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Topological LA-groups and LA-rings

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Abstract. We introduce the notion of topological LA-groups and topological LA-rings which are some generalizations of topological groups and topological rings respectively. We extend some characterizations of topological groups and rings to topological LA-groups and topological LA-rings.

1. Introduction

Kazim and Naseerudin [4] have introduced the concept of *LA-semigroups*, i.e., groupoids whose elements satisfy the *left invertive law*: $(ab)c = (cb)a$. Such groupoids also are known as Abel-Grassmann's groupoids or AG-groupoids (see [2]). Many interesting results on LA-semigroups one can find in [5], [6] and [7]. Some authors studied also *left almost groups* (*LA-groups*), i.e., LA-semigroups in which for every $a \in G$ there exists $e \in G$ such that $ea = a$ and $a^{-1} \in G$ such that $a^{-1}a = e$. LA-rings are studied by T. Shah and I. Rehman (cf. [9]).

In this paper we introduced the notion of topological LA-groups and topological LA-rings. Furthermore we established some of properties regarding products, quotient and subgroups of a topological LA-group. In case of topological LA-ring we prove that the product of any family of topological LA-rings is again a topological LA-ring and an LA-subring of a topological LA-ring is again a topological LA-ring.

2. Preliminaries

A *topological group* is a group $(G, *)$ with a topology τ such that the group operations $G \times G \to G : (x, y) \to x * y$ and $G \to G : x \to x^{-1}$ are continuous

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or the map $G \times G \to G : (x, y) \to x * y^{-1}$ is continuous. For topological group one may consult [3] and [8].

Definition 2.1. A non empty set G is called a *topological LA-group* if
(a) $(G, *)$ is an LA-group,
(b) (G, τ) is a topological space,
(c) LA-group operation $* : G \times G \to G$ and the inversion function $i : G \to G$ defined by $i(x) = x^{-1}$ are continuous.

The condition (c) can be replaced by
$(c)'$ The mapping $(x, y) \to x * y^{-1}$ of $G \times G$ onto G is continuous.

Example 2.2. Let G be an LA-group. It is easy to verify that the condition (a) is true in the discrete (respectively indiscrete) topology on G. Consequently G is an LA-topological group. In this manner any LA-group may be considered as a topological LA-group in the discrete (respectively indiscrete) topology. □

The following theorem is a generalization of Proposition 3.2 from [3].

Theorem 2.3. *Let G be a topological LA-group. Then*
(1) *the right translation $r_a : x \to xa$ is homeomorphism,*
(2) *the left translation $l_a : x \to ax$ is homeomorphism and*
(3) *the inversion mapping $i : x \to x^{-1}$ is homeomorphism.*

Proof. (1) Let $x = y$. This implies $xa = ya$ which shows that $r_a(x) = r_a(y)$, which shows that r_a is well-defined.

Let $r_a(x) = r_a(y)$. This implies $xa = ya$. Since G is cancellative, so $x = y$, so r_a is one-to-one.

For each $x \in G$ there exist $xa^{-1} \in G$ such that $r_a(xa^{-1}) = (xa^{-1})a = (aa^{-1})x = ex = x$ implies that r_a is onto. Thus r_a is bijective.

Let U be any neighbourhood of $r_a(x) = xa$. Since G is a topological LA-group, so the mapping $* : G \times G \to G$ is continuous and for any neighbourhood U of $r_a(x) = xa$ there exists neighbourhoods V and W of x and a (respectively) such that $V * W \subseteq U$.

Now $r_a(V) = V * a \subseteq V * W$. So, $r_a(V) \subseteq V * W \subseteq U$. Thus $r_a(V) \subseteq U$. Since x is an arbitrary element of G, the mapping r_a is continuous.

Let U be any neighbourhood of $r_a^{-1}(x) = xa^{-1}$. Since G is a topological LA-group, the mapping $* : G \times G \to G$ is continuous. Hence for any neighbourhood U of $r_a^{-1}(x) = xa^{-1}$ there exists neighbourhoods V and W of x and a^{-1} respectively such that $V * W \subseteq U$.

