_i)\wedge\bigwedge_{j=1}^n f(b_j')\Big\}\wedge\beta\Big)\vee\alpha\\ &= \Big(\sup\Big\{\Big(\bigwedge_{i=1}^m f(b_i)\wedge\beta\Big)\wedge\Big(\bigwedge_{j=1}^n f(b_j')\wedge\beta\Big)\Big\}\wedge\beta\Big)\vee\alpha\\ &\leqslant \Big(\sup\Big\{\bigwedge_{i=1}^m\Big(f(a_ib_i)\vee\alpha\Big)\wedge\bigwedge_{j=1}^n\Big(f(a_j'b_j')\vee\alpha\Big)\Big\}\wedge\beta\Big)\vee\alpha\\ &= \Big(\sup\Big\{\Big(\bigwedge_{i=1}^m f(a_ib_i)\wedge\beta\Big)\wedge\Big(\bigwedge_{j=1}^n f(a_j'b_j')\wedge\beta\Big)\Big\}\wedge\beta\Big)\vee\alpha\\ &\leqslant \Big(\sup\Big\{\Big(f\big(\textstyle\sum_{i=1}^m a_ib_i\big)\vee\alpha\Big)\wedge\Big(f\big(\textstyle\sum_{j=1}^n a_j'b_j'\big)\vee\alpha\Big)\Big\}\wedge\beta\Big)\vee\alpha\\ &\leqslant (f(x)\wedge\beta)\vee\alpha.\end{aligned}$$

This implies that $\mathcal{R}\odot_\alpha^\beta f \leqslant (f\wedge\beta)\vee\alpha$.

Conversely, assume that f satisfies (7) and (8). Then, by Lemma 3.13, it satisfies (1) and (3). To show that f satisfies (4) let $x, y \in R$ and $a_i, b_i, c_j, d_j, z \in R$ be such that $xy+\sum_{i=1}^m a_ib_i+z = \sum_{j=1}^n c_jd_j+z$. Then we have

$$f(xy)\vee\alpha \geqslant (f(xy)\wedge\beta)\vee\alpha \geqslant (\mathcal{R}\odot_\alpha^\beta f)(xy)$$

$$= \Big(\sup\Big\{\bigwedge_{i=1}^{m}\Big(\mathcal{R}(a_i)\wedge f(b_i)\Big)\wedge\bigwedge_{j=1}^{n}\Big(\mathcal{R}(c_j)\wedge f(d_j)\Big)\Big\}\wedge\beta\Big)\vee\alpha$$
$$= \Big(\sup\Big\{\bigwedge_{i=1}^{m} f(b_i)\wedge\bigwedge_{j=1}^{n} f(d_j)\Big\}\wedge\beta\Big)\vee\alpha$$
$$\geqslant (f(y)\wedge\beta)\vee\alpha \geqslant f(y)\wedge\beta$$

because $xy + 0y + z = xy + z$.

This shows that f satisfies (4). So f is a fuzzy left h-ideal with thresholds (α,β) of R.

For fuzzy right h-ideals the proof is similar. □

Theorem 3.15. *A fuzzy subset f of a hemiring R is a fuzzy h-quasi-ideal with thresholds (α,β) of R if and only if f satisfies* (6) *and* (7)*.*

Proof. Proof is straightforward because by Lemma 3.13, (1) and (3) are equivalent to (7). □

Theorem 3.16. *Every fuzzy left h-ideal with thresholds (α,β) of a hemiring R is a fuzzy h-quasi-ideal with thresholds (α,β) of R.*

Proof. Proof is straightforward because (8) implies (6). □

Theorem 3.17. *Every fuzzy h-quasi-ideal with thresholds (α,β) of R is a fuzzy h-bi-ideal with thresholds (α,β) of R.* □

Lemma 3.18. *If f and g are fuzzy right and left h-ideals with thresholds (α,β) of R respectively, then $f\odot_\alpha^\beta g\leqslant f\wedge_\alpha^\beta g$.*

