Quasigroups and Related Systems **19** (2011), 239 – 264

Right product quasigroups and loops

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Abstract. Right groups are direct products of right zero semigroups and groups and they play a significant role in the semilattice decomposition theory of semigroups. Right groups can be characterized as associative right quasigroups (magmas in which left translations are bijective). If we do not assume associativity we get right quasigroups which are not necessarily representable as direct products of right zero semigroups and quasigroups. To obtain such a representation, we need stronger assumptions which lead us to the notion of *right product quasigroup*. If the quasigroup component is a (one-sided) loop, then we have a *right product* (*left, right*) *loop*.

We find a system of identities which axiomatizes right product quasigroups, and use this to find axiom systems for right product (left, right) loops; in fact, we can obtain each of the latter by adjoining just one appropriate axiom to the right product quasigroup axiom system.

We derive other properties of right product quasigroups and loops, and conclude by showing that the axioms for right product quasigroups are independent.

1. Introduction

In the semigroup literature (e.g., [1]), the most commonly used definition of *right group* is a semigroup $(S;\cdot)$ which is right simple (*i.e.*, has no proper right ideals) and left cancellative (*i.e.*, $xy = xz \implies y = z$). The structure of right groups is clarified by the following well-known representation theorem (see [1]):

Theorem 1.1. *A semigroup $(S;\cdot)$ is a right group if and only if it is isomorphic to a direct product of a group and a right zero semigroup.* □

There are several equivalent ways of characterizing right groups. One of particular interest is the following: a right group is a semigroup $(S;\cdot)$ which is also a *right quasigroup*, that is, for each $a, b \in S$, there exists a

2010 Mathematics Subject Classification: Primary: 20N02; Secondary: 20N05, 08A50
Keywords: right quasigroup, right product quasigroup, right product loop, axiomatization, axiom independence, word problem, reproductive equation
The second author was supported by the Ministry of Education and Science of Serbia, grants 174008 and 174026

unique $x \in S$ such that $ax = b$. In a right quasigroup $(S;\cdot)$, one can define an additional operation $\backslash : S \times S \to S$ as follows: $z = x\backslash y$ is the unique solution of the equation $xz = y$. Then the following equations hold.

$$x\backslash xy = y \qquad \text{(Q1)} \qquad\qquad x(x\backslash y) = y \qquad \text{(Q2)}$$

Conversely, if we now think of S as an algebra with two binary operations then we have an equational definition.

Definition 1.2. An algebra $(S;\cdot,\backslash)$ is a *right quasigroup* if it satisfies $(Q1)$ and $(Q2)$. An algebra $(S;\cdot,/)$ is a *left quasigroup* if it satisfies

$$xy/y = x \qquad \text{(Q3)} \qquad\qquad (x/y)y = x \qquad \text{(Q4)}$$

An algebra $(S;\cdot,\backslash,/)$ is a *quasigroup* if it is both a right quasigroup and a left quasigroup.

(We are following the usual convention that juxtaposition binds more tightly than the division operations, which in turn bind more tightly than an explicit use of $\cdot$. This helps avoid excessive parentheses.)

From this point of view, a group is an associative quasigroup with $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$. If $(S;\cdot,\backslash)$ is a right group viewed as an associative right quasigroup, then its group component has a natural right division operation $/$. This operation can be extended to all of S as follows. We easily show that $x\backslash x = y\backslash y$ for all $x, y \in S$, and then define $e = x\backslash x$, $x^{-1} = x\backslash e$, and $x/y = xy^{-1}$. Note that in the right zero semigroup component of S, we have $xy = x\backslash y = x/y = y$.

If one tries to think of a right quasigroup as a "nonassociative right group", one might ask if there is a representation theorem like Theorem 1.1 which expresses a right quasigroup as a direct product of a quasigroup and a right zero semigroup. This is clearly not the case.

Example 1.3. On the set $S = \{0, 1\}$, define operations $\cdot, \backslash : S \times S \to S$ by $x \cdot 0 = x\backslash 0 = 1$ and $x \cdot 1 = x\backslash 1 = 0$. Then $(S;\cdot,\backslash)$ is a right quasigroup which is neither a quasigroup nor a right zero semigroup, and since $|S| = 2$, $(S;\cdot,\backslash)$ is also not a product of a quasigroup and right zero semigroup. $\square$

For another obstruction to a representation theorem, note that if an algebra which is a direct product of a quasigroup and a right zero semigroup possesses a right neutral element, then the right zero semigroup component is trivial and the algebra is, in fact, a right loop (see below). However, there are right quasigroups with neutral elements which are not right loops.

Example 1.4. Let $\mathbb{N}$ be the set of natural numbers and define

$$x \cdot y = x\backslash y = \begin{cases} y & \text{if } x < y \\ x - y & \text{if } x \geq y \end{cases}$$

Then $(\mathbb{N}; \cdot, \backslash)$ is a right quasigroup, 0 is a neutral element, and $0 \cdot 1 = 1 = 2 \cdot 1$. Since $\cdot$ is not a quasigroup operation, it follows from the preceding discussion that $(\mathbb{N}; \cdot, \backslash)$ is not a direct product of a quasigroup and a right zero semigroup. □

Simply adjoining a right division operation $/$ to a right quasigroup does not fix the problem; for instance, in either Example 1.3 or 1.4, define $x/y = 0$ for all x, y.

In this paper, we will investigate varieties of right quasigroups such that there is indeed a direct product decomposition.

Definition 1.5. A quasigroup $(S; \cdot, \backslash, /)$ is a $\{$*left loop, right loop, loop*$\}$ if it satisfies the identity $\{\ x/x = y/y,\ x\backslash x = y\backslash y,\ x\backslash x = y/y\ \}$.

An algebra $(S; \cdot, \backslash, /, e)$ is a *pointed quasigroup* if $(S; \cdot, \backslash, /)$ is a quasigroup. A pointed quasigroup is a $\{$*quasigroup with an idempotent, left loop, right loop, loop*$\}$ if the distinguished element e is $\{$an idempotent ($ee = e$), left neutral ($ex = x$), right neutral ($xe = x$), neutral ($ex = xe = x$)$\}$.