Now as $r_a^{-1}(V) = V * a^{-1} \subseteq V * W$, we have $r_a^{-1}(V) \subseteq V * W \subseteq U$. Thus $r_a^{-1}(V) \subseteq U$. As x is an arbitrary element of G, the mapping r_a^{-1} is continuous. Hence r_a is a homeomorphism.

(2) The proof is analogous to (1).

(3) Let $i(x) = i(y)$. Then $x^{-1} = y^{-1}$. Now $e = yy^{-1} = yx^{-1}$, which implies $ex = (yx^{-1})x$ and therefore by left invertive law we have $x = (xx^{-1})y = ey = y$ and hence i is one-to-one.

For each $x \in G$ there exist $x^{-1} \in G$ such that $i(x^{-1}) = (x^{-1})^{-1} = x$, so i is onto.

Since G is a topological LA-group, i is continuous. Also $i^{-1}(x) = x^{-1}$ is continuous because i is one-to-one. □

Remark 2.4. The mappings $x \mapsto a(xa^{-1})$, $x \mapsto a^{-1}(xa)$, $x \mapsto (ax)a^{-1}$, $x \mapsto (a^{-1}x)a$ are homeomorphisms as composition of two homeomorphisms $x \mapsto xa(xa^{-1})$ and $x \mapsto ax(a^{-1}x)$.

Remark 2.5. In topological groups we obtain only one homeomorphism axa^{-1}, but in the case of topological LA-groups we obtain distinct homeomorphisms $a(xa^{-1})$, $a^{-1}(xa)$, $(ax)a^{-1}$ etc.

Corollary 2.6. *Let E be open and F be closed in a topological LA-group G and A be any subset of G. Then for $a \in G$*

(1) *aE, Ea, E^{-1} are open,*

(2) *aF, Fa, F^{-1} are closed and AE, EA are open.*

Proof. The mappings in Theorem2.3 are homeomorphisms, so (1) is obvious.

Since $AE = \cup_{a \in A} aE$, $EA = \cup_{a \in A} Ea$, and the union of open sets is open, therefore (2) is established. □

3. Topological LA-groups

In this section we define topological LA-groups and give some characterizations of such LA-groups.

3.1. Construction of a new topological LA-group from old

We can always construct a new topological LA-group from old ones. A product of topological LA-groups permits us the construction of a new topological LA-group from the given ones and also permits the reduction of the study of relatively complicated topological LA-groups to the investigation of their simple constituents.

The following theorem is a generalization of Proposition 3.12 from [3].

Theorem 3.1. *Let A be an index set. For each $\alpha \in A$, let G_α be a topological LA-group. Then $G = \prod\limits_{\alpha \in A} G_\alpha$ endow with product topology, is also a topological LA-group.*

Proof. To prove that G is a topological LA-group, we have to show that the onto mapping $* : G \times G \to G$; $(x, y) \mapsto xy^{-1}$ is continuous.

Let W be a neighbourhood of xy^{-1} in G, then there exists an open set U such that $xy^{-1} \in U \subseteq W$, where $U = \prod_{\alpha \in A} U_\alpha$ with U_α is an open neighbourhood of $x_\alpha y_\alpha^{-1}$ in G_α. Since $(x_\alpha, y_\alpha) \mapsto x_\alpha y_\alpha^{-1}$ is continuous for each $\alpha \in A$, so there exists neighbourhoods V_{α_i}, $V_{\alpha_i}'^{-1}$ of x_{α_i} and y_{α_i} respectively such that $V_{\alpha_i} V_{\alpha_i}'^{-1} \subseteq U_{\alpha_i}$for each $1 \leqslant i \leqslant n$. Now let $V = \prod\limits_{\alpha \in A} V_\alpha$ and $V' = \prod\limits_{\alpha \in A} V_\alpha'$, then V and V' are neighbourhoods of x and y respectively. This means $VV'^{-1} = \prod(V_{\alpha_i} V_{\alpha_i}'^{-1}) \subseteq \prod U_\alpha = U \subseteq W$. This proves the theorem. □

Now we give the following definition.

Definition 3.2. Let G be a topological LA-group and H be an LA-subgroup of G. Then H endow with relative topology, is a topological LA-group called *topological LA-subgroup* of G.

