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How and why Moufang loops behave like groups

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Abstract. This paper gives a brief survey of recent progress in terms of developing results on loops, particularly Moufang loops, and how certain loops behave very similarly to groups.

1. Introduction

To make the paper accessible to a broader audience, we will start off by looking at the connections between Moufang loops and groups that admit triality summarizing the similarities of these two structures. Much of what we present here is already known. The main aim here is to give a brief survey showing that different classes of loops, especially Moufang loops, hold many of the characteristics as do groups.

Many properties of groups have recently been established for special classes of loops. Some of these classes require the classification of finite simple groups. The particular algebraic properties presented here will include

1. the Lagrange property,
2. Sylow's Theorems,
3. Hall's Theorem, and
4. the property of a loop having its commutant as a (normal) subloop.

Such results which are elementary for groups are more sophisticated for loops.

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2. Loops

A non-empty set L with a binary operation is called a *loop* if there exists a two-sided identity $1 \in L$ such that for any $a, b \in L$ the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in L$. The loop L is said to be *power associative* if the subloop generated by any element of L is a cyclic subgroup. Furthermore, it is said to be *diassociative* if any two of its elements generate a group.

A *Moufang loop* is a loop L that satisfies one (equivalently all) of the following identities:

$$(xy)(zx) = x((yz)x),$$
$$((xy)x)z = x(y(xz)),$$
$$z(x(yx)) = ((zx)y)x.$$

These weaker forms of the associative law, also called the *Moufang identities*, were first introduced by Ruth Moufang. The equivalence of these identities can be found in Lemma 3.1 of [2].

Even in the absence of associativity, Moufang loops still capture many of the properties that hold for groups. It was proven by Ruth Moufang [33] that every Moufang loop is diassociative. From this it follows that every element x of a Moufang loop of odd order has a unique square root which will be denoted by $x^{1/2}$.

We will define permutations R_x and L_x for an arbitrary loop L in the following way:

$$aR_x = ax,$$
$$aL_x = xa$$

for any $a \in L$. Here R_x is called the *right translation* of L by x. Similarly, L_x is called the *left translation* of L by x. The *multiplication group* of L, denoted $\mathrm{Mlt}(L)$, is the group of permutations generated by these maps:

$$\langle R_x, L_x \mid x \in L \rangle .$$

For any loop L the *inner mapping group* is the group of elements in $\mathrm{Mlt}(L)$ that stabilize the identity. It has been shown in [2] that the inner mapping group is generated by $T_x = L_x^{-1}R_x$, $R_xR_yR_{xy}^{-1}$, and $L_xL_yL_{xy}^{-1}$. A subloop K of L is *normal* in L if it is invariant under any inner mapping.

The set of all elements of a loop L which associate in any order with every pair of elements of L is called the *nucleus* of L and is denoted by

Nuc(L). Here the nucleus of L is a subgroup of L and if L is a Moufang loop then its nucleus is a normal subgroup [2, p. 114, Theorem 2.1].

A *CC-loop,* namely a conjugacy closed loop, was originally defined [20] to be a loop in which the left and right multiplications are closed under conjugations. That is, for any elements x and y, there exist elements a and b such that $L_x^{-1}L_yL_x = L_a$ and $R_x^{-1}R_yR_x = R_b$. A loop L is an *extra loop* if L is both conjugacy closed and a Moufang loop. It was F. Fenyves [6, 7] who showed that those CC-loops that are also diassociative are precisely the extra loops. Another type of Moufang loop, due to H.O. Pflugfelder [35], is an M_k *loop* which is a Moufang loop L for which $L/\mathrm{Nuc}(L)$ is of exponent $k-1$.

A Moufang loop can be formed by looking at the split octonions generated by *Zorn vector matrices* over a field F

$$\mathrm{Oct}(F) = \left\{ \left(\begin{array}{cc} a & \vec{v} \\ \vec{u} & b \end{array} \right) \middle| \; a, b \in F, \ \vec{u}, \vec{v} \in F^3 \right\}.$$

Here

$$\left(\begin{array}{cc} a & \vec{v} \\ \vec{u} & b \end{array} \right) + \left(\begin{array}{cc} c & \vec{\alpha} \\ \vec{\beta} & d \end{array} \right) = \left(\begin{array}{cc} a+c & \vec{v}+\vec{\alpha} \\ \vec{u}+\vec{\beta} & b+d \end{array} \right)$$

$$\left(\begin{array}{cc} a & \vec{v} \\ \vec{u} & b \end{array} \right)\left(\begin{array}{cc} c & \vec{\alpha} \\ \vec{\beta} & d \end{array} \right) = \left(\begin{array}{cc} ac+\vec{v}\cdot\vec{\beta} & a\vec{\alpha}+d\vec{v}-\vec{u}\times\vec{\beta} \\ b\vec{\beta}+c\vec{u}+\vec{v}\times\vec{\alpha} & bd+\vec{u}\cdot\vec{\alpha} \end{array} \right)$$

where $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ are the usual dot product and cross product. With the quadratic form $q\left(\left(\begin{array}{cc} a & \vec{v} \\ \vec{u} & b \end{array} \right)\right) = ab - \vec{v}\cdot\vec{u}$, such a matrix M is invertible if and only if $q\,(M) \neq 0$. The set of invertible matrices of norm one forms a nonassociative Moufang loop and is called the special linear loop denoted by SLL(F). The center of SLL(F_q) is $\{\pm I\}$ and the central quotient is the *Paige loop* $P(q)$. This central quotient is also sometimes denoted by PSLL(F_q).

Finally, a loop L is said to be a (left) Bol loop if it satisfies the identity $x(y(xz)) = (x(yx))z$ for all $x, y, z \in L$. One can see that every Moufang loop is a Bol loop. A right Bol loop is similarly defined in the sense that it satisfies the identity $((zx)y)x = z((xy)x)$ for all $x, y, z \in L$. In particular, a loop is both left Bol and right Bol if and only if it is a Moufang loop.

3. Groups with triality

Here we look at the correspondence between Moufang loops and groups with triality to better understand why Moufang loops behave like groups. The following definition is due to S. Doro [5].