Proof. Let $x\in R$. If $(f\odot_\alpha^\beta g)(x)=\alpha$, then $f\odot_\alpha^\beta g\leqslant f\wedge_\alpha^\beta g$. Otherwise, there exist elements $a_i, b_i, c_j, d_j, z\in R$ such that $x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z$. Then for all such expressions we have

$$(f\odot_\alpha^\beta g)(x) = \Big(\sup\Big\{\bigwedge_{i=1}^{m}\Big(f(a_i)\wedge g(b_i)\Big)\wedge\bigwedge_{j=1}^{n}\Big(f(a_j')\wedge g(b_j')\Big)\Big\}\wedge\beta\Big)\vee\alpha$$
$$= \left(\sup\left\{\begin{array}{l}\bigwedge_{i=1}^{m}\Big\{\Big(f(a_i)\wedge\beta\Big)\wedge\Big(g(b_i)\wedge\beta\Big)\Big\}\wedge\\ \bigwedge_{j=1}^{n}\Big\{\Big(f(a_j')\wedge\beta\Big)\wedge\Big(g(b_j')\wedge\beta\Big)\Big\}\end{array}\right\}\wedge\beta\right)\vee\alpha$$
$$\leqslant \left(\sup\left\{\begin{array}{l}\bigwedge_{i=1}^{m}\Big\{\Big(f(a_ib_i)\vee\alpha\Big)\wedge\Big(g(a_ib_i)\vee\alpha\Big)\Big\}\wedge\\ \bigwedge_{j=1}^{n}\Big\{\Big(f(a_j'b_j')\vee\alpha\Big)\wedge\Big(g(a_j'b_j')\vee\alpha\Big)\Big\}\end{array}\right\}\wedge\beta\right)\vee\alpha$$

$$= \left(\sup \left\{ \begin{array}{c} \bigwedge\limits_{i=1}^{m} \left\{ \Big(f\left(a_i b_i\right) \wedge \beta \Big) \wedge \Big(g\left(a_i b_i\right) \wedge \beta \Big) \right\} \wedge \\ \bigwedge\limits_{j=1}^{n} \left\{ \Big(f(a'_j b'_j) \wedge \beta \Big) \wedge \Big(g(a'_j b'_j) \wedge \beta \Big) \right\} \end{array} \right\} \wedge \beta \right) \vee \alpha$$

$$\leqslant \left(\sup \left\{ \max \left\{ \begin{array}{l} f(\sum_{i=1}^{m} a_i b_i), g(\sum_{i=1}^{m} a_i b_i), \\ f(\sum_{j=1}^{n} a'_j b'_j), g(\sum_{j=1}^{n} a'_j b'_j), \alpha \end{array} \right\} \right\} \wedge \beta \right) \vee \alpha$$

$$\leqslant (f \wedge_{\alpha}^{\beta} g)(x),$$

because $\min\{f(x), f(y), \beta\} \leqslant \{f(x+y, \alpha)\}$. □

4. h-hemiregular hemirings

In this section we characterize h-hemiregular hemirings by the properties of their h-ideals, h-quasi-ideals and h-bi-ideals with thresholds (α, β).

Theorem 4.1. *For a hemiring R the following conditions are equivalent:*

(i) *R is h-hemiregular,*

(ii) *$f \wedge_{\alpha}^{\beta} g = f \odot_{\alpha}^{\beta} g$ for every fuzzy right and left h-ideals f and g with thresholds (α, β) of R, respectively.*

Proof. $(i) \Rightarrow (ii)$ Let $x \in R$, then there exists $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Now for all $x, a_i, b_i, c_j, d_j, z \in R$ such that $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z$, we have

$$(f \odot_{\alpha}^{\beta} g)\,(x) = \left(\sup \left\{ \bigwedge_{i=1}^{m} \Big(f\left(a_i\right) \wedge g\left(b_i\right) \Big) \wedge \bigwedge_{j=1}^{n} \Big(f(a'_j) \wedge g(b'_j) \Big) \right\} \wedge \beta \right) \vee \alpha$$

$$\geqslant (f(xa) \wedge f(xa') \wedge g(x) \wedge \beta) \vee \alpha \geqslant (f(x) \wedge g(x) \wedge \beta) \vee \alpha = (f \wedge_{\alpha}^{\beta} g)(x)$$

because $x + xax + z = xa'x + z$.

