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Indicators of quasigroups

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Abstract. We present some useful conditions which are necessary for isotopy of two quasigroups of the same finite order.

Let $Q = \{1, 2, 3, \ldots, n\}$ be a finite set, S_n – the set of all permutations of Q. The multiplication (composition) of permutations φ and ψ of Q is defined as $\varphi\psi(x) = \varphi(\psi(x))$. All permutations will be written in the form of cycles and cycles will be separated by points, e.g.

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (132.45.6.)$$

By a *cyclic type* of a permutation $\varphi \in S_n$ we mean the sequence $l_1, l_2, \ldots, l_n$, where l_i denotes the number of cycles of the length i. In this case we will write

$$C(\varphi) = \{l_1, l_2, ..., l_n\}.$$

Obviously, $\sum\limits_{i=1}^{n} i \cdot l_i = n$.

Definition 1. By the *indicator* of a permutation φ of type $C(\varphi) = \{l_1, l_2, ..., l_n\}$ we mean the polynomial

$$w(\varphi) = x_1^{l_1}\, x_2^{l_2} \cdots x_n^{l_n}\,.$$

For example, for $\varphi = (123.45.6.)$ we have $C(\varphi) = \{1, 1, 1, 0, 0, 0\}$ and $w(\varphi) = x_1x_2x_3$; for $\psi = (1.2536.47.80.9.)$, $C(\psi) = \{2, 2, 0, 1, 0, 0, 0, 0, 0, 0\}$ and $w(\psi) = x_1^2x_2^2x_4$.

As it is well-known, two permutations $\varphi, \psi \in S_n$ are *conjugate* if there exists a permutation $\rho \in S_n$ such that

$$\rho\varphi\rho^{-1} = \psi.$$

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Theorem 1. (Theorem 5.1.3 in [4]) *Two permutations are conjugate if and only if they have the same cyclic type.* □

As a consequence we obtain

Corollary 1. *Conjugated permutations have the same indicators.* □

As it is well-known, two quasigroups $Q(\circ)$ and $Q(\cdot)$ are *isotopic* if there are three permutations α, β, γ of Q such that

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y)\,. \tag{1}$$

In the case $\alpha = \beta = \gamma$ we say that quasigroups are *autotopic*.

A *track* (or a *right middle translation*) of a quasigroup $Q(\cdot)$ is a permutation φ_i of Q satisfying the identity

$$x \cdot \varphi_i(x) = i,$$

where $i \in Q$. Each quasigroup can be identified with the set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of all its tracks (cf. [2]).

Tracks of $Q(\cdot)$ will be denoted by φ_i, track of $Q(\circ)$ by ψ_1. Similarly, left and right translations of $Q(\cdot)$ will be denoted by L_a and R_a, left and right translations of $Q(\circ)$ by L_a° and R_a°.

Proposition 1. (cf. [2]) *Tracks of isotopic quasigroups satisfying* (1) *are connected by the formula*

$$\varphi_{\gamma(i)} = \beta\psi_i\alpha^{-1}. \tag{2}$$

Similar results hold for left and right translations.

Theorem 2. *Left and right translations of isotopic quasigroups satisfying* (1) *are connected by the conditions*

$$L_{\alpha(a)} = \gamma L_a^\circ \beta^{-1}, \qquad R_{\beta(b)} = \gamma R_b^\circ \alpha^{-1}. \tag{3}$$

Proof. Indeed, putting $x = a$ we obtain $\gamma L_a^\circ(y) = L_{\alpha(a)}\beta(y)$ for every $y \in Q$, which implies $\gamma L_a^\circ \beta^{-1} = L_{\alpha(a)}$. Similarly, putting in (1) $y = b$ we obtain $R_{\beta(b)} = \gamma R_b^\circ \alpha^{-1}$. □

Corollary 2. *For autotopic quasigroups we have*

$$\varphi_{\alpha(i)} = \alpha\psi_i\alpha^{-1}, \qquad L_{\alpha(a)} = \alpha L_a^\circ \alpha^{-1}, \qquad R_{\alpha(b)} = \alpha R_b^\circ \alpha^{-1}. \tag{4}$$

Consider the following three matrices:

$$\Phi = [\varphi_{ij}]\,, \qquad L = [L_{ij}]\,, \qquad R = [R_{ij}]\,,$$

where $\varphi_{ij} = \varphi_i \varphi_j^{-1}$, $L_{ij} = L_i L_j^{-1}$, $R_{ij} = R_i R_j^{-1}$ for all $i, j \in Q$. Obviously, $\varphi_{ii}(x) = L_{ii}(x) = R_{ii}(x) = x$ and $\varphi_{ij}(x) \neq x$, $L_{ij}(x) \neq x$, $R_{ij}(x) \neq x$ for all $i, j, x \in Q$ and $i \neq j$.