Definition 3.1. Suppose G is a group, $S \leqslant \mathrm{Aut}(G)$, $S = \langle \sigma, \rho \rangle \cong S_3$ with $|\sigma| = 2$ and $|\rho| = 3$. The pair (G, S) is called a *group with triality* if $[G, S] = G$ and for every $g \in G$ the identity

$$[g, \sigma][g, \sigma]^{\rho}[g, \sigma]^{\rho^2} = 1$$

holds.

Here $[g, \sigma] = g^{-1}g^{\sigma}$ for all $g \in G$. So if $a = [g, \sigma]$ for some $g \in G$ then it follows that $a^{\sigma} = a^{-1}$.

If G is a group with triality $S = \langle \sigma, \rho \rangle$ and $L = \{[g, \sigma] \mid g \in G\}$ then, from [5], L is a Moufang loop associated with G under the binary operation

$$a \centerdot b = \left(a^{-1}\right)^{\rho} b \left(a^{-1}\right)^{\rho^2} = a^{-\rho} b a^{-\rho^2}.$$

Suppose that G is a group with triality S where L is the corresponding Moufang loop. If K is a subloop of L then there is a subgroup

$$G_L(K) = \left\langle K, K^{\rho}, K^{\rho^2} \right\rangle = \langle K, K^{\rho} \rangle$$

of G, which is also invariant under S, such that K is the Moufang loop associated with $G_L(K)$. Moreover, this group is minimal with respect to these properties. To see that K is the Moufang loop associated with $G_L(K)$, note that for any $g \in K$ there exists an element $g^{\rho} \in G_L(K)$ such that $[g^{\rho}, \sigma] = g^{-\rho}g^{-\rho^2} = g$. Therefore, K is contained in the Moufang loop associated with $G_L(K)$. For the other direction, let $a_1 \cdots a_{n-1}a_n \in G_L(K)$ where $a_i \in K \cup K^{\rho}$ for all $1 \leqslant i \leqslant n$. Here $[a_1 \cdots a_{n-1}a_n, \sigma] = a_n^{-1}[a_1 \cdots a_{n-1}, \sigma]a_n^{\sigma}$. If $a_n = g^{\rho} \in K^{\rho}$ then $[a_1 \cdots a_{n-1}a_n, \sigma] = g^{-\rho}[a_1 \cdots a_{n-1}, \sigma]g^{-\rho^2} = g \centerdot [a_1 \cdots a_{n-1}, \sigma]$. Whereas, if $a_n = g \in K$ then

$$\begin{aligned}
[a_1 \cdots a_{n-1}a_n, \sigma] &= g^{-1}[a_1 \cdots a_{n-1}, \sigma]g^{-1} \\
&= g^{\rho}g^{\rho^2}[a_1 \cdots a_{n-1}, \sigma]g^{\rho}g^{\rho^2} \\
&= g^{-1} \centerdot [a_1 \cdots a_{n-1}, \sigma] \centerdot g^{-1}.
\end{aligned}$$

So by induction on n, the Moufang loop associated with $G_L(K)$ is contained in K and is therefore equal to K.

It was shown by S. Doro [5] that for any Moufang loop L there exists a group G with triality S such that L is the corresponding Moufang loop. However, such a group is not necessarily unique.

Lemma 3.2. ([5], Lemma 1) *If L is a Moufang loop and G is a triality group of L then L^{ρ^2} is a right transversal of $C_G(\sigma)$ in G. So every element in G can be written uniquely as xg^{ρ^2} where $x \in C_G(\sigma)$ and $g \in L$ and $|G| = |C_G(\sigma)| \cdot |L|$. Similarly, every element in G can be written uniquely as yh^{ρ} where $y \in C_G(\sigma)$ and $h \in L$.* □

Definition 3.3. A *Latin square design* $\mathcal{D}$ is a pair $\mathcal{D} = (P, A)$ of points $P = P(\mathcal{D})$ and lines $A = A(\mathcal{D})$ (subsets of P) such that

(i) P is the disjoint union of three parts R, C, E called the *components of* P with $|R| = |C| = |E|$,

(ii) every line $l \in A$ contains exactly three points, meeting each of the components R, C, E exactly once,

(iii) any pair of points from different components belong to exactly one line.

A Latin square design $\mathcal{D}$ may be constructed using a loop L. If L is a loop satisfying the inverse property, namely it has two-sided inverses such that $x^{-1}(xy) = y = (yx)x^{-1}$ and $(xy)^{-1} = y^{-1}x^{-1}$ for any $x, y \in L$, then the Latin square design will have central automorphisms (as defined below) for certain elements of L. Moreover, it will have central automorphisms for every element of L if and only if L is a Moufang loop. If L is a Moufang loop, this design can then be used to construct a triality group whose corresponding Moufang loop is isomorphic to L.

Let the point set be three disjoint copies of L, $P = L_R \cup L_C \cup L_E$, where there is a bijection $L \longrightarrow L_R$ given by $a \mapsto a_R$ and similarly for L_C and L_E. In particular 1_R, 1_C, and 1_E are distinct elements of P where 1 denotes the identity of L. Furthermore, let the line set A be the set of triples $\{a_R, b_C, c_E\}$ where $a, b, c \in L$ with $(ab)c = 1$ in L. A question that arises is: does there exist a copy of S_3 in $\mathrm{Aut}(\mathcal{D})$ that permutes these three copies of L? We will now show that Moufang loops correspond to Latin square designs admitting certain types of automorphisms that permute the three copies of L.

Choose a component, say L_E, of the design and an element $x_E \in L_E$. There exists a unique reflection on $L_R \cup L_C \cup \{x_E\}$ that swaps the two components L_R and L_C, fixing the point $x_E \in L_E$ and stabilizing all of the lines that contain x_E. Denote this reflection by σ_{x_E}. Here

$$\sigma_{x_E} : a_R \mapsto (a^{-1}x^{-1})_C, \quad a_C \mapsto (x^{-1}a^{-1})_R$$

for any $a_R \in L_R$ and $a_C \in L_C$. We will now show that σ_{x_E} can be uniquely extended to the entire set P forming an automorphism of $\mathcal{D}$ if and only if $x((ba)x) = (xb)(ax)$ for all $a, b \in L$.