Thus $(f \odot_{\alpha}^{\beta} g)\,(x) \geqslant (f \wedge_{\alpha}^{\beta} g)(x)$. But by Lemma 3.18, $(f \odot_{\alpha}^{\beta} g)\,(x) \leqslant (f \wedge_{\alpha}^{\beta} g)(x)$, hence $f \wedge_{\alpha}^{\beta} g = f \odot_{\alpha}^{\beta} g$.

$(ii) \Rightarrow (i)$ Let A and B be right and left h-ideals of R,respectively. Then by Theorem 3.8, χ_A is fuzzy right and χ_B is fuzzy left h-ideal with thresholds (α, β) of R. By hypothesis $\chi_A \odot_{\alpha}^{\beta} \chi_B = \chi_A \wedge_{\alpha}^{\beta} \chi_B$ implies $(\chi_{\overline{AB}} \wedge \beta) \vee \alpha = (\chi_{A \cap B} \wedge \beta) \vee \alpha$. Hence $A \cap B = \overline{AB}$. So, R is h-hemiregular. □

Lemma 4.2. [25] *Let R be a hemiring. Then the following conditions are equivalent:*

(i) *R is h-hemiregular,*

(ii) *$B = \overline{BRB}$ for every h-bi-ideal B of R,*

(iii) *$Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.* □

Theorem 4.3. *For a hemiring R, the following conditions are equivalent:*

(i) *R is h-hemiregular,*

(ii) $(f \wedge \beta) \vee \alpha \leqslant (f \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta f)$ *for every fuzzy h-bi-ideal f with thresholds (α, β) of R,*

(iii) $(f \wedge \beta) \vee \alpha \leqslant (f \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta f)$ *for every fuzzy h-quasi-ideal f with thresholds (α, β) of R.*

Proof. $(i) \Rightarrow (ii)$ Let $x \in R$, then there exists $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Now for all $x, a_i, b_i, c_j, d_j, z \in R$ such that $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n c_j d_j + z$, we have

$$
\begin{aligned}
&(f \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta f)(x) = \\
&= \left(\sup \left\{ \begin{array}{c} \bigwedge_{i=1}^m \left\{ (f \odot_\alpha^\beta \mathcal{R})(a_i) \wedge f(b_i) \right\} \wedge \\ \bigwedge_{j=1}^n \left\{ (f \odot_\alpha^\beta \mathcal{R})(a_j') \wedge f(b_j') \right\} \end{array} \right\} \wedge \beta \right) \vee \alpha \\
&\geqslant \left(\left\{ (f \odot_\alpha^\beta \mathcal{R})(xa) \wedge (f \odot_\alpha^\beta \mathcal{R})(xa') \wedge f(x) \right\} \wedge \beta \right) \vee \alpha \\
&= \left(\left\{ \begin{array}{c} \bigwedge_{i=1}^m \left(\left(\sup \left\{ \bigwedge_{i=1}^m f(a_i) \wedge \bigwedge_{j=1}^n f(a_j') \right\} \wedge \beta \right) \vee \alpha \right) \wedge \\ \bigwedge_{j=1}^n \left(\left(\sup \left\{ \bigwedge_{i=1}^m f(a_i) \wedge \bigwedge_{j=1}^n f(a_j') \right\} \wedge \beta \right) \vee \alpha \right) \wedge f(x) \end{array} \right\} \wedge \beta \right) \vee \alpha \\
&\geqslant \left(\left\{ \begin{array}{c} \{ (\min\{ f(xax), f(xa'x) \} \wedge \beta) \vee \alpha \} \wedge \\ \{ (\min\{ f(xax), f(xa'x) \} \wedge \beta) \vee \alpha \} \end{array} \right\} \wedge \beta \right\} \vee \alpha \geqslant (f(x) \wedge \beta) \vee \alpha
\end{aligned}
$$

since $xa + xaxa + za = xa'xa + za$ and $xa' + xaxa' + za' = xa'xa' + za'$.