Definition 1.6. Let $T = \{\cdot, \backslash, /\}$ be the language of quasigroups and M a further (possibly empty) set of operation symbols disjoint from T. The language $\hat{T} = T \cup M$ is *an extended language of quasigroups*.

The language $T_1 = \{\cdot, \backslash, /, e\}$, obtained from T by the addition of a single constant, is *the language of loops*.

Note that we have two different algebras under the name "loop". They are equivalent and easily transformed one into the other. When we need to distinguish between them we call the algebra $(S; \cdot, \backslash, /)$ satisfying $x\backslash x = y/y$ "the loop in the language of quasigroups" while the algebra $(S; \cdot, \backslash, /, e)$ satisfying identities $ex = xe = x$ is called "the loop in the language of loops". Analogously we do for left and right loops.

Definition 1.7. Let $\mathbb{V}$ be a class of quasigroups. An algebra is a *right product* $\mathbb{V}$*–quasigroup* if it is isomorphic to $Q \times R$, where $Q \in \mathbb{V}$ and R is a right zero semigroup.

In particular, when $\mathbb{V}$ is the class $\{\mathbf{Q}, \mathbf{L}\Lambda, \mathbf{R}\Lambda, \Lambda\}$ of all $\{$*quasigroups, left loops, right loops, loops*$\}$ (in the language of quasigroups) then $\{\mathbf{RPQ}, \mathbf{RPL}\Lambda, \mathbf{RPR}\Lambda, \mathbf{RP}\Lambda\}$ denote the class of all right product $\mathbb{V}$–quasigroups.

If $\mathbb{V}$ is the class $\{\mathbf{pQ}, \mathbf{Qi}, \mathbf{eQ}, \mathbf{Qe}, \mathbf{Q1}\}$ of all $\{$*pointed quasigroups, quasigroups with an idempotent, left loops, right loops, loops*$\}$ (in the language of loops), then $\{$**RPpQ**, **RPQi**, **RPeQ**, **RPQe**, **RPQ1**$\}$ denote the class of all right product $\mathbb{V}$–quasigroups.

We wish to view these classes as varieties of algebras. In order to make sense of this, we need to adjust the type of right zero semigroups to match that of (equational) quasigroups. We adopt the convention suggested above.

Convention 1.8. A right zero semigroup is considered to be an algebra in $\hat{T}$ satisfying $x\backslash y = x/y = xy = y$ for all x, y.

This convention agrees with the one used in [7, 8]. Different definitions of $\backslash$ and $/$ in right zero semigroups would affect the form of the axioms for right product quasigroups.

We also denote the class of all (pointed) right zero semigroups by $\mathcal{R}$ ($p\mathcal{R}$). Then, in the language of universal algebra, the variety of all right product $\mathbb{V}$–quasigroups is a product $\mathbb{V} \otimes \mathcal{R}$ of independent varieties $\mathbb{V}$ and $\mathcal{R}$ (see [20]).

Definition 1.9. If t is a term, then $\{\mathrm{head}(t), \mathrm{tail}(t)\}$ is the $\{$first, last$\}$ variable of t.

The following is an immediate consequence of Definition 1.7 and Convention 1.8.

Theorem 1.10. *Let u, v be terms in a language extending $\{\cdot, \backslash, /\}$. Then the equality $u = v$ is true in all right product $\mathbb{V}$–quasigroups if and only if* $\mathrm{tail}(u) = \mathrm{tail}(v)$ *and $u = v$ is true in all $\mathbb{V}$–quasigroups.* □

In particular:

Corollary 1.11. *Let s, t, u be terms in a language extending $\{\cdot, \backslash, /\}$. If $s = t$ is true in all $\mathbb{V}$–quasigroups then $s \circ u = t \circ u$ ($\circ \in \{\cdot, \backslash, /\}$) is true in all right product $\mathbb{V}$–quasigroups.* □

We conclude this introduction with a brief discussion of the sequel and some notation conventions. In §2, we will consider the problem of axiomatizing the varieties introduced by the Definition 1.7. In §3 we consider various properties of right product (pointed) quasigroups and loops. Finally, in §4, we verify the independence of the axioms.

We should mention some related work by Tamura *et al* and others. [18, 19, 4, 21]. An "M-groupoid", defined by certain axioms, turns out to be a direct product of a right zero semigroup and a magma with a neutral element. The axiomatic characterization of these in [18, 19] is of a somewhat different character than ours; besides the fact that they did not need to adjust signatures since they did not consider quasigroups, their axioms are also not entirely equational.

2. Axioms

We now consider the problem of axiomatizing **RPQ**, the class of all right product quasigroups. One approach to axiomatization is the standard method of Knoebel [6], which was used in [7, 8]. It turns out that the resulting axiom system consists of 14 identities, most of which are far from elegant. Another way is via independence of **Q** and $\mathcal{R}$. Using the term $\alpha(x, y) = xy/y$ (see [20, Proposition 0.9]), we get these axioms:

$$xx/x = x$$

$$(xy/y)(uv/v)/(uv/v) = xv/v \qquad (xy \cdot uv)/uv = (xu/u)(yv/v)$$

$$(x\backslash y)(u\backslash v)/(u\backslash v) = (xu/u)\backslash(yv/v) \quad (x/y)(u/v)/(u/v) = (xu/u)/(yv/v)$$

which we also find to be somewhat complicated. Instead, we propose a different scheme, which we call system (A):

$$x\backslash xy = y \tag{A1}$$

$$x \cdot x\backslash y = y \tag{A2}$$

$$x/y \cdot y = xy/y \tag{A3}$$

$$(x/y \cdot y)/z = x/z \tag{A4}$$

$$xy/z \cdot z = x(y/z \cdot z) \tag{A5}$$

We now prove that system (A) axiomatizes the variety of right product quasigroups. It is not difficult to use the results of [5] to prove this, but instead we give a somewhat more enlightening self-contained proof. We start with an easy observation.

Lemma 2.1. *Every right product quasigroup satisfies system* (A).