Suppose σ_{x_E} does extend to some automorphism of $\mathcal{D}$. We continue to denote this extension by σ_{x_E}. Now consider the lines

$$\{((ax)^{-1})_R, ((xb)^{-1})_C, ((xb)(ax))_E\}, \{b_R, a_C, (a^{-1}b^{-1})_E\} \in A.$$

Since

$$\begin{aligned} \sigma_{x_E} : b_R &\mapsto (b^{-1}x^{-1})_C = ((xb)^{-1})_C \\ a_C &\mapsto (x^{-1}a^{-1})_R = ((ax)^{-1})_R, \end{aligned}$$

the involution σ_{x_E} must swap the lines $\{((ax)^{-1})_R, ((xb)^{-1})_C, ((xb)(ax))_E\}$ and $\{b_R, a_C, (a^{-1}b^{-1})_E\}$. Thus, in order for $\sigma_{x_E} \in \mathrm{Aut}(\mathcal{D})$, σ_{x_E} must swap $((ba)^{-1})_E$ and $((xb)(ax))_E$. Since a and b are arbitrary, we get that

$$\sigma_{x_E} : a_E \mapsto (x(a^{-1}x))_E = ((xa^{-1})x)_E.$$

Note that this uniquely determines the extension. So from this, σ_{x_E} must map $((ba)^{-1})_E$ to $(x((ba)x))_E = ((x(ba))x)_E$. But since

$$\sigma_{x_E} : ((ba)^{-1})_E \mapsto ((xb)(ax))_E,$$

$x((ba)x) = (xb)(ax)$ for all $a, b \in L$. Therefore, if $\sigma_{x_E} \in \mathrm{Aut}(\mathcal{D})$ for any $x \in L$ then L is a Moufang loop.

With the Moufang identities, it can easily be shown that $\sigma_x \in \mathrm{Aut}(\mathcal{D})$ for any $x \in P$. These reflections are called *triality reflections*. The triality reflection σ_x is often referred to as a central automorphism with center x. Let H be the group generated by these triality reflections, namely, $H = \langle \sigma_x \mid x \in P \rangle$. If we let G be the normal subgroup of H, of index six, that stabilizes the three components of $\mathcal{D}$ then G, which is sometimes called the *rotation subgroup,* is a group with triality $S = \langle \sigma_{1_R}, \sigma_{1_C} \rangle$. Here H is sometimes thought of as a triality group [8, 16, 25, 37] which is equivalent but different from Doro's definition. If we let $\sigma = \sigma_{1_E}$ and $\rho = \sigma_{1_C}\sigma_{1_R}$ then

$$[g, \sigma] = g^{-1}\sigma g\sigma : a_R \mapsto (xa)_R,\ \ a_C \mapsto (ax)_C,\ \ a_E \mapsto (x^{-1}ax^{-1})_E$$

where x_E is the image of 1_E under g. Moreover,

$$[g, \sigma][g, \sigma]^{\rho}[g, \sigma]^{\rho^2} = 1$$

and under the binary operation

$$a \centerdot b = a^{-\rho}ba^{-\rho^2}$$

the set $\{[g, \sigma] \mid g \in G\}$ is isomorphic to the original Moufang loop L.

4. Lagrange's Theorem

Definition 4.1. A loop L is said to have the *weak Lagrange property* if, for each subloop K of L, $|K|$ divides $|L|$. It has the *strong Lagrange property* if every subloop K of L has the weak Lagrange property.

From this point on by "the Lagrange property" or "Lagrange's Theorem" we mean the strong Lagrange property.

Unlike for groups, (right) cosets of a subloop of a loop need not partition the loop. Nevertheless, for certain classes of loops the Lagrange property still holds.

Definition 4.2. If K is a loop then the *left, middle*, and *right nucleus* of K are defined, respectively, as:

$$\begin{aligned} N_\lambda(K) &= \{a \in K \mid aR_{xy} = aR_xR_y \text{ for all } x, y \in K\}\,, \\ N_\mu(K) &= \{a \in K \mid aL_yR_x = aR_xL_y \text{ for all } x, y \in K\}\,, \\ N_\rho(K) &= \{a \in K \mid aL_{yx} = aL_xL_y \text{ for all } x, y \in K\}\,. \end{aligned}$$

It immediately follows from the definition that if $H \leqslant K$ is a subgroup of $N_\lambda(K)$ then its right cosets partition K. Likewise, if H is a subgroup of $N_\rho(K)$ then its left cosets partition K. If H is a subgroup of $N_\mu(K)$ then, using the fact that $H(hx) = Hx$ and $(xh)H = xH$ for all $h \in H$ and all $x \in K$, it can easily be shown that both its left cosets and its right cosets partition K.

Definition 4.3. A K subloop of a loop L is a *characteristic subloop* if every automorphism of L takes K onto K.

It was K. Kunen [31] who then showed the following.

Lemma 4.4. *Suppose L is a CC-loop and that K is a characteristic subloop of L. Then any two right cosets of K in L are either equal or disjoint.* $\square$

From this, we get the following theorem for those CC-loops that satisfy certain conditions.

Theorem 4.5. *If L is finite CC-loop and K is a proper subloop of L, then $|K|$ divides $|L|$ if either of the following hold.*

1. *K is a group and the Sylow p-subgroups of K are commutative for each prime p.*
2. *K is a characteristic subloop of L.* $\square$

However, many loops L satisfy the Lagrange property even though there exist subloops whose cosets do not partition L. Originally, back in 1958, R. Bruck [2] reduced this problem to simple loops by proving the following lemma.

Lemma 4.6. *Let L be a loop with a normal subloop N such that both N and L/N have the Lagrange property. Then L itself satisfies the Lagrange property.* $\square$

From this, along with A.S. Basarab's result that $L/\mathrm{Nuc}(L)$ is an abelian group for any finite CC-loop L, it follows that the Lagrange property holds for any finite CC-loop.