$(ii) \Rightarrow (iii)$ This is straightforward.

$(iii) \Rightarrow (i)$ Let Q be any h-quasi ideal of R, then by Theorem 3.8, χ_Q is h-quasi-ideal with thresholds (α, β) o R.

Now by the given condition $(\chi_Q \wedge \beta) \vee \alpha \leqslant (\chi_Q \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta \chi_Q) = \chi_{\overline{QRQ}}$ implies $Q \subseteq \overline{QRQ}$. Also $\overline{QRQ} \subseteq \overline{RQ} \cap \overline{QR} = Q$. Thus$Q = \overline{QRQ}$. Therefore, by Lemma 4.2, R is h-hemiregular. $\square$

Theorem 4.4. *For a hemiring R, the following conditions are equivalent:*

(i) *R is h-hemiregular,*

(ii) $f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g \odot_\alpha^\beta f$ *for every fuzzy h-bi-ideal f and fuzzy h-ideal g with thresholds (α, β) of R,*

(iii) $f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g \odot_\alpha^\beta f$ *for every fuzzy h-quasi-ideal f and fuzzy h-ideal g with thresholds (α, β) of R.*

Proof. $(i) \Rightarrow (ii)$ Let f be any fuzzy h-bi-ideal and g any fuzzy h-ideal with thresholds (α, β) of R. Since R is h-hemiregular, so for any $a \in R$

there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Now for all $a, a_i, b_i, c_j, d_j, z \in R$ such that $a + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z$, we have

$$\left(f \odot_\alpha^\beta g \odot_\alpha^\beta f\right)(a) = \left[\sup\left\{\begin{array}{c} \bigwedge\limits_{i=1}^{m}\left(\left(f \odot_\alpha^\beta g\right)(a_i) \wedge f(b_i)\right) \wedge \\ \bigwedge\limits_{j=1}^{n}\left(\left(f \odot_\alpha^\beta g\right)(a'_j) \wedge f(b'_j)\right) \end{array}\right\} \wedge \beta\right] \vee \alpha$$

$$\geqslant \left(\{(f \odot_\alpha^\beta g)(ax_1) \wedge f(a) \wedge (f \odot_\alpha^\beta g)(ax_2)\} \wedge \beta\right) \vee \alpha$$

because $a + ax_1a + z = ax_2a + z$

$$= \left(\left[\begin{array}{c} \left\{\left[\sup\left\{\begin{array}{c} \bigwedge\limits_{i=1}^{m}\left(f(c_i) \wedge g(d_i)\right) \wedge \\ \bigwedge\limits_{j=1}^{n}\left(f(c'_j) \wedge g(d'_j)\right) \end{array}\right\} \wedge \beta\right] \vee \alpha\right\} \wedge f(a) \\ \left\{\left[\sup\left\{\begin{array}{c} \bigwedge\limits_{i=1}^{m}\left(f(p_i) \wedge g(q_i)\right) \wedge \\ \bigwedge\limits_{j=1}^{n}\left(f(p'_j) \wedge g(q'_j)\right) \end{array}\right\} \wedge \beta\right] \vee \alpha\right\} \end{array}\right] \wedge \beta\right) \vee \alpha$$

for all possible expressions $ax_1 + \Sigma_{i=1}^{m} c_i d_j + z = \Sigma_{j=1}^{n} c'_j d'_j + z$ and $ax_2 + \Sigma_{i=1}^{m} p q_i + z = \Sigma_{j=1}^{n} p'_j q'_j + z$

$$\geqslant \{f(a) \wedge g(x_1ax_1) \wedge g(x_1ax_2) \wedge g(x_2ax_1) \wedge g(x_2ax_2) \wedge \beta\} \vee \alpha$$
$$\geqslant \{f(a) \wedge g(a) \wedge \beta\} \vee \alpha = (f \wedge_\alpha^\beta g)(a)$$

because $ax_1 + ax_1ax_1 + zx_1 = ax_2ax_1 + zx_1$ and $ax_2 + ax_1ax_2 + zx_2 = ax_2ax_2 + zx_2$.