Proof. The quasigroup axioms (Q3) and (Q4) trivially imply (A3)–(A5), and so quasigroups satisfy (A). For each (Ai), the tails of both sides of the equation coincide. By Theorem 1.10, we have the desired result. □

In an algebra $(S;\cdot,\backslash,/)$ satisfying system (A), define a new term operation $\star : S\times S\to S$ by

$$x\star y = xy/y = x/y\cdot y \tag{$\star$}$$

for all $x,y\in S$. Here the second equality follows from (A3), and we will use it freely without reference in what follows.

Lemma 2.2. *Let* $(S;\cdot,\backslash,/)$ *be an algebra satisfying system* (A)*. Then for all* $x,y,z\in S$*,*

$$(xy)\star z = x(y\star z) \tag{1}$$

$$(x\backslash y)\star z = x\backslash(y\star z) \tag{2}$$

$$(x/y)\star z = x/(y\star z) \tag{3}$$

Proof. Equation (1) is just (A5) rewritten. Replacing y with $x\backslash y$ and using (A1), we get (2). Finally, for (3), we have

$$\begin{aligned} x/(y\star z) &= (x\star y\star z)/(y\star z) = [(x/y\cdot y)\star z]/(y\star z)\\ &= [(x/y)(y\star z)]/(y\star z) = (x/y)\star y\star z = (x/y)\star z\,, \end{aligned}$$

using (A4) in the first equality, (1) in the third, and the rectangular property of $\star$ in the fifth. □

Lemma 2.3. *Let* $(S;\cdot,\backslash,/)$ *be an algebra satisfying system* (A)*. Then* $(S;\star)$ *is a rectangular band.*

Proof. Firstly,

$$(x\star y)\star z = (x/y\cdot y)/z\cdot z = x/z\cdot z = x\star z\,, \tag{4}$$

using (A4). Replacing x with $x/(y\star z)$ in (1), we get

$$[(x/(y\star z))y]\star z = x/(y\star z)\cdot(y\star z) = x\star(y\star z)\,. \tag{5}$$

Thus,

$$x\star z = (x\star(y\star z))\star z = ([(x/(y\star z))y]\star z)\star z = [(x/(y\star z))y]\star z = x\star(y\star z)\,, \tag{6}$$

using (4), (5), (4) again and (5) once more. Together, (4) and (6) show that $(S;\star)$ is a semigroup satisfying $x\star y\star z = x\star z$.

What remains is to show the idempotence of $\star$. Replace x with x/x in (1) and set $y = z = x$, we have

$$(x/x)(x \star x) = (x/x \cdot x) \star x = (x \star x) \star x = x \star x\,,$$

using (4), and so

$$x \star x = (x/x)\backslash(x \star x) = (x/x)\backslash(x/x \cdot x) = x\,,$$

using (A1) in the first and third equalities. $\square$

Let $(S; \cdot, \backslash, /)$ be an algebra satisfying system (A). By Lemma 2.3, $(S; \star)$ is a rectangular band, and so $(S; \star)$ is isomorphic to the direct product of a left zero semigroup and a right zero semigroup [1]. It will be useful to make this explicit. Introduce translation maps in the semigroup $(S; \star)$ as follows

$$\ell_x(y) := x \star y =: (x)r_y\,,$$

so that the left translations $\ell_x : S \to S$ act on the left and the right translations $r_y : S \to S$ act on the right. Let $L = \langle \ell_x | x \in S\rangle$ and $R = \langle r_x | x \in S\rangle$. Then L is a left zero transformation semigroup, that is, $\ell_x \ell_y = \ell_x$, while R is a right zero transformation semigroup, that is, $r_x r_y = r_y$. Since $\ell_x = \ell_{x\star y}$ and $r_y = r_{x\star y}$ for all $x, y \in S$, it follows easily that the map $S \to L \times R; x \mapsto (\ell_x, r_x)$ is an isomorphism of semigroups.

Now we define operations $\cdot$, $\backslash$ and $/$ on R and L. Firstly, we define $\cdot, \backslash, / : R \times R \to R$ by

$$r_x \cdot r_y := r_x \backslash r_y := r_x / r_y := r_y\,.$$

For later reference, we formally record the obvious.

Lemma 2.4. *Let* $(S; \cdot, \backslash, /)$ *be an algebra satisfying system* (A). *With the definitions above,* $(R; \cdot, \backslash, /)$ *is a right zero semigroup.* $\square$

It follows from Lemma 2.1 that $(R; \cdot, \backslash, /)$ is an algebra satisfying system (A).

Lemma 2.5. *Let* $(S; \cdot, \backslash, /)$ *be an algebra satisfying system* (A). *The mapping* $S \to R; x \mapsto r_x$ *is a surjective homomorphism of such algebras.*

Proof. Firstly,

$$(x)r_{yz} = x \star (yz) = x \star [y(z \star z)] = x \star (yz) \star z = x \star z = xr_z = (x)(r_y \cdot r_z)\,,$$

using (1) in the third equality and $(S; \star)$ being a rectangular band in the fourth equality. Similar arguments using (2) and (3) give $r_{y\backslash z} = r_y \backslash r_z$ and $r_{y/z} = r_y / r_z$, respectively. The surjectivity is clear. $\square$

Next, we define $\cdot, \backslash, / : L \times L \to L$ by

$$(\ell_x \cdot \ell_y)(z) = \ell_x(z) \cdot \ell_y(z)$$
$$(\ell_x \backslash \ell_y)(z) = \ell_x(z) \backslash \ell_y(z)$$
$$(\ell_x / \ell_y)(z) = \ell_x(z) / \ell_y(z)$$

for all $x, y, z \in S$.