Looking deeper into this problem, specifically along the lines of Moufang loops, G. Glauberman [17, (1968)] showed any finite Moufang loop of odd order satisfies the Lagrange property. It was then further shown by O. Chein, M. Kinyon, A. Rajah, and P. Vojtěchovský [3] that if L is an M_k loop where k is even then L must satisfy the Lagrange property. The outline of their proof goes as follows. If k is even then the nucleus of L, say N, is a normal subloop with L/N of exponent $k-1$ which is odd. Thus L/N satisfies the Lagrange property. Hence, since N is associative, by Lemma 4.6, L itself satisfies the Lagrange property.

But this just raised the question as to whether or not the Lagrange property is valid for all Moufang loops. From Lemma 4.6 Lagrange's Theorem holds for all finite Moufang loops if and only if it holds for finite simple Moufang loops.

Since Lagrange's Theorem certainly holds for finite associative simple Moufang loops, one now only has to show that it carries over to the finite nonassociative simple Moufang loops. Nonassociative Moufang loops exist and it was L. Paige [34] who first obtained the result that the projective special linear loops are simple.

Theorem 4.7. *If L is a Paige loop, namely $L = SLL(q)/\{\pm I\}$, then L is a simple nonassociative Moufang loop.* $\square$

An outline of the proof of Theorem 4.7 goes as follows. Nonidentity elements of the form $\begin{pmatrix} 1 & \vec{v} \\ \mathbf{0} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{0} \\ \vec{u} & 1 \end{pmatrix} \in \mathrm{SLL}(q)$ are all conjugate and together generate all elements of the form $\begin{pmatrix} 1 & \vec{v} \\ \vec{u} & 1 \end{pmatrix}$. From this it follows that they generate the entire set of elements in $P(q)$. So it is enough to show that a nontrivial normal subloop would have to contain a nontrivial element of the form $\begin{pmatrix} 1 & \vec{v} \\ \mathbf{0} & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \mathbf{0} \\ \vec{u} & 1 \end{pmatrix}$. Now if $N \leqslant \mathrm{SLL}(q)$ is a nontrivial normal subloop strictly greater than $\{\pm I\}$ then it must contain a nonidentity element of the form $\begin{pmatrix} 1 & \vec{v} \\ \vec{u} & 1 \end{pmatrix}$. It then follows that every normal subloop of $\mathrm{SLL}(q)$ which properly contains the center $\{\pm I\}$ contains a nontrivial element of the form $\begin{pmatrix} 1 & \vec{v} \\ \mathbf{0} & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \mathbf{0} \\ \vec{u} & 1 \end{pmatrix}$. Since these generate the entire loop, $N = \mathrm{SLL}(F)$.

It now becomes a natural question to ask for a classification of all nonassociative simple Moufang loops where the Paige loops form an infinite family of these. It was first shown by S. Doro [5] that if L is a finite nonassociative simple Moufang loop then its multiplication group is a finite simple group that admits triality. With this fact, Martin Liebeck [32] classified the finite simple nonassociative Moufang loops by using the classification of finite simple groups to show that the only finite simple groups that admit triality are the simple groups $D_4(q)$. These groups are associated with the Paige loops, $P(q)$. J.I. Hall [24] later strengthened this result by using a similar technique and proved that a locally finite simple Moufang loop is either a group or is isomorphic to $\mathrm{PSLL}(F)$, where F is a locally finite field. A field is locally finite if any finite subset generates a finite subfield. Such a field has characteristic p, is algebraic over its prime subfield, and is therefore a subfield of the algebraic closure of F_p.

Thus it is now clear that the Lagrange property holds for Moufang loops if it holds for the finite Paige loops, $P(q)$. The verification that this holds The first Paige loop $P(2)$ for which the Lagrange property was checked was done by M.L. Merlini Giuliani and C. Polcino Milies [23] and separately by P. Vojtěchovský [38].

Lagrange's Theorem for the full family of Paige loops was then checked by A. Grishkov and A. Zavarnitsine [22], and independently, by S.M. Gagola III and J.I. Hall [16]. A. Grishkov and A. Zavarnitsine used the fact that every maximal subloop K of $P(q)$ is associated with a particular subgroup G of $D_4(q) = P\Omega_8^+(q)$. They used P. B. Kleidman's result [30] of the classification of maximal subgroups of $\mathrm{Aut}(P\Omega_8^+(q))$ to first find the possible candidates for G. With this, they then determined the possible orders for K, thus showing that $|K|\,||P(q)|$. In [16], S.M. Gagola III and J.I. Hall determined all of the maximal subgroups of $D_4(q) : S_3 = P\Omega_8^+(q) : S_3$ using P. B. Kleidman's result [30]. It was then determined which of these maximal subgroups, M, of $P\Omega_8^+(q) : S_3$ satisfied $P\Omega_8^+(q) : S_3 = P\Omega_8^+(q).\langle D \cap M\rangle$ where D is the set of triality reflections in $P\Omega_8^+(q) : S_3$. With this information it follows that if K is a subloop of $P(q)$ which is associated with M then the number of triality reflections in M is $3|K| = |D \cap M|$ which divides $|D| = 3|P(q)|$. This implies that if K is a maximal subloop of $P(q)$ then $|K|\,||P(q)|$.

Later in [15] all of the subloops of the unit octonions were categorized. In particular, all of the finite maximal subloops of the unit octonions were described. With the order of each of these maximal subloops dividing the order of the Paige loop, Lagrange's Theorem for Moufang loops immediately follows as a corollary.

5. Sylow's Theorems

After Lagrange's Theorem was established for finite Moufang loops [16, 22], it then becomes natural to ask to what extent Sylow's Theorems carry over to finite Moufang loops.

Definition 5.1. Let p be a prime. A finite loop L is said to be a *p-loop* if its order is a power of the prime p.

In some types of loops this is equivalent to saying that every element of L has an order that is a power of p. The following theorem was first proven by Glauberman and Wright in [18] and [19].

Theorem 5.2. *If L is a finite Moufang loop and p is a prime, then the following are equivalent:*

1. $|L|$ *is a power of* p.
2. *The order of every element of* L *is a power of* p. □

These statements are also equivalent in other classes of loops. Since the Lagrange property holds for finite CC-loops, it immediately follows that if L is a power-associative CC-loop and p is a prime then L is a p-loop if and only if every element in L has an order that is a power of p.