Theorem 3.3. *An LA-subgroup H of a topological LA-group G is a topological LA-subgroup.*

Proof. Let G be a topological LA-group and H be an LA-subgroup of G. Then H is endowed with relative topology induced from G. Since the mapping $(x, y) \mapsto xy^{-1}$ of $G \times G$ onto G is continuous, so its restriction from $H \times H$ onto H is also continuous. Let a, b be two elements of H and let $ab^{-1} = c$. Every neighbourhood W' of the element c in H can be obtained as the intersection with H of some neighbourhood W of c in G, i.e., $W' = H \cap W$. Since G is a topological LA-group, so there exists neighbourhoods U and V of a, b such that $UV^{-1} \subseteq W$. Now $U' = H \cap U$ and $V' = H \cap V$ are the relative neighbourhoods of a and b in H. Thus we have $U'V'^{-1} \subseteq W$ and also $U'V'^{-1} \subseteq H$. Hence $U'V'^{-1} \subseteq W'$ and H is a topological LA-subgroup. □

3.2. Topological factor LA-groups

Let G be a topological LA-group and H is an LA-subgroup of G. Then G/H denotes the set of all cosets Ha, $a \in G$. Let φ be a canonical mapping of G onto G/H. With the help of φ we can define a topology on G/H as follows: A subset A' of G/H is open if and only if $\varphi^{-1}(A')$ is an open subset of G. This topology in G/H is called the *quotient topology* and G/H, endowed with quotient topology, is called the *quotient space*.

The following theorem is a generalization of Proposition 3.8 from [3].

Theorem 3.4. *Let G be a topological LA-group and H be an LA-subgroup of G. Let G/H be the quotient space endowed with the quotient topology and φ be the canonical mapping of G onto G/H, then*

(1) *φ is homomorphism,*
(2) *φ is continuous,*
(3) *φ is open.*

Proof. (1) Let $x, y \in G$, then $\varphi(xy) = H(xy) = (HH)(xy) = (Hx)(Hy) = \varphi(x)\varphi(y)$.

(2) φ is continuous by the definition of quotient topology.

(3) Let U be open in G. We have to prove that $\varphi(U)$ is open in G/H. That is, $\varphi^{-1}(\varphi(U))$ is open in G. But $\varphi^{-1}(\varphi(U)) = \{g : g \in uH \text{ for some } u \in U\} = UH$, which is open. Hence φ is open. □

The following theorem is a generalization of Proposition 3.10(ii) from [3].

Theorem 3.5. *Let G be a topological LA-group and H be an LA-subgroup of G. Then G/H endowed with the quotient topology, is a topological LA-group.*

Proof. To prove that G/H is a topological LA-group we have to show that the mapping $* : (x', y') \to x'y'^{-1}$ of $G/H \times G/H$ onto G/H is continuous.

Let W be an open neighbourhood of $x'y'^{-1}$, where $x' = xH$ and $y' = yH$ and $x, y \in G$. Clearly $\varphi^{-1}(W)$ is open in G and $x'y'^{-1} \in \varphi^{-1}(W)$.

Since G is a topological LA-group, so there exists open sets U and V such that $x \in U$, $y^{-1} \in V^{-1}$ and $xy^{-1} \in UV^{-1} \subseteq \varphi^{-1}(W)$. Since by Theorem 3.4 φ is continuous and open homomorphism so $x'y'^{-1} \in \varphi(U)(\varphi(V))^{-1} \subset \varphi(\varphi^{-1}(W))$, which implies $x'y'^{-1} \in \varphi(U)(\varphi(V))^{-1} \subset W$.

As by theorem 3.4 φ is open, so $\varphi(U)$ and $\varphi((V))^{-1} = \varphi(V^{-1})$ are open because U and V are open. Thus G/H is a topological LA-group. □

Definition 3.6. A topological LA-group G is said to be *homogeneous* if for all $x, y \in G$, there exists a homeomorphism $f : G \to G$ such that $f(x) = y$.

The following theorem is a generalization of Proposition 3.14 from [3].