$(ii) \Rightarrow (iii)$ is straightforward.

$(iii) \Rightarrow (i)$ Let f be fuzzy h-quasi-ideal and $\mathcal{R}$ be fuzzy h-ideal with thresholds (α, β) *of* R. Then by hypothesis, $f \wedge_\alpha^\beta \mathcal{R} \leqslant f \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta f$ implies $(f \wedge \beta) \vee \alpha \leqslant f \odot_\alpha^\beta \mathcal{R} \odot_\alpha^\beta f$. Then, by Theorem 4.3, R is h-hemiregular. □

Theorem 4.5. *For a hemiring R, the following conditions are equivalent:*

(i) *R is h-hemiregular,*

(ii) *$f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g$ for every fuzzy h-bi-ideal f and fuzzy left h-ideal g with thresholds (α, β) of R,*

(iii) *$f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g$ for every fuzzy h-quasi-ideal f and fuzzy left h-ideal g with thresholds (α, β) of R,*

(iv) *$f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g$ for every fuzzy right h-ideal f and fuzzy h-bi-ideal g with thresholds (α, β) of R,*

(v) *$f \wedge_\alpha^\beta g \leqslant f \odot_\alpha^\beta g$ for every fuzzy right h-ideal f and fuzzy h-quasi-ideal g with thresholds (α, β) of R,*

(vi) *$f \wedge_\alpha^\beta g \wedge_\alpha^\beta h \leqslant f \odot_\alpha^\beta g \odot_\alpha^\beta h$ for every fuzzy right h-ideal f, fuzzy h-bi-ideal g and fuzzy left h-ideal h with thresholds (α, β) of R,*

(vii) *$f \wedge_\alpha^\beta g \wedge_\alpha^\beta h \leqslant f \odot_\alpha^\beta g \odot_\alpha^\beta h$ for every fuzzy right h-ideal f, fuzzy h-quasi-ideal g and fuzzy left h-ideal h with thresholds (α, β) of R.*

Proof. $(i) \Rightarrow (ii)$ Let f be any fuzzy h-bi-ideal and g any fuzzy left h-ideal with thresholds (α, β) of R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Now for all $a, a_i, b_i, c_j, d_j, z \in R$ such that $a + \sum_{i=1}^m a_ib_i + z = \sum_{j=1}^n c_jd_j + z$, we have

$$\begin{aligned}
&(f \odot_\alpha^\beta g)(a) = \\
&= \left(\sup\left\{ \bigwedge_{i=1}^m \Big(f(a_i) \wedge g(b_i)\Big) \wedge \bigwedge_{j=1}^n \Big(f(a'_j) \wedge g(b'_j)\Big)\right\} \wedge \beta\right) \vee \alpha \\
&\geqslant (\{f(a) \wedge g(x_1a) \wedge g(x_2a)\} \wedge \beta) \vee \alpha \quad \text{because } a + ax_1a + z = ax_2a + z \\
&\geqslant (\{f(a) \wedge g(a)\} \wedge \beta) \vee \alpha = (f \wedge_\alpha^\beta g)(a).
\end{aligned}$$

So, $f \odot_\alpha^\beta g \geqslant f \wedge_\alpha^\beta g$.

$(ii) \Rightarrow (iii)$ is straightforward.

$(iii) \Rightarrow (i)$ Let f be any fuzzy right h-ideal and g be any fuzzy left h-ideal with thresholds (α, β) of R. Since every fuzzy right h-ideal with thresholds (α, β) is fuzzy h-quasi-ideal with thresholds (α, β), so by (iii) we have $f \odot_\alpha^\beta g \geqslant f \wedge_\alpha^\beta g$. But by Lemma 3.18, $f \odot_\alpha^\beta g \leqslant f \wedge_\alpha^\beta g$. Hence $f \odot_\alpha^\beta g = f \wedge_\alpha^\beta g$ for every fuzzy right h-ideal f with thresholds (α, β) of R, and for every fuzzy left h-ideal g with thresholds (α, β) of R. Thus by Theorem 4.1, R is h-hemiregular.