Having established Lagrange's Theorem for certain loops, in particular Moufang loops, one can then define a Sylow p-subloop just as they are defined for groups. For loops for which Lagrange's Theorem holds, the index of a subloop H of a loop L is $|L|/|P|$ denoted by $[L:H]$.

Definition 5.3. A p-subloop, P, of a Moufang loop, L, is called a *Sylow p-subloop* if the index is not divisible by the prime p. We will denote the set of all Sylow p-subloops of L as $Syl_p(L)$.

G. Glauberman [18] first approached Sylow's Theorem for Moufang loops of odd order by looking at the multiplication group of a p-loop and proving the following lemma.

Lemma 5.4. *Let p be a prime and let H be a finite Moufang loop of odd order. If P is a p-subloop of H then* $\mathrm{Mlt}(P)$ *is a p-group.* □

Using Lemma 5.4, G. Glauberman proved the existence of Sylow p-subloops in Moufang loops of odd order in the following way. Let H be a Moufang loop of odd order and suppose that P is a p-subloop of H that is not a Sylow p-subloop of H. If $G \leqslant \mathrm{Mlt}(H)$ is the set of elements of $\mathrm{Mlt}(H)$ that stabilize P then G contains a copy of $\mathrm{Mlt}(P)$. By Sylow's Theorems for groups, G contains a Sylow p-subgroup, say Q, that contains $\mathrm{Mlt}(P)$. Since $p|[H:P]$, it can be shown that $p|[\mathrm{Mlt}(H):Q]$. With this G. Glauberman then proved that there is a subloop of H, namely $K = \{g \in H \mid R_g L_g \in N_{\mathrm{Mlt}(H)}(Q)\}$, such that $P \trianglelefteq K$ and $p|[K:P]$. With this one can create a p-subloop of K that properly contains P. Therefore, by induction, H must contain a Sylow p-subloop.

Later the Sylow Theorems were shown to hold for other Moufang loops, namely, finite extra loops. It was first F. Fenyves [7] who showed that if L is an extra loop then for any $x \in L$, $x^2 \in \mathrm{Nuc}(L)$. O. Chein and

D. Robinson [4] proved later that extra loops are exactly those Moufang loops with the squares in the nucleus. It then immediately follows that Sylow's Theorems hold for finite extra loops provided that p is an odd prime. In 2004, M. Kinyon and K. Kunen [26] proved the following theorem, showing that this is true for extra loops even when p is equal to 2.

Theorem 5.5. *Suppose that L is a finite extra loop and p is prime.*

1. *If Q is a p-subloop of L then there exists $P \in Syl_p(L)$ containing Q.*
2. $|Syl_p(L)| \equiv 1$ *(mod p) with* $|Syl_p(L)|\,\big|\,|\mathrm{Nuc}(L)|$.
3. *If $P_1, P_2 \in Syl_p(L)$ then there exists an $x \in \mathrm{Nuc}(L)$ such that $P_1 T_x = P_2$, so that P_1 and P_2 are isomorphic.* □

It was then shown in [11] that Sylow's Theorems, with the possible exception of conjugacy, hold in all finite Moufang loops for the prime $p = 2$. For if L is a finite Moufang loop and G is a corresponding group with triality $S = \langle \sigma, \rho \rangle$, then for every 2-subloop $Q \leqslant L$ there exists a 2-subgroup $H \leqslant G$ that is invariant under σ satisfying $Q = [H, \sigma]$. Furthermore, if P is a Sylow 2-subgroup of G that is invariant under σ then $[P, \sigma]$ is a Sylow 2-subloop of L showing that every 2-subloop of L is embedded in a Sylow 2-subloop of L. By using the fact that there are an odd number of Sylow 2-subgroups of G that are invariant under σ, one can also show that the number of Sylow 2-subloops of L is also congruent to one modulo two.

This then brings up the question as to whether or not Sylow's Theorems hold for other primes in finite Moufang loops.

A. Grishkov and A. Zavarnitsine [21] noted if K is a finite Moufang loop then proving the existence of a Sylow p-subloop of K amounts to proving the existence of a related S-invariant subgroup of a corresponding group G with triality $S = \langle \sigma, \rho \rangle$. Unfortunately, one cannot guarantee the existence of an S-invariant Sylow p-subgroup of G. Thus A. Grishkov and A. Zavarnitsine were only able to prove only that G possesses a sufficiently large S-invariant p-subgroup whose corresponding Moufang subloop in K is a Sylow p-subloop of K for certain primes p.

From [15] one can see that any element of the Paige loop $P(q)$ will have an order that divides $\frac{q^2-1}{gcd(q+1,2)}$. But $|P(q)| = \frac{q^3(q^4-1)}{gcd(q+1,2)}$ and any prime divisor of $\frac{q^2+1}{gcd(q+1,2)}$ is not a prime divisor of $q^3(q^2-1)$. It then follows that for every prime divisor p of $\frac{q^2+1}{gcd(q+1,2)}$ there does not exist an element of $P(q)$ whose order is p.

Definition 5.6. We say that p is a *Sylow prime* for a finite Moufang loop K if $p \nmid \frac{q^2+1}{gcd(q+1,2)}$ for all q for which a composition factor of L is isomorphic to the Paige loop $P(q)$.

With this definition, A. Grishkov and A. Zavarnitsine [21] proved the following theorem.