Lemma 2.6. *Let* $(S; \cdot, \backslash, /)$ *be an algebra satisfying system* (A). *With the definitions above,* $(L; \cdot, \backslash, /)$ *is a quasigroup.*

Proof. Equations (Q1) and (Q2) follow immediately from the definitions together with (A1) and (A2). By (A3), it remains to prove, say, (Q3). For all $x, y, z \in S$,

$$((\ell_x \cdot \ell_y)/\ell_y)(z) = (\ell_x(z) \cdot \ell_y(z))/\ell_y(z) = (x \star z) \star (y \star z) = x \star z = \ell_x(z)\,,$$

where we have used the fact that $(S; \star)$ is a rectangular band in the third equality. $\square$

Lemma 2.7. *Let* $(S; \cdot, \backslash, /)$ *be an algebra satisfying system* (A). *The mapping* $S \to L; x \mapsto \ell_x$ *is a surjective homomorphism of such algebras.*

Proof. For all $x, y, z \in S$, we compute

$$\begin{aligned}\ell_x(z) \cdot \ell_y(z) &= (x \star z)(y \star z) = (x \star y \star z)(y \star z) = [(x(y \star z))/(y \star z)](y \star z)\\ &= (x(y \star z)) \star y \star z = x(y \star z \star y \star z) = x[y \star z] = (xy) \star z\\ &= \ell_{xy}(z)\,,\end{aligned}$$

where we use rectangularity of $\star$ in the second equality, (1) in the fifth, idempotence of $\star$ in the sixth and (1) in the seventh. Next, if we replace y with $x\backslash y$ and use (A1), we get $\ell_{x\backslash y}(z) = \ell_x(z)\backslash \ell_y(z)$. Finally,

$$\ell_x(z)/\ell_y(z) = (x \star z)/(y \star z) = ((x \star z)/y) \star z = (x/y) \star z = \ell_{x/y}(z)\,,$$

using (3) in the second equality and (A5) in the third. $\square$

We now turn to the main result of this section.

Theorem 2.8. *An algebra* $(S; \cdot, \backslash, /)$ *is a right product quasigroup if and only if it satisfies* (A).

Proof. The necessity is shown by Lemma 2.1. Conversely, if $(S;\cdot,\backslash,/)$ satisfies (A), then by Lemmas 2.5 and 2.7, the mapping $S\to L\times R; x\mapsto(\ell_x,r_x)$ is a surjective homomorphism. This map is, in fact, bijective, since as already noted, it is an isomorphism of rectangular bands. By Lemmas 2.4 and 2.6, $L\times R$ is a right product quasigroup, and thus so is S. □

Remark 2.9. There are other choices of axioms for right product quasigroups. For instance, another system equivalent to (A) consists of (A1), (A2), (A3) and the equations

$$xx/x = x \quad \text{(B1)} \qquad\qquad (xy\cdot(z/u))/(z/u) = x(yu/u) \quad \text{(B2)}\,.$$

Call this system (B). We omit the proof of the equivalence of systems (A) and (B). One can use the results of [5] to prove the system (B) variant of Theorem 2.8 as follows: (A1) and (A2) trivially imply the equations

$$x(x\backslash y) = x\backslash xy \qquad \text{(A3}'\text{)}$$

$$x\backslash xx = x \quad \text{(B1}'\text{)} \qquad (x\backslash y)\backslash((x\backslash y)\cdot zu) = (x\backslash xz)u \quad \text{(B2}'\text{)}\,.$$

By [5], (A3), (A3′), (B1), (B1′), (B2′) and (B2′) axiomatize the variety of *rectangular quasigroups*, each of which is a direct product of a left zero semigroup, a quasigroup and a right zero semigroup. By (A1) and (A2), the left zero semigroup factor must be trivial, and so a system satisfying system (B) must be a right product quasigroup. □

We conclude this section by considering other varieties of right product quasigroups. Utilizing [9] we get:

Theorem 2.10. *Let $\mathbb{V}$ be a variety of quasigroups axiomatized by additional identities:*

$$s_i = t_i \qquad (\mathrm{V}_i)$$

$(i\in I)$ in an extended language $\hat{T}$ and let z be a variable which does not occur in any s_i, t_i. Then the variety $\boldsymbol{RP}\mathbb{V}$ of right product $\mathbb{V}$–quasigroups can be axiomatized by system (A) *together with* (*for all $i\in I$*)*:*

$$s_i z = t_i z\,. \qquad (\hat{\mathrm{V}}_i)$$

Proof. Both $\mathbb{V}$–quasigroups and right zero semigroups satisfy system (A) and all $(\hat{\mathrm{V}}_i)$, $i\in I$, and thus so do their direct products i.e., right product $\mathbb{V}$–quasigroups.

Conversely, if an algebra satisfies system (A), it is a right product quasigroup by Theorem 2.8. Since all ($\hat{\mathrm{V}}_i$) are satisfied, the quasigroup factor has to satisfy them, too. But in quasigroups, the identities ($\hat{\mathrm{V}}_i$) are equivalent to the identities (V_i) and these define the variety $\mathbb{V}$. □

Theorem 2.11. *Theorem* 2.10 *remains valid if we replace* ($\hat{\mathrm{V}}_i$) *by any of the following families of identities:*

$$s_i\backslash z = t_i\backslash z$$
$$s_i/z = t_i/z$$
$$z/(s_i\backslash z) = (z/t_i)\backslash z$$
$$s_i = (t_i \cdot \mathrm{tail}(s_i))/\,\mathrm{tail}(s_i)$$
$$s_i = (t_i/\,\mathrm{tail}(s_i)) \cdot \mathrm{tail}(s_i)$$
$$s_i = t_i \quad (\text{if } \mathrm{tail}(s_i) = \mathrm{tail}(t_i)).$$
□

Example 2.12. Adding associativity $x \cdot yz = xy \cdot z$ to system (A) gives yet another axiomatization of *right groups.* □

Example 2.13. Right product commutative quasigroups are right product quasigroups satisfying $xy \cdot z = yx \cdot z$. However, commutative right product quasigroups are just commutative quasigroups. □

Obviously:

Corollary 2.14. *If the variety* $\mathbb{V}$ *of quasigroups is defined by the identities* $s_i = t_i$ $(i \in I)$ *such that* $\mathrm{tail}(s_i) = \mathrm{tail}(t_i)$ *for all* $i \in I$, *then the class of all right product quasigroups satisfying identities* $s_i = t_i (i \in I)$ *is the class of all right product* $\mathbb{V}$*–quasigroups.*

If $\mathrm{tail}(s_i) \neq \mathrm{tail}(t_i)$ *for some* $i \in I$, *then the class of all right product quasigroups satisfying identities* $s_i = t_i$ $(i \in I)$ *is just the class of all* $\mathbb{V}$*–quasigroups.* □

Example 2.15. The variety **RPpQ** is defined by adding a constant to the language of quasigroups, not by any extra axioms. □

Example 2.16. The variety **RPQi** of all *right product quasigroups with an idempotent* may be axiomatized by system (A) and $ee = e$. □

Corollary 2.17. *A right product quasigroup is a right product left loop iff it satisfies any (and hence all) of the following axioms:*

$$(x/x)y = y \tag{LL1}$$

$$(x/x)z = (y/y)z \tag{LL2}$$

$$(x \circ y)/(x \circ y) = y/y \tag{LL3}$$

where $\circ$ *is any of the operations* $\cdot, \backslash, /$.