Theorem 3.7. *Let G be a topological LA group and H be a subgroup of G. Then the topological LA-group G/H is a homogeneous space.*

Proof. Let $x' = Hx$, $y' = Hy$ and $g \in G$ be such that $g = yx^{-1}$. Define the mapping $f_g : x' = Hx \mapsto H(gx)$ for all $x' \in G/H$.

Let $Hx = Hy$, then $g(Hx) = g(Hy)$ implies $H(gx) = H(gy)$ and hence $f_g(Hx) = f_g(Hy)$. Thus the mapping is well-defined.

Let $f_g(Hx) = f_g(Hy)$. Then $H(gx) = H(gy)$ and $g(Hx) = g(Hy)$. Hence $Hx = Hy$ and so f_g is one-to-one.

For each $x' = Hx \in G/H$ there exists $H\{(g^{-1}e)x\} \in G/H$ such that

$$\begin{aligned} f_g(H\{(g^{-1}e)x\}) &= H\{g((g^{-1}e)x)\} = H\{g((xe)g^{-1})\} \\ &= H\{(xe)gg^{-1}\} = H\{(xe)e\} = H\{(ee)x\} = Hx, \end{aligned}$$

which shows that f_g is onto.

Let U be any neighbourhood of $f_g(Hx) = H(gx)$. Since G/H is a topological LA-group, so the mapping $* : G/H \times G/H \to G/H$ is continuous and thus for any neighbourhood U of $f_g(x) = H(gx) = Hg * Hx$ there exists neighbourhoods V and W of Hg and Hx respectively such that $V * W \subseteq U$.

Now $f_g(V) = f_g(HS) = H(gS)$, so $f_g(V) = Hg * HS$ implies $f_g(V) \subseteq W * V \subseteq U$. As x is an arbitrary element of G, we see that f_g is continuous.

Now let U be any neighbourhood of $f_g^{-1}(Hx) = H(g^{-1}e)x = H(g^{-1}e) * Hx$. Since G/H is a topological LA-group, so for any neighbourhood U of $f_g^{-1}(Hx)$ there exists neighbourhoods V and W of $H(g^{-1}e)$ and Hx respectively such that $V * W \subseteq U$.

Now $f_g^{-1}(W) = f_g^{-1}(HS)$ so $f_g^{-1}(W) = H\{(g^{-1}e)S\}$ implies $f_g^{-1}(W) = H(g^{-1}e) * HS$ and this means $f_g^{-1}(W) \subseteq V * W \Rightarrow f_g^{-1}(W) \subseteq V * W \subseteq U$. Hence $f_g^{-1}(W) \subseteq U$ and therefore f_g^{-1} is continuous. Thus we concluded that f_g^{-1} is a homeomorphism.

Clearly
$f_g(x') = f_g(Hx) = H(gx) = H((yx^{-1})x) = H((xx^{-1})y) = Hy = y'$,
which shows that G/H is a homogeneous space. □

The following theorem is a generalization of Proposition 3.4 from [3].

Theorem 3.8. *For a topological LA-group G, the following statements are equivalent:*

(1) *G is a T_0-space,*
(2) *G is a T_1-space,*
(3) *G is a T_2-space,*
(4) *$\cap U = \{e\}$,where U is a fundamental system of neighbourhood of the identity e.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in G$, $x \neq y$. (1) implies that for at least one of x and y, there exists an open neighbourhood P of x such that $y \notin P$. Since $x \in P$, so $xx^{-1} \in Px^{-1}$, i.e., $e \in Px^{-1}$ and $Px^{-1} = V$ is an open neighbourhood of e.

Now $V \cap V^{-1} = Q$ is an open symmetric neighbourhood of e, so $e \in Q$, which implies $ey \in Qy$. Hence $y \in Qy$. Now $x \notin Qy$ because if $x \in Qy$ then $x^{-1} \in y^{-1}Q$ $(Q = Q^{-1})$ and $x^{-1} \in y^{-1}Q \subset y^{-1}V\ x^{-1} \subset y^{-1}(Px^{-1}) = P(y^{-1}x^{-1})$ but this implies that

$$e = x^{-1}x \in (P(y^{-1}x^{-1}))x = (y^{-1}x^{-1})(Px).$$

Thus, by medial law,

$$e \in (y^{-1}P)(x^{-1}x) = (y^{-1}P)e = (eP)y^{-1} = Py^{-1}.$$

Hence,

$$y = ey \in (Py^{-1})y = (yy^{-1})P = eP = P,$$

which is a contradiction.