Theorem 5.7. *Let K be a finite Moufang loop and let p be a prime. Let G be a group with triality such that K is its corresponding Moufang loop. Then the following conditions are equivalent:*

(i) K has a Sylow p-subloop,

(ii) p is a Sylow prime for K,

(iii) G has an S-invariant p-subgroup Q such that its corresponding Moufang loop is in $Syl_p(K)$. □

By the Lagrange property, the order of a maximal p-subloop of a finite Moufang loop K is certainly bounded by the product of the orders of Sylow p-subloops of the composition factors of K for which p is a Sylow prime. This bound is actually achieved for all finite Moufang loops K. Let p be a prime and K be a finite Moufang loop where K_i, $1 \leqslant i \leqslant n$, are the composition factors of K. If p is a Sylow prime for K_i then let $P_i \in Syl_p(K_i)$ otherwise let $P_i = 1$. We say that a subloop of L is a *quasi-Sylow p-subloop* if it has order $\prod_{i=1}^{n} |P_i|$. With this, A. Grishkov and A. Zavarnitsine [21] showed that Sylow's Theorem can be reformulated as follows:

Theorem 5.8. *Every finite Moufang loop has a quasi-Sylow p-subloop for all primes p.* □

But the question still stands - for what primes p is a p-subloop of a finite Moufang loop guaranteed to be contained in a Sylow p-subloop?

In [10] it was shown that if G is a finite group with triality $S = \langle \sigma, \rho \rangle$ where L is its corresponding Moufang loop then for any prime p and any p-subloop $P \leqslant L$ the subgroup $G_L(P) = \left\langle P, P^\rho, P^{\rho^2} \right\rangle \leqslant G$ is a p-subgroup. Like for the case where $p = 2$, the following theorem holds for all primes.

Theorem 5.9. ([10], Theorem 4.5) *Let G be a finite group with triality $S = \langle \sigma, \rho \rangle$ where L is the corresponding Moufang loop. If P is a Sylow p-subloop of L then $G_L(P)$ is contained in a σ-invariant Sylow p-subgroup of G, say Q, satisfying $Q \cap L = P$.* □

So if K is a finite Moufang loop and p is a Sylow prime for K then one can better understand the Sylow p-subloops of K by looking at the Sylow p-subgroups of G that are invariant under σ.

Theorem 5.10. ([10], Theorem 4.6) *If L is a finite Moufang loop, p is a Sylow prime for L, and P is a p-subloop of L then there exists a Sylow p-subloop of L that contains P.* □

From the last two theorems it immediately follows that for any p-subloop of a finite Moufang loop L, say P, the triality group $G_L(P)$ is contained in a σ-invariant Sylow p-subgroup of a triality group of L. One can use this to get the following theorem.

Theorem 5.11. *If L is a finite Moufang loop, p is a Sylow prime for L, and $P_1, P_2 \in Syl_p(L)$ then there exists an element $x \in C_{G(L)}(\sigma)$ such that $\langle P_1 \rangle^x = \langle P_2 \rangle \leqslant G(L)$.* □

From this theorem it follows that all of the Sylow p-subloops of a finite Moufang loop are isomorphic provided p is a Sylow prime.

Suppose that p is a Sylow prime for a finite Moufang loop L. It is now natural to ask if the number of Sylow p-subloops of L is congruent to one modulo p as is the case when L is a group.

Theorem 5.12. ([14], Theorem 1.1) *If L is a finite Moufang loop and p is a Sylow prime for L then $|Syl_p(L)| \equiv 1(mod\ p)$.* □

One can first show that this holds for the finite simple Moufang loops. From [11] it is known that if $p = 2$ then the number of Sylow p-subloops is odd. However, if p is an odd prime then

$$|Syl_p(P(q))| = \begin{cases} (q^4+q^2+1)|Syl_p(\mathrm{SL}_2(q))| & \text{if } p \mid q \\ (q^2-q+1)|Syl_p(\mathrm{SL}_2(q))| & \text{if } p \mid (q-1) \\ q^2(q^2+q+1)|Syl_p(\mathrm{SL}_2(q))| & \text{if } p \mid (q+1) \end{cases}$$

so that in all cases $|Syl_p(L)| \equiv 1(\mathrm{mod}\ p)$. This leads to the following theorem.

Theorem 5.13. ([14], Theorem 5.4) *Let p be a prime and let L be a finite Moufang loop that has a normal subloop, N, of index p. If $P \in Syl_p(L)$ then the number of Sylow p-subloops of L that contain $P \cap N$ is congruent to 1 modulo p.* □

Theorem 5.12 can then be proven inductively on the order of the Moufang loop L. If a counterexample has a normal subloop, N, whose index is not a power of p then one can create a smaller counterexample by looking at subloops A, $N \leqslant A \leqslant L$, where $A/N \in Syl_p(L/N)$. Otherwise, if it has a normal subloop of index which is a power of p and therefore a normal subloop with index p then it would immediately follow from Theorem 5.13 that $|Syl_p(L)| \equiv 1 \pmod p$.

For Moufang loops, it is known that the cardinality of $Syl_p(L)$ doesn't always divide $|L|$. For example, if L is the Paige loop $P(2)$ then $|L| = 120$ whereas $|Syl_3(L)| = 28$. To better understand the set $Syl_p(L)$, it is natural to examine a group G with triality $S = \langle \sigma, \rho \rangle$ which is associated with L and then determine how the elements of $\{G_L(P) \leqslant G \mid P \in Syl_p(L)\}$ are permuted by the elements of $C_G(S) \leqslant G$.

Theorem 5.14. *Suppose that G is a finite group with triality $S = \langle \sigma, \rho \rangle$ where L is the corresponding Moufang loop. If $3 \nmid |L|$ and $P_1, P_2 \in Syl_p(L)$ then there exists an element $y \in C_G(S)$ such that $G_L(P_1)^y = G_L(P_2)$.* □

Theorem 5.15. *Let G be a finite group with triality $S = \langle \sigma, \rho \rangle$ where its corresponding Moufang loop L is simple. If p is a Sylow prime of L with $P_1, P_2 \in Syl_p(L)$ then there exists an element $y \in C_G(S)$ such that $G_L(P_1)^y = G_L(P_2)$.* □

From these theorems we get the following corollary which gives us a better understanding of the cardinality of $Syl_p(L)$.

Corollary 5.16. *Suppose that G is a finite group with triality $S = \langle \sigma, \rho \rangle$ where L is the corresponding Moufang loop. If $3 \nmid |L|$ or if L is simple with p being a Sylow prime of L, then $|Syl_p(L)| \big| |C_G(S)|$.* □

Now if we let $Gr_p(L)$ be the product of all normal subloops of a finite Moufang loop L for which p is a Sylow prime, then from [21] we get that p is a Sylow prime for $Gr_p(L)$ and the set of Sylow p-subloops of $Gr_p(L)$ is equal to the set of quasi-Sylow p-subloops of L. So from this it follows that Theorems 5.10, 5.12 and 5.14 all carry over to quasi-Sylow p-subloops.