Proof. In a quasigroup, identities (LL1), (LL2) and (LL3) are equivalent to each other and to $x/x = y/y$, and so a quasigroup satisfying either axiom is a left loop. Conversely, in a left loop with left neutral element e, we have $e = x/x$, and so (LL1), (LL2) and (LL3) hold. Thus a quasigroup satisfies either (and hence all) of (LL1), (LL2), (LL3) if and only if it is a left loop.

On the other hand, (LL1), (LL2) and (LL3) trivially hold in right zero semigroups by Convention 1.8. Putting this together, we have the desired result. □

In the language of loops we have:

Corollary 2.18. *A right product quasigroup is a right product left loop if and only if it satisfies the identity* $ex = x$. □

Similarly:

Corollary 2.19. *A right product quasigroup is a right product right loop if and only if it satisfies any (and hence all) of the following axioms:*

$$x(y\backslash y) \cdot z = xz$$

$$(x\backslash x)z = (y\backslash y)z$$

$$(x \circ y)\backslash(x \circ y) = y\backslash y$$

where $\circ$ *is any of the operations* $\cdot, \backslash, /$. □

Corollary 2.20. *A right product quasigroup is a right product right loop (in the language of loops) iff it satisfies the identity* $xe \cdot y = xy$. □

Corollary 2.21. *A right product quasigroup is a right product loop if and only if it satisfies any (and hence all) of the following axioms:*

$$(x\backslash x)y = y \tag{L}$$
$$x(y/y) = xy/y$$
$$x(y/y) = (x/y)y$$
$$(x\backslash x)z = (y/y)z$$
$$(x \circ y)\backslash(x \circ y) = y/y$$
$$(x \circ y)/(x \circ y) = y\backslash y$$

where $\circ$ is any of the operations $\cdot, \backslash, /$. □

Corollary 2.22. *A right product quasigroup is a right product loop (in the language of loops) iff it satisfies both $ex = x$ and $xe \cdot y = xy$.* □

3. Properties of right product (pointed) quasigroups

Calling upon the tools of universal algebra, we now examine some properties of right product quasigroups. We will use the following standard notation.

Definition 3.1.

E_S	–	the subset of all idempotents of S.
$\mathrm{Sub}(S)$	–	the lattice of all subalgebras of S.
$\mathrm{Sub}^0(S)$	–	the lattice of all subalgebras of S with the empty set adjoined as the smallest element (used when two sub–algebras have an empty intersection).
$\mathrm{Con}(S)$	–	the lattice of all congruences of S.
$\mathrm{Eq}(S)$	–	the lattice of all equivalences of S.
$\mathrm{Hom}(S,T)$	–	the set of all homomorphisms from S to T.
$\mathrm{End}(S)$	–	the monoid of all endomorphisms of S.
$\mathrm{Aut}(S)$	–	the group of all automorphisms of S.
$\mathrm{Free}(\mathbb{V},n)$	–	the free algebra with n generators in the variety $\mathbb{V}$.
$\mathrm{Var}(\mathbb{V})$	–	the lattice of all varieties of a class $\mathbb{V}$ of algebras.

In addition, R_n will denote the unique n-element right zero semigroup – which also happens to be free. However, note that in the language of loops, the free right zero semigroup generated by n elements is R_{n+1}.

3.1. The word problem

Using a well-known result of Evans [3] we have the following corollary of Theorem 1.10:

Corollary 3.2. *The word problem for right product $\mathbb{V}$–quasigroups is solvable if and only if it is solvable for $\mathbb{V}$–quasigroups.* $\square$

In particular:

Corollary 3.3. *The word problem for* $\{\boldsymbol{RPQ}, \boldsymbol{RPL}\Lambda, \boldsymbol{RPR}\Lambda, \boldsymbol{RP}\Lambda\}$ *is solvable.* $\square$

Likewise:

Corollary 3.4. *The word problem for* $\{\boldsymbol{RPpQ}, \boldsymbol{RPQi}, \boldsymbol{RPeQ}, \boldsymbol{RPQe}, \boldsymbol{RPQ1}\}$ *is solvable.* $\square$

3.2. Properties of right product quasigroups and loops

The following corollaries are special cases of results in universal algebra (see [20]).

Corollary 3.5. *For all $Q, Q' \in \boldsymbol{Q}$ and $R, N \in \mathcal{R}$:*

1. $E_{Q\times R} = E_Q \times R$,
   in particular:
   - *$Q \times R$ have idempotents if and only if Q have them.*
   - *$E_{Q\times R}$ is subalgebra of $Q \times R$ if and only if E_Q is subalgebra of Q.*
   - *$Q \times R$ is a groupoid of idempotents if and only if Q is.*
2. $\mathrm{Sub}^0(Q \times R) = (\mathrm{Sub}(Q) \times (\mathbf{2}^R \setminus \{\emptyset\})) \cup \{\emptyset\}$.
3. $\mathrm{Con}(Q \times R) = \mathrm{Con}(Q) \times \mathrm{Eq}(R)$.
4. $\mathrm{Hom}(Q \times R,\, Q' \times N) = \mathrm{Hom}(Q, Q') \times N^R$.
5. $\mathrm{End}(Q \times R) = \mathrm{End}(Q) \times R^R$.
6. $\mathrm{Aut}(Q \times R) = \mathrm{Aut}(Q) \times S_{|R|}$. $\square$

Having a distinguished element changes the properties of a variety radically. For example, if $e = (i, j)$ is a distinguished element of the right product pointed quasigroup $Q \times R$ then there is always the smallest subalgebra $< i > \times\{j\}$. So, in case of right product pointed quasigroups, the results analogous to Corollary 3.5 are actually somewhat different in character.