(2) $\Rightarrow$ (3) Let $x, y \in G$, $x \neq y$. By (2) G is a T_1-space, so $\{x\}$ is a closed set and therefore $P = G\backslash\{x\}$ is an open neighbourhood of y, thus $y \in P$, which implies $y^{-1}y \in y^{-1}P$, this means $e \in y^{-1}P$ and hence $y^{-1}P$ is an open neighbourhood of e by Theorem 2.3.

Let V be an open neighbourhood of e such that $VV^{-1} \subset y^{-1}P$. Then Vy is an open neighbourhood of y. Let $Q = G\backslash\overline{Vy}$, an open set and $x \in Q$.

Otherwise $x \in \overline{Vy}$ and hence by the definition of closure $Vy \cap Vx \neq \emptyset$.

But this shows that $x \in (ye)(VV^{-1}) \subset (ye)(y^{-1}P)$, which implies that $x \in (yy^{-1})(eP) = eP$ and hence $x \in P$, a contradiction. Clearly $Q \cap Vy = \emptyset$ gives $y \in Vy$ and so $x \in Q$. This proves (3).

(3) $\Rightarrow$ (4) Let $x \in U$ for each U in $\{U\}$ and assume $x \neq e$. Then (3) shows that there exists a neighbourhood P of e such that $x \notin P$. But then there exists a U in $\{U\}$ such that $U \subset P$. We have a contradiction that $x \in U \subset P$ and $x \notin P$. Hence $x = e$ and (4) is satisfied.

(4) $\Rightarrow$ (1) Let $x \neq y$. Then $xy^{-1} \neq e$ and hence by (4) there exists a U in $\{U\}$ such that $xy^{-1} \notin U$. Thus Uy being a neighbourhood of y and $x \notin Uy$. This proves (1). □

4. Topological LA-rings

The following definition of a topological ring is taken from [1].

Definition 4.1. A *topological ring* is a ring R with a topology τ such that the additive group of the ring R is topological group in topology τ and the one of the following equivalent conditions is satisfied:

(a) the maps $R \times R \to R : (x, y) \to xy$ is continuous, (multiplication condition (MC)),

(b) for any two elements $x, y \in R$ and arbitrary neighborhood U of the element xy there exist neighborhoods V and W of elements x and y respectively such that $VW \subset U$.

Definition 4.2. An LA-ring $(R, +, \cdot$ is called a *topological LA-ring* if

(a) $(R, +)$ is an LA-group,

(b) (R, τ) is a topological space,

(c) the algebraic operations defined in R are continuous in topological space R, i.e., the mappings $(a, b) \to a - b$ and $(a, b) \to a \cdot b$ of the topological space $R \times R$ to the topological space R are continuous. In greater detail: for arbitrary elements $a, b \in R$ and for arbitrary neighbourhoods W and W' of the elements $a - b$ and ab respectively, there exist neighbourhoods U and V of a and b such that $U - V \subset W$ and $UV \subset W'$.

Example 4.3. By the virtue of above definition the additive LA-group of any topological LA-ring is a topological LA-group. Conversely, if R is a topological LA-group, then R could be transformed into the LA-ring by the definition of zero multiplication on R, i.e., setting $a.b = 0$ for any $a, b \in R$. In doing so, the condition (MC) is fulfilled, and hence R is transformed into a topological LA-ring. In this manner every LA-group may be considered as a topological LA-ring with zero multiplication.

Theorem 4.4. *Let R be a topological LA-ring, then for each $r \in R$, the functions $\phi_r : x \to rx$ and $\psi_r : x \to xr$ are continuous from R to R.*

Proof. Let U be any neighbourhood of $\varphi_r(x) = xr$. Since R is a topological LA-ring so the mapping $* : R \times R \to R$ is continuous so for any neighbourhood U of $\varphi_r(x) = xr$ there exists neighbourhoods V and W of x and r respectively such that $V * W \subseteq U$

Now

$$\varphi_r(V) = V * r \subseteq V * W \subseteq U.$$

As x is an arbitrary element of R, so φ_r is continuous.