6. Hall's Theorem

A loop L is said to be *solvable* if there exists a series of subloops

$$L = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_k = l$$

such that for all $1 \leqslant i \leqslant k$, $N_i \trianglelefteq N_{i-1}$ and N_{i-1}/N_i is an abelian group. Let π be a set of primes. We say that L is a *π-loop* if it is finite and any prime factor of $|L|$ is contained in π. If Q is a π-subloop of a Moufang loop L whose index is not divisible by any primes in π then Q is called a *Hall π-subloop* of L.

Definition 6.1. Let H be a finite Moufang loop of odd order. Define a binary operation $\circ : H \times H \longrightarrow H$ by $x \circ y = x^{1/2}yx^{1/2}$ for $x, y \in H$. Then, under this new binary operation, H forms a loop that is power-associative. G. Glauberman denoted this loop by $H(\frac{1}{2})$.

Lemma 6.2. ([18], Theorem 3) *Let π be a set of primes and let K be a solvable π-subloop of a finite Moufang loop H of odd order. Then $K(\frac{1}{2})$ is a solvable π-loop and* $\mathrm{Mlt}(K)$ *is a solvable π-group.* □

Theorem 6.3. *Every finite Moufang loop of odd order is solvable.* □

In [18, Theorem 16], G. Glauberman proved this inductively by using $H(\frac{1}{2})$ along with its commutator-associator subloop and showing that if H is a finite simple Moufang loop of odd order then it is an abelian group.

Theorem 6.4. *Let π be a set of primes. Every π-subloop of an odd ordered Moufang loop H is contained in a Hall π-subloop of H.* □

Since H is a solvable loop, from Lemma 6.2, $H(\frac{1}{2})$ is also solvable. In [17, Theorem 8] G. Glauberman showed that the Hall π-condition holds for $H(\frac{1}{2})$. That is, every π-subloop of $H(\frac{1}{2})$ is contained in some Hall π-subloop of $H(\frac{1}{2})$. Then, by inducting on $|H|$, G. Glauberman [18, Theorem 11] finished off by showing that every Hall π-subloop of $H(\frac{1}{2})$ is a Hall π-subloop of H.

P. Hall's Theorem was then later proved to also hold for extra loops. Since $L/\mathrm{Nuc}(L)$ is an abelian group for any finite CC-loop L, a CC-loop L is solvable if and only if $\mathrm{Nuc}(L)$ is solvable. So it immediately follows that an extra loop is solvable if and only if its nucleus is solvable. With this M. Kinyon and K. Kunen [26] proved the following theorem.

Theorem 6.5. *Let π be a set of primes and let L be a finite solvable extra loop. Then*

1. *there exists a Hall π-subloop of L,*
2. *any π-subloop of L is contained in a Hall π-subloop of L,*

3. *if Q_1 and Q_2 are Hall π-subloops of L then there exists $x \in L$ such that $Q_1 T_x = Q_2$.* □

But since 1966, it has been an open question as to whether or not P. Hall's Theorem holds for all finite Moufang loops. For starters, Theorem 5.2 for p-loops carries over to π-loops for those Moufang loops whose composition factors are not Paige loops.

Lemma 6.6. *Suppose L is a finite Moufang loop where none of its composition factors are isomorphic to a Paige loop $P(q)$. Then it is a π-loop if and only if every element in L has an order that is a product of primes in π.*

Proof. If L contains an element whose order is not a product of primes in π then, by the Lagrange property, L is not a π-loop.

If L is not a π-loop then there exists a composition factor of L, say K, which is not a π-loop. Furthermore, since K is not isomorphic to a Paige loop, K must be a finite simple group. Thus, K contains an element of prime order not belonging to π. Thus the same is true for L. □

Corollary 6.7. *If L is a finite solvable Moufang loop then it is a π-loop if and only if every element in L has an order that is a product of primes in π.* □

Since groups with triality are naturally connected with Moufang loops, it is reasonable to look at the properties that hold amongst the triality groups of solvable Moufang loops.

If G is a group with triality S and $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a series of subgroups that are invariant under S then there exists a normal series $1 = L_0 \trianglelefteq L_1 \trianglelefteq \cdots \trianglelefteq L_n = L$ of Moufang loops such that L_i is the Moufang loop corresponding to G_i for $0 \leqslant i \leqslant n$. Moreover, L_{i+i}/L_i is the Moufang loop corresponding to G_{i+1}/G_i for $0 \leqslant i \leqslant n-1$. However, S may centralize G_{i+1}/G_i in which case its corresponding Moufang loop is 1.

Theorem 6.8. ([12], Theorem 3.6) *Suppose G is a finite group with triality where L is its corresponding Moufang loop. If $K \leqslant L$ is a subloop that is solvable then $G_L(K)$ is a solvable subgroup of G.* □

Theorem 6.9. ([12], Theorem 3.7) *Suppose L is a finite Moufang loop and G is a triality group associated with L. If $K \leqslant L$ is a solvable subloop then K is a π-subloop if and only if $G_L(K)$ is a π-subgroup of G.* □

A theorem of P. Hall's asserts that a group is solvable if and only if it contains a Hall π-subloop for any set of primes π. With the previous theorems in place one can readily establish the fact that P. Hall's Theorem extends to all finite Moufang loops.

Theorem 6.10. *Suppose L is a finite Moufang loop. Then L is solvable if and only if for every divisor n of $|L|$ such that $gcd(n, |L|/n) = 1$, L has a subloop of order n.*

Proof. Let L be a finite Moufang loop of minimal order such that it is not solvable and it has a subloop of order n for every divisor n of $|L|$ with $\gcd(n, |L|/n) = 1$. By P. Hall's Theorem, we know that L is nonassociative. Therefore, if L is simple then it is a Paige loop, $P(q)$. From [10] and [11], we know that $P(q)$ does not contain any Sylow p-subloop for any prime $p\Big|\frac{q^2+1}{\gcd(q+1,2)}$. Therefore, L is not simple and contains a nontrivial normal subloop, say N. Here both N and L/N have the property of containing a Hall π-subloop for any set of primes π. By minimality of L, both N and L/N are solvable. Hence L is solvable which forms a contradiction.