Corollary 3.6. *For all $Q, Q' \in \boldsymbol{pQ}$ and $R, N \in \mathcal{R}$ with a distinguished element j:*

1. $E_{Q\times R} = E_Q \times R$,
   in particular:
   - *$Q \times R$ has idempotents if and only if Q has them.*
   - *$E_{Q\times R}$ is subalgebra of $Q \times R$ if and only if E_Q is subalgebra of Q.*
   - *$Q \times R$ is a groupoid of idempotents if and only if Q is.*
2. $\mathrm{Sub}(Q \times R) = \mathrm{Sub}(Q) \times \{Y \subseteq R \mid j \in Y\} \simeq \mathrm{Sub}(Q) \times \mathbf{2}^{R\smallsetminus\{j\}}$.
3. $\mathrm{Con}(Q \times R) = \mathrm{Con}(Q) \times \mathrm{Eq}(R)$.
4. $\mathrm{Hom}(Q \times R, Q' \times N) = \mathrm{Hom}(Q, Q') \times \{f : R \to N \mid f(j) = j\} \simeq \mathrm{Hom}(Q, Q') \times N^{R\smallsetminus\{j\}}$.
5. $\mathrm{End}(Q \times R) = \mathrm{End}(Q) \times \{f : R \to R \mid f(j) = j\} \simeq \mathrm{End}(Q) \times R^{R\smallsetminus\{j\}}$.
6. $\mathrm{Aut}(Q \times R) = \mathrm{Aut}(Q) \times \{f \in S_R \mid f(j) = j\} \simeq \mathrm{Aut}(Q) \times S_{|R|-1}$. □

Corollary 3.7. *If $\mathbb{V}$ is one of $\boldsymbol{Q}, \boldsymbol{L}\Lambda, \boldsymbol{R}\Lambda, \Lambda$, then*
$$\mathrm{Free}(\boldsymbol{RP}\mathbb{V}, n) \simeq \mathrm{Free}(\mathbb{V}, n) \times R_n.$$ □

Corollary 3.8. *If $\mathbb{V}$ is one of $\boldsymbol{pQ}, \boldsymbol{Qi}, \boldsymbol{eQ}, \boldsymbol{Qe}, \boldsymbol{Q1}$, then*
$$\mathrm{Free}(\boldsymbol{RP}\mathbb{V}, n) \simeq \mathrm{Free}(\mathbb{V}, n) \times R_{n+1}.$$ □

Corollary 3.9. *If $\mathbb{V}$ is one of the above varieties of (pointed) quasigroups, then* $\mathrm{Var}(\boldsymbol{RP}\mathbb{V}) \simeq \mathrm{Var}(\mathbb{V}) \times \mathbf{2}$. □

All cases suggested by Corollary 3.5(1) can actually occur. In the examples below, right product quasigroups are in fact quasigroups and thus we display the Cayley tables of the multiplication only.

Example 3.10. Tables 1 give a right product quasigroup with no idempotents (on the left) and an idempotent right product quasigroup (on the right).

$\cdot$	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

$\cdot$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Table 1: A right product quasigroup with no idempotents and an idempotent right product quasigroup.

Example 3.11. Tables 2 give a right product quasigroup in which E_S is not a subalgebra (on the left) and a right product quasigroup in which E_S is a nontrivial subalgebra (on the right).

$\bullet$	0	1	2	3
0	0	2	1	3
1	3	1	2	0
2	1	3	0	2
3	2	0	3	1

$\bullet$	0	1	2	3	4	5
0	0	2	1	3	5	4
1	2	1	0	5	4	3
2	1	0	2	4	3	5
3	3	5	4	0	2	1
4	5	4	3	2	1	0
5	4	3	5	1	0	2

Table 2: E_S is not closed, E_S is a nontrivial subalgebra.

Moreover, we have the following. These are immediate consequences of well understood properties of quasigroups and semigroups.

Theorem 3.12. *Let $S = Q \times R$ be a right product quasigroup. Then:*

1. *If Q_m $(m \in M)$ is the (possibly empty) family of all maximal subquasigroups of Q then $Q_m \times R$ $(m \in M)$, $Q \times (R \setminus \{r\})$ $(r \in R)$ is the family of all maximal right product subquasigroups of S.*

2. *There are $|R|$ maximal subquasigroups of S. They are all mutually isomorphic and of the form $Q \times \{r\}$ $(r \in R)$.*

From now on we assume $E_S \neq \emptyset$ (i.e., $E_Q \neq \emptyset$).

3. *If E_S is subalgebra then it is the largest subalgebra of idempotents of S.*

4. *There are $|E_S|$ maximal left zero subsemigroups of S. They are all singletons $\{e\}$ $(e \in E_S)$.*

5. *There are* $|E_Q|$ *maximal right zero subsemigroups of* S. *They are all mutually isomorphic and of the form* $\{i\} \times R$ $(i \in E_Q)$.

From now on we assume $E_Q = \{i\}$.

6. $E_S = \{i\} \times R$ *is the unique largest subband of* S, *which happens to be a right zero semigroup.*

7. *If the quasigroup* Q *is a left loop then the left neutral* i *of* Q *is the only idempotent of* Q *and:*
   – $E_S = \{a/a \mid a \in S\}$.
   – *The element* $e \in S$ *is a left neutral if and only if it is an idempotent.*
   – *The maximal subquasigroups*
   $$Q \times \{r\} = Se = \{x \in S \mid x/x = e\} \ \ (e \in E_S, e = (i, r))$$
   are maximal left subloops of S.
   – *For all* $e \in E_S$ $S \simeq Se \times E_S$ *and the isomorphism is given by* $f(x) = (xe/e, x/x)$.

8. *If the quasigroup* Q *is a right loop then the right neutral* i *of* Q *is the only idempotent of* Q *and:*
   – $E_S = \{a\backslash a \mid a \in S\}$.
   – S *has a right neutral if and only if* $|R| = 1$ *and then the right neutral is unique. In this case* S *is a right loop.*
   – *The maximal subquasigroups*
   $$Q \times \{r\} = Se = \{x \in S \mid x\backslash x = e\} \ \ (e \in E_S, e = (i, r))$$
   are maximal right subloops of S.
   – *For all* $e \in E_S$ $S \simeq Se \times E_S$ *and the isomorphism is given by* $f(x) = (xe, x\backslash x)$.