Similarly we can prove theorem for ψ_r. □

Theorem 4.5. *Let A be an index set. For each $\alpha \in A$, let R_α be a topological LA-ring. Then $R = \prod_{\alpha\in A} R_\alpha$ endow with the product topology, is also a topological LA-ring.*

Proof. As R is a LA-ring so $(R, +)$ is a topological group, so $* : (x, y) \to x - y$ is continuous. We have to check the continuity of : $(x, y) \to xy$ only.

Let W be a neighbourhood of xy in R, then there exists an open set U such that $xy \in U \subseteq W$, where $U = \prod_{\alpha\in A} U_\alpha$ and U_α is an open neighbourhood of $x_\alpha y_\alpha$ in R_α. Since $(x_\alpha, y_\alpha) \to x_\alpha y_\alpha$ is continuous for each $\alpha \in A$, so there exists neighbourhoods $V_{\alpha_i}, V'^{-1}_{\alpha_i}$ of x_{α_i} and y_{α_i} respectively such that $V_{\alpha_i} V'^{-1}_{\alpha_i} \subseteq U_{\alpha_i}$ for each $i = 1, 2, \ldots, n$. Now let $V = \prod_{\alpha\in A} V_\alpha$ and $V' = \prod_{\alpha\in A} V'_\alpha$, then V and V' are neighbourhoods of x, y respectively. This implies $VV'^{-1} = \prod(V_{\alpha_i} V'^{-1}_{\alpha_i}) \subseteq \prod U_\alpha = U \subseteq W$. This proves the theorem. □

We finish our work by the following

Theorem 4.6. *An LA-subring S of a topological LA-ring R is a topological LA-subring.*

Proof. Let R be a topological LA-ring and S be an algebraic LA-subring of R. Then S is endowed with relative topology induced from R. Since the mappings : $(x, y) \to x - y$ and $(x, y) \to xy$ of $R \times R$ are continuous so their restriction from $S \times S$ into S is also continuous.

Let a, b be two elements of S and let $ab^{-1} = c$. Every neighbourhood W' of the element c in H can be obtained as the intersection with S of some neighbourhood W of c in G. i.e., $W' = H \cap W$. Since R is a topological LA-ring so there exists neighbourhoods U and V of a, b such that $UV^{-1} \subseteq W$. Now $U' = S \cap U$ and $V' = S \cap V$ are the relative neighbourhoods of a and b in S. Thus we have $U'V'^{-1} \subseteq W$ and also $U'V'^{-1} \subseteq H$. Hence $U'V'^{-1} \subseteq W'$. Hence S is a topological LA-subring. □

References

[1] **V. I. Arnautov, S. T. Glavatsky and A. V. Mikhalev**, *Introduction to the theory of topological rings and modules*, Marcel Dekker, New York, 1996.

[2] **P. Holgate**, *Groupoids satisfying a simple invertive law*, Math. Stud. **61** (1992), 101 − 106.

[3] **T. Hussain:**, *Introduction to topological groups*, W. B. Saunders Company, 1966.

[4] **M. A. Kazim and M. Naseerudin**, *On almost semigroups*, Alig. Bull. Math. **2** (1972), 1 − 7.

[5] **Q. Mushtaq and Q. Iqbal**, *Decomposition of a locally associative LA-semigroups*, Semigroup Forum **41** (1991), 155 − 64.

[6] **Q. Mushtaq and M. S. Kamran**, *On LA-semigroup with weak associative law*, Scientific Khyber. **1** (1989), 69 − 71.

[7] **Q. Mushtaq and S. M. Yusuf:**, *On LA-semigroups*, Alig. Bull. Math. **8** (1978), 65 − 70.

[8] **L. Pontrjagin**, *Topological groups*, Princeton University Press, 1946.

[9] **T. Shah and I. Rehman**, *On LA-rings of finitely nonzero functions*, Int. J. Contemp. Math. Sci. **5** (2010), 209 − 222.

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