Similarly we can show that $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$.

$(i) \Rightarrow (vi)$ Let f be a fuzzy right h-ideal, g be a fuzzy h-bi-ideal and h be a fuzzy left h-ideal with thresholds (α, β) of R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Now for all $a, a_i, b_i, c_j, d_j, z \in R$ such that $a + \sum_{i=1}^m a_ib_i + z = \sum_{j=1}^n c_jd_j + z$, we have

$$\begin{aligned}
&(f \odot_\alpha^\beta g \odot_\alpha^\beta h)(a) = \\
&= \left(\sup\left\{ \bigwedge_{i=1}^m \Big((f \odot_\alpha^\beta g)(a_i) \wedge h(b_i)\Big) \wedge \bigwedge_{j=1}^n \Big((f \odot_\alpha^\beta g)(a'_j) \wedge h(b'_j)\Big)\right\} \wedge \beta\right) \vee \alpha \\
&\geqslant (\{(f \odot_\alpha^\beta g)(a) \wedge h(x_1a) \wedge h(x_2a)\} \wedge \beta) \vee \alpha \\
&\geqslant \left(\left\{\left(\sup\left\{ \bigwedge_{i=1}^m \Big(f(a_i) \wedge g(b_i)\Big) \wedge \bigwedge_{j=1}^n \Big(f(a'_j) \wedge g(b'_j)\Big)\right\} \wedge \beta\right) \vee \alpha\right\} \wedge h(x_1a) \wedge h(x_2a) \wedge \beta\right) \vee \alpha \\
&\geqslant (f(ax_1) \wedge f(ax_2) \wedge g(a) \wedge h(x_1a) \wedge h(x_2a) \wedge \beta) \vee \alpha \\
&\geqslant (f(a) \wedge g(a) \wedge h(a) \wedge \beta) \vee \alpha = (f \wedge_\alpha^\beta g \wedge_\alpha^\beta h)(a).
\end{aligned}$$

$(vi) \Rightarrow (vii)$ is straightforward.

$(vii) \Rightarrow (i)$ Let f be a fuzzy right h-ideal, and h be a fuzzy left h-ideal with thresholds (α, β) of R. Then

$$f \wedge_{\alpha}^{\beta} h = f \wedge_{\alpha}^{\beta} \mathcal{R} \wedge_{\alpha}^{\beta} h \leqslant f \odot_{\alpha}^{\beta} \mathcal{R} \odot_{\alpha}^{\beta} h \leqslant f \odot_{\alpha}^{\beta} h.$$

But $f \odot_{\alpha}^{\beta} h \leqslant f \wedge_{\alpha}^{\beta} h$ always. Hence $f \odot_{\alpha}^{\beta} h = f \wedge_{\alpha}^{\beta} h$ for every fuzzy right h-ideal f and for every fuzzy left h-ideal h with thresholds (α, β) of R. Thus by Theorem 4.1, R is h-hemiregular. □

5. h-intra-hemiregular hemirings

In this section we characterize h-intra-hemiregular hemirings and hemirings which are both h-hemiregular and h-intra-hemiregular in terms of their fuzzy ideals with thresholds (α, β).