9. *If the quasigroup* Q *is a loop then:*
   – *The element* $e \in S$ *is a left neutral if and only if it is an idempotent.*
   – S *has a right neutral if and only if* $|R| = 1$ *and then the right neutral is unique. In this case* S *is a loop.*
   – *The maximal subquasigroups*

$$Q \times \{r\} = Se = \{x \in S \mid x\backslash x = x/x = e\} \quad (e \in E_S, e = (i,r))$$

are maximal subloops of S.

– *For all* $e \in E_S$ $S \simeq Se \times E_S$ *and the isomorphism is given by* $f(x) = (xe, x/x)$. □

Theorem 3.13. *Let* $S = Q \times R$ *be a right product pointed quasigroup with a distinguished element* $e = (i, j)$. *Then:*

1. *If* Q_m $(m \in M)$ *is the (possibly empty) family of all maximal pointed subquasigroups of* Q *then* $Q_m \times R$, $Q \times (R \setminus \{r\})$, *where* $r \in R \setminus \{j\}$, *is the family of all maximal right product pointed subquasigroups of* S.

2. $Se = Q \times \{j\}$ *is the largest pointed subquasigroup of* S.

From now on we assume $E_S \neq \emptyset$ *(i.e.,* $E_Q \neq \emptyset$*).*

3. *If* E_S *is subalgebra then it is the largest subalgebra of idempotents of* S.

4. $Se \cap E_S = \{e\}$ *is the largest pointed left zero subsemigroup of* S *if and only if* $e \in E_S$.

5. $E_S = \{i\} \times R$ *is the largest pointed right zero subsemigroup of* S *if and only if* $i \in E_Q$.

6. $S \simeq Se \times E_S$ *and the isomorphism is given by* $f(x) = (xe/e, ex/x)$.

From now on we assume $E_Q = \{i\}$.

7. $E_S = \{i\} \times R$ *is the unique largest pointed subband of* S, *which happens to be a pointed right zero semigroup.*

8. *If the element* i *is the left neutral of* Q, *it is the only idempotent of* Q *and:*

   – $E_S = \{a/a \mid a \in S\}$.

   – *The element* $a \in S$ *is a left neutral if and only if it is an idempotent.*

   – $Se = \{x \in S \mid x/x = e\}$ *is the largest left subloop of* S.

   – *The isomorphism* $S \simeq Se \times E_S$ *is given by* $f(x) = (xe/e, x/x)$.

9. *If the element i is the right neutral of Q, it is the only idempotent of Q and:*
   - $E_S = \{a\backslash a \mid a \in S\}$.
   - *S has a right neutral if and only if $|R| = 1$ and then the right neutral is e. In this case S is a right loop.*
   - *$Se = \{x \in S \mid x\backslash x = e\}$ is the largest right subloop of S.*
   - *The isomorphism $S \simeq Se \times E_S$ is given by $f(x) = (xe, x\backslash x)$.*

10. *If the element i is the two–sided neutral of Q, it is the only idempotent of Q and:*
    - *The element $a \in S$ is a left neutral if and only if it is an idempotent.*
    - *S has a right neutral if and only if $|R| = 1$ and then the right neutral is e. In this case S is a loop.*
    - *$Se = \{x \in S \mid x\backslash x = x/x = e\}$ is the largest subloop of S.*
    - *The isomorphism $S \simeq Se \times E_S$ is given by $f(x) = (xe, x/x)$.* □

3.3. The equation xa=b

Since a right product quasigroup is a right quasigroup the equation $ax = b$ has the unique solution $x = a\backslash b$. For the equation $xa = b$, the situation is not so clear.

We solve the equation $xa = b$ using the notion of *reproductivity.* The related notion of *reproductive general solution* was defined by E. Schröder [17] for Boolean equations and studied by L. Löwenheim [10, 11] who also introduced the term “reproductive". More recently, S. B. Prešić made significant contributions to the notion of reproductivity [13, 14, 15]. For an introduction to reproductivity, see S. Rudeanu [16].

Definition 3.14. Let $S \neq \emptyset$ and $F : S \longrightarrow S$. The equation $x = F(x)$ is *reproductive* if for all $x \in S$ $F(F(x)) = F(x)$.

The most significant properties of reproductivity are:

Theorem 3.15. *A general solution of the reproductive equation $x = F(x)$ is given by: $x = F(p)$ $(p \in S)$.* □

Theorem 3.16 (S. B. Prešić). *Every consistent equation has an equivalent reproductive equation.* □

We now apply these results to our equation $xa = b$.

Theorem 3.17.

1. *In right product quasigroups, the equation* $xa = b$ *is consistent if and only if* $(b/a)a = b$.
2. *In right product {left, right} loops, the consistency of* $xa = b$ *is equivalent to* $\{a/a = b/b, a\backslash a = b\backslash b\}$.
3. *If the equation* $xa = b$ *consistent, then it is equivalent to the reproductive equation* $x = (b/a)x/x$*, and thus its general solution is given by* $x = (b/a)p/p$ $(p \in S)$*. There are exactly* $|R|$ *distinct solutions.*
4. *If a right product quasigroup* $S = Q \times R$ *has idempotents, then the general solution of the consistent equation* $xa = b$ *may be given in the form* $x = (b/a)e/e \quad (e \in E_S)$.
5. *If the quasigroup* Q *has a unique idempotent, then any idempotent* $e \in E_S$ *defines the unique solution* $x = (b/a)e/e$ *of* $xa = b$.
6. *In a right product right loop, the general solution of* $xa = b$ *may be simplified to* $x = (b/a)e \quad (e \in E_S)$.

Proof. (1) If the equation is consistent then there is at least one solution $x = c$. It follows that $b = ca$ and $(b/a)a = (ca/a)a = ca = b$. If we assume that $(b/a)a = b$ then $x = b/a$ is one solution of the equation $xa = b$, which therefore must be consistent.