Now let L be a finite solvable Moufang loop of minimal order such that there exists a set of primes, π, where L does not contain a Hall π-subloop. Let G be a triality group of minimal order such that L is its corresponding Moufang loop. Since L is solvable, by Theorem 6.8, G is solvable. Since G is a solvable group with triality S, it contains a minimal normal elementary abelian p-group that is invariant under S, say M. Let P be the Moufang loop associated with M. Assume that $P = 1$. Then G/M is a triality group whose corresponding Moufang loop is isomorphic to L. But G is a minimal group with triality whose corresponding Moufang loop is L. Hence, $P \neq 1$ and G/M is a triality group whose corresponding Moufang loop, L/P, is of smaller order than L. By minimality of L, L/P contains a Hall π-subloop, say K/P. Note that $p \notin \pi$ because otherwise K would then be a Hall π-subloop of L. Also note that $L = K$ because if $L \neq K$ then, by minimality of L, K would contain a Hall π-subloop which would also be a Hall π-subloop of L. Since $L = K$, $p \nmid |L/P|$. Hence, there exists a complement to M in G that is invariant under S and has a corresponding Moufang loop which is a Hall π-subloop of L. Therefore, by contradiction, every finite solvable Moufang loop contains a Hall π-subloop for any set of primes π. $\square$

7. Normality of the center/commutant

For a group, the center and commutant coincide and end up being a normal subgroup.A question that may sound rather easy but has been around for over thirty years is the following. When is the commutant of a Moufang loop a normal subloop? We will start this section with some basic terminology.

Definition 7.1. The *center* $Z(L)$ of a loop L is the set of elements in L that commute and associate with all elements from L. That is, the set of elements in the nucleus of L that commute with each element in L.

The nucleus of a loop is an associative subloop making the center an abelian subgroup [2, 36]. Since a subloop $N \leqslant L$ is normal when $xN = Nx$, $x(yN) = (xy)N$, and $(Nx)y = N(xy)$ for any $x, y \in L$, it immediately follows that the center, like for groups, is normal.

Definition 7.2. The *commutant* $\mathbf{C}(L)$ of a loop L, sometimes called the *centrum*, is the set of elements in L that commute with each element in L. Namely,

$$\mathbf{C}(L) = \{a \in L \mid ax = xa \textit{ for all } x \in L\}.$$

The commutant of a loop L is not necessarily a subloop, nevertheless, $Z(L) = \mathrm{Nuc}(L) \cap \mathbf{C}(L)$. It is however true that the commutant is a subloop for certain loops. In [36], the following lemma is proven:

Lemma 7.3. *The commutant* $\mathbf{C}(L)$ *is a subloop if* L *is a Moufang loop.* □

Even though this is not generally true for all loops, M. Kinyon and J.D. Phillips [28] showed that if every element of the commutant of a Bol loop has finite odd order then the commutant is indeed a subloop. In 1964, G. Glauberman [17] showed that a finite Bol loop has odd order if and only if every element has odd order. So from these results one gets the following corollary.

Corollary 7.4. *The commutant of a finite Bol loop of odd order is a subloop.* □

This was then strengthened later [29] by M. Kinyon, J.D. Phillips, and P. Vojtěchovský who observed that if L is a finite Bol loop of order $2k$ where k is odd then $\mathbf{C}(L)$ is a subloop of L. This was done by noting that if L has an element of order two, say a, then that would be its only element of order two. Otherwise, L would contain an abelian subgroup of order four with $4 \nmid |L|$. So $\mathbf{C}(L) = aC \cup C$ where $C = \{x \in \mathbf{C}(L) \mid 2 \nmid |x|\}$. With this $\mathbf{C}(L)$ is closed under multiplication and is a subloop of L. Furthermore, they showed that for any integer $n > 2$, there exists a Bol loop of order $4n$ such that its commutant is not a subloop.

But it was back in 1978 when S. Doro [5] proposed a question asking under what conditions is the commutant of a Moufang loop a normal subloop.

It is easy to see that one gets $\mathbf{C}(L)T_x = \mathbf{C}(L)$ for any $x \in L$. But in order for $\mathbf{C}(L)$ to be normal in L, the following equalities must also hold for any $x, y \in L$:

$$(xy)^{-1}\,(x(y\mathbf{C}(L))) = \mathbf{C}(L)$$
$$((\mathbf{C}(L)x)y)\,(xy)^{-1} = \mathbf{C}(L).$$

Using the connection between groups with triality and Moufang loops, one can show that a Moufang loop L is an abelian group if and only if the minimal triality group G associated with L is abelian. Now let L be any Moufang loop, not necessarily commutative, and define a subset, N, of the minimal corresponding triality group G in the following way:

$$N = \left\{ xg^{\rho} \in G \;\middle|\; \begin{array}{l} x \in C_G(\sigma),\ g \in \mathbf{C}(L),\ x^{-\rho}x^{\rho^2} = c^3 \text{ for some } c \in \mathbf{C}(L), \\ \text{and } [z^{\rho}x, \sigma] = x^{-1}zx \in z \centerdot \mathbf{C}(L) \text{ for any } z \in L \end{array} \right\}$$

By showing that for any $x \in L$ and any $a \in \mathbf{C}(L)$ the map $R_x R_a R_x^{-1} R_a^{-1}$ stabilizes the left (and right) cosets of $\mathbf{C}(L)$ in L, it can be shown that N itself is a triality group whose corresponding Moufang loop is the commutant of L. Furthermore, it can also be proven that N is a normal subgroup of G. With this information one can obtain the following theorem.

Theorem 7.5. ([13], Theorem 1.1) *For any Moufang loop L, if $\mathbf{C}(L)$ is the commutant of L then $\mathbf{C}(L) \trianglelefteq L$.* □

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