Theorem 5.1. *A hemiring R is h-intra-hemiregular if and only if $f \wedge_{\alpha}^{\beta} g \leqslant f \odot_{\alpha}^{\beta} g$ for every fuzzy left h-ideal f and for every fuzzy right h-ideal g with thresholds (α, β) of R.*

Proof. Let R be an h-intra-hemiregular and f be a fuzzy left h-ideal and g a fuzzy right h-ideal with thresholds (α, β) of R. As R is h-intra-hemiregular so for every $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$. Then

$$\begin{aligned}(f \odot_{\alpha}^{\beta} g)(x) &= \Big(\sup \Big\{ \bigwedge_{i=1}^{m} f(a_i) \wedge \bigwedge_{i=1}^{m} g(b_i) \wedge \bigwedge_{j=1}^{n} f(a_j') \wedge \bigwedge_{j=1}^{n} g(b_j') \Big\} \wedge \beta \Big) \vee \alpha \\ &\geqslant \Big(f(a_i x) \wedge f(b_j x) \wedge g(x a_i') \wedge g(x b_j') \wedge \beta \Big) \vee \alpha,\end{aligned}$$

because $x + \sum_{i=1}^{m} (a_i x)(x a_i') + z = \sum_{j=1}^{n} (b_j x)(x b_j') + z$.

Conversely assume that A and B are left and right h-ideals of R, respectively. Then, by Theorem 3.8, the characteristic functions $\chi_A$and χ_B are respectively fuzzy left h-ideal and fuzzy right h-ideal with thresholds (α, β). Then by hypothesis $\chi_A \wedge_{\alpha}^{\beta} \chi_B \leqslant \chi_A \odot_{\alpha}^{\beta} \chi_B$ implies $(\chi_{A \cap B} \wedge \beta) \vee \alpha \leqslant (\chi_{\overline{AB}} \wedge \beta) \vee \alpha$. Hence $A \cap B \subseteq \overline{AB}$. Thus, by Lemma 2.11, R is h-intra-hemiregular. □

Theorem 5.2. *The following conditions are equivalent for a hemiring R:*

(i) *R is both h-hemiregular and h-intra-hemiregular,*

(ii) *$(f \wedge \beta) \vee \alpha = f \odot_{\alpha}^{\beta} f$ for every fuzzy h-bi-ideal f with thresholds (α, β) of R,*

(iii) *$(f \wedge \beta) \vee \alpha = f \odot_{\alpha}^{\beta} f$ for every fuzzy h-quasi-ideal f with thresholds (α, β) of R.*

Proof. $(i) \Rightarrow (ii)$ Let f be a fuzzy h-bi-ideal with thresholds (α, β) of R and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, there exist elements $a_1, a_2, p_i, p_i', q_j, q_j', z \in R$ such that

$$x+\sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_1x)+\sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_2x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_1x) +\sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_2x)+z=\sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_1x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_2x) +\sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_1x)+\sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_2x)+z$$

(cf. Lemma 5.6 in [25]).

$$\begin{aligned} f\odot_\alpha^\beta f(x) &= \Big(\sup\Big\{\bigwedge_{i=1}^{m}\Big(f(a_i)\wedge f(b_i)\Big)\wedge\bigwedge_{j=1}^{n}\Big(f(a_j')\wedge f(b_j')\Big)\Big\}\wedge\beta\Big)\vee\alpha \\ &\geqslant \left(\begin{array}{l} \bigwedge_{j=1}^{n}\Big(f(xa_2q_jx)\wedge f(xq_j'a_1x)\wedge f(xa_1q_jx)\wedge f(xq_j'a_2x)\Big)\wedge \\ \bigwedge_{i=1}^{m}\Big(f(xa_1p_ix)\wedge f(xp_i'a_1x)\wedge f(xa_2p_ix)\wedge f(xp_i'a_2x)\Big)\wedge\beta \end{array}\right)\vee\alpha \\ &\geqslant (\{(f(x)\wedge\beta)\vee\alpha\}\wedge\beta)\vee\alpha=(f(x)\wedge\beta)\vee\alpha. \end{aligned}$$