Theorem 3. *For isotopic quasigroups $Q(\circ)$ and $Q(\cdot)$ with the isotopy of the form* (1) *we have*

$$\varphi_{\gamma(i)\gamma(j)} = \beta \psi_{ij} \beta^{-1}, \qquad L_{\alpha(i)\alpha(j)} = \gamma L^{\circ}_{ij} \gamma^{-1}, \qquad R_{\beta(i)\beta(j)} = \gamma R^{\circ}_{ij} \gamma^{-1}.$$

Proof. Indeed, using (2) we obtain

$$\varphi_{\gamma(i)\gamma(j)} = \varphi_{\gamma(i)} \varphi^{-1}_{\gamma(j)} = (\beta \psi_i \alpha^{-1})(\beta \psi_j \alpha^{-1})^{-1} = \beta \psi_i \psi_j^{-1} \beta^{-1} = \beta \psi_{ij} \beta^{-1}.$$

In a similar way, using (3), we obtain the other two equations. □

Definition 2. By the *indicator of the matrix* Φ we mean the polynomial

$$w(\Phi) = \sum_{i=1}^{n} w(\Phi_i)\,,$$

where $\Phi_i = \{\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}\}$ and $w(\Phi_i) = \sum\limits_{j=1,\, j\neq i}^{n} w(\varphi_{ij})$.

Indicators of the matrices L and M are defined analogously.

Example 1. Consider two quasigroups defined by the following tables:

$\cdot$	1	2	3	4	5	6
1	4	1	6	2	5	3
2	5	3	2	6	4	1
3	2	6	5	3	1	4
4	3	5	1	4	6	2
5	6	2	4	1	3	5
6	1	4	3	5	2	6

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	5	6	4	3
3	3	5	4	2	6	1
4	4	6	2	3	1	5
5	5	4	6	1	3	2
6	6	3	1	5	2	4

For the quasigroup $Q(\cdot)$ we have:

$$\begin{array}{lll} \varphi_1 = (126.354.) & \varphi_2 = (146523.) & \varphi_3 = (1634.2.5.) \\ \varphi_4 = (1.2536.4.) & \varphi_5 = (15642.3.) & \varphi_6 = (13245.6.). \end{array}$$

Thus,

$$\begin{array}{lll} \varphi_{11} = (1.2.3.4.5.6.) & \varphi_{12} = (15.24.36.) & \varphi_{13} = (13.26.45.) \\ \varphi_{14} = (12.34.56.) & \varphi_{15} = (164.235.) & \varphi_{16} = (146.253.). \end{array}$$

Consequently,

$$w(\varphi_{11}) = x_1^6, \quad w(\varphi_{12}) = w(\varphi_{13}) = w(\varphi_{14}) = x_2^3, \quad w(\varphi_{15}) = w(\varphi_{16}) = x_3^2.$$

Hence $w(\Phi_1) = 3x_2^3 + 2x_3^2$.

By analogous computations we can see that for this quasigroup

$$w(\Phi) = w(L) = w(R) = 6(3x_2^3 + 2x_3^2).$$

For the second quasigroup we obtain:

$$w(\Phi) = (2x_2x_4 + x_2^3 + 2x_6) + (x_2^3 + 2x_3^2 + 2x_6) + 2(x_2x_4 + x_3^2 + 3x_6) + 2(2x_3^2 + 3x_6),$$

$$w(L) = w(R) = 2(x_2x_4 + 4x_6) + 4(2x_2x_4 + x_3^2 + 2x_6)\,. \qquad \Box$$

As a consequence of our Theorem 3 and Corollary 1 we obtain

Theorem 4. *Isotopic quasigroups have the same indicators of the matrices* Φ*,* L *and* R*.* $\Box$

This theorem shows that quasigroups from the above example are not isotopic.

Corollary 3. *For quasigroups of order* n *isotopic to a group we have* $w(\Phi) = w(\Phi_1)$*.*

Proof. In [2] it is proved that for a quasigroup isotopic to a group all its Φ_i are groups isomorphic to Φ_1. Hence $w(\Phi_i) = w(\Phi_1)$ for every $i \in Q$. $\Box$

There are examples proving that the converse statement is not true.

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