This implies that $f\odot_\alpha^\beta f \geqslant (f\wedge\beta)\vee\alpha$

On the other hand, if $x+\Sigma_{i=1}^m a_ib_i+z=\Sigma_{j=1}^n a_j'b_j'+z$, we have

$$\begin{aligned} (f(x)\wedge\beta)\vee\alpha &= ((f(x)\wedge\beta)\vee\alpha)\vee\alpha=((f(x)\vee\alpha)\wedge\beta)\vee\alpha \\ &\geqslant (f(\Sigma_{i=1}^m a_ib_i)\wedge f(\Sigma_{j=1}^n a_j'b_j')\wedge\beta)\vee\alpha \qquad \text{by (3)} \\ &\geqslant \Big(\bigwedge_{i=1}^{m} f(a_ib_i)\wedge\bigwedge_{j=1}^{n} f(a_j'b_j')\wedge\beta\Big)\vee\alpha \\ &\geqslant \Big(\bigwedge_{i=1}^{m}\Big(f(a_i)\wedge f(b_i)\Big)\wedge\bigwedge_{i=1}^{m}\Big(f(a_i)\wedge f(b_i)\Big)\wedge\beta\Big)\vee\alpha. \end{aligned}$$

Thus

$$\begin{aligned} (f\odot_\alpha^\beta f)(x) &= \Big(\sup\Big\{\bigwedge_{i=1}^{m}\Big(f(a_i)\wedge f(b_i)\Big)\wedge\bigwedge_{j=1}^{n}\Big(f(a_j')\wedge f(b_j')\Big)\Big\}\wedge\beta\Big)\vee\alpha \\ &\leqslant (f(x)\wedge\beta)\vee\alpha. \end{aligned}$$

Consequently $f\odot_\alpha^\beta f=(f\wedge\beta)\vee\alpha$.

$(ii) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (i)$ Let Q be an h-quasi-ideal of R. Then χ_Q is a fuzzy h-quasi-ideal with thresholds (α, β) of R. Thus by hypothesis

$$[\chi_Q\wedge\beta]\vee\alpha=\chi_Q\odot_\alpha^\beta\chi_Q=[\chi_Q\odot\chi_Q\wedge\beta]\vee\alpha=[\chi_{\overline{Q^2}}\wedge\beta]\vee\alpha.$$

Then it follows $Q=\overline{Q^2}$. Hence by Lemma 2.12, R is both h-hemiregular and h-intra-hemiregular. □

Theorem 5.3. *The following conditions are equivalent for a hemiring R:*

(i) *R is both h-hemiregular and h-intra-hemiregular,*

(ii) *$f\wedge_\alpha^\beta g\leqslant f\odot_\alpha^\beta g$ for all fuzzy h-bi-ideals f and g with thresholds (α,β) of R,* (*iii*) $f \wedge_{\alpha}^{\beta} g \leqslant f \odot_{\alpha}^{\beta} g$ *for every fuzzy h-bi-ideal f and every fuzzy h-quasi-ideals f with thresholds* (α, β) *of R,*

(*iv*) $f \wedge_{\alpha}^{\beta} g \leqslant f \odot_{\alpha}^{\beta} g$ *for every fuzzy h-quasi-ideal f and every fuzzy h-bi-ideals f with thresholds* (α, β) *of R,*

(*v*) $f \wedge_{\alpha}^{\beta} g \leqslant f \odot_{\alpha}^{\beta} g$ *for all fuzzy h-quasi-ideals f and g with thresholds* (α, β) *of R.*

Proof. $(i) \Rightarrow (ii)$ Analogously as in previous proof.

$(ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are straightforward.

$(v) \Rightarrow (i)$ Let f be a fuzzy left h-ideal and g be a fuzzy right h-ideal with thresholds (α, β) of R. Then f and g are fuzzy h-bi-ideals with thresholds (α, β) of R. So by hypothesis $f \wedge_{\alpha}^{\beta} g \leqslant f \odot_{\alpha}^{\beta} g$ but $f \wedge_{\alpha}^{\beta} g \geqslant f \odot_{\alpha}^{\beta} g$ by Lemma 3.18. Thus $f \wedge_{\alpha}^{\beta} g = f \odot_{\alpha}^{\beta} g$. Hence by Theorem 4.1, R is h-hemiregular. On the other hand by hypothesis we also have $f \wedge_{\alpha}^{\beta} g \leqslant g \odot_{\alpha}^{\beta} f$. By Theorem 5.1, R is h-intra-hemiregular. □

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