(2) Assume S is a right product left loop. If $xa = b$ is consistent then $b/b = xa/xa = a/a$. For the converse assume $S = Q \times R$ for some left loop Q with the left neutral i and a right zero semigroup R. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then $(i, a_2) = (a_1/a_1, a_2/a_2) = a/a = b/b = (b_1/b_1, b_2/b_2) = (i, b_2)$ i.e., $a_2 = b_2$ which is equivalent to the consistency of $xa = b$.

Now assume S is a right product right loop. If $xa = b$ is consistent then $b\backslash b = xa\backslash xa = a\backslash a$. For the converse assume $S = Q \times R$ for some right loop Q with the right neutral i and a right zero semigroup R. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then $(i, a_2) = (a_1\backslash a_1, a_2\backslash a_2) = a\backslash a = b\backslash b = (b_1\backslash b_1, b_2\backslash b_2) = (i, b_2)$ i.e., $a_2 = b_2$ which is equivalent to the consistency of $xa = b$.

(3) Let the equation $xa = b$ be consistent. Then $(b/a)x/x = (xa/a)x/x = xx/x = x$. Conversely, if $x = (b/a)x/x$ then $xa = ((b/a)x/x)a =$

$(b/a)a = b$. Therefore, equations $xa = b$ and $x = (b/a)x/x$ are equivalent. The form of the later equation is $x = F(x)$ where $F(x) = (b/a)x/x$. Also, $F(F(x)) = ((b/a) \cdot F(x))/F(x) = ((b/a)((b/a)x/x))/((b/a)x/x) = (b/a)x/x = F(x)$ so equation $x = F(x)$ is reproductive. Its general solution is $x = F(p) = (b/a)p/p (p \in S)$.

Without loss of generality we may assume that S is $Q \times R$ for some quasigroup Q and a right zero semigroup R. Let $a = (a_1, a_2), b = (b_1, b_2), x = (x_1, x_2)$ and $p = (p_1, p_2)$. The consistency of $xa = b$ reduces to $(b_1, b_2) = (b/a)a = ((b_1/a_1)a_1, a_2) = (b_1, a_2)$ i.e., $a_2 = b_2$. In that case, the solutions of $xa = b$ are $x = (b/a)p/p = ((b_1/a_1)p_1/p_1, (b_2/a_2)p_2/p_2) = (b_1/a_1, p_2)$. Evidently, the number of different solutions of $xa = b$ is $|R|$.

(4) Let $S = Q \times R$ and let i be an idempotent of Q. For every $p = (p_1, p_2)$ there is an idempotent $e = (i, p_2)$ of S such that $x = (b/a)p/p = (b_1/a_1, p_2) = ((b_1/a_1)i/i, (b_2/a_2)p_2/p_2) = (b/a)e/e$.

(5) If the idempotent $i \in Q$ is unique then E_S has exactly $|E_S| = |R|$ idempotents, just as many as the equation $xa = b$ has solutions.

(6) Let S be a right product right loop and $e = (i, r)$. Then

$$\begin{aligned} x &= (b/a)e/e = ((b_1/a_1)i/i, (b_2/a_2)r/r) = (b_1/a_1, r) \\ &= ((b_1/a_1)i, (b_2/a_2)r) = (b_1/a_1, b_2/a_2)(i, r) = (b/a)e\,. \end{aligned}$$

The proof is complete. □

3.4. Products of sequences of elements including idempotents

We use $\varrho(a_i, a_{i+1}, \dots, a_j)$ to denote *the right product* i.e., the product of $a_i, \dots, a_j$ with brackets associated to the right. More formally, $\varrho(a_i) = a_i$ $(1 \leqslant i \leqslant n)$ and $\varrho(a_i, a_{i+1}, \dots, a_j) = a_i \cdot \varrho(a_{i+1}, \dots, a_j)$ $(1 \leqslant i < j \leqslant n)$.

Further, we define $\varrho(\pm a_n) = a_n$ and

$$\varrho(a_i, a_{i+1}, \dots, a_j, \pm a_n) = \begin{cases} \varrho(a_i, \dots, a_j); & \text{if } j = n \\ \varrho(a_i, \dots, a_j, a_n); & \text{if } j < n. \end{cases}$$

In short, a_n should appear in the product $\varrho(a_i, \dots, a_j, \pm a_n)$, but only once.

The following is an analogue of Theorem 2.4 from [7].

Theorem 3.18. *Let $a_1, \dots, a_n$ $(n > 0)$ be a sequence of elements of the right product left loop S, such that $a_{p_1}, \dots, a_{p_m}$ $(1 \leqslant p_1 < \dots < p_m \leqslant n;$ $0 \leqslant m \leqslant n)$ comprise exactly the idempotents among $a_1, \dots, a_n$. Then $\varrho(a_1, \dots, a_n) = \varrho(a_{p_1}, \dots, a_{p_m}, \pm a_n)$.*

Proof. The proof is by induction on n.
(1) $n = 1$.
If $m = 0$ then $\varrho(a_1) = a_1 = \varrho(\pm a_1)$.
If $m = 1$ then $\varrho(a_1) = a_1 = \varrho(a_1, \pm a_1)$.
(2) $n > 1$.
If $a_1 \in E_S$ (i.e., $1 < p_1$) then $\varrho(a_1, \dots, a_n) = a_1 \cdot \varrho(a_2, \dots, a_n) = \varrho(a_2, \dots, a_n)$ which is equal to $\varrho(a_{p_1}, \dots, a_{p_m}, \pm a_n)$ by the induction argument.
If $a_1 \notin E_S$ (i.e., $1 = p_1$) then, using induction argument again, we get $\varrho(a_1, \dots, a_n) = a_1 \cdot \varrho(a_2, \dots, a_n) = a_{p_1} \cdot \varrho(a_{p_2}, \dots, a_{p_m}, \pm a_n) = \varrho(a_{p_1}, a_{p_2}, \dots, a_{p_m}, \pm a_n)$. □

Analogously to $\varrho(\dots)$, we use $\lambda(a_i, \dots, a_{j-1}, a_j)$ to denote *the left product* i.e., the product of $a_i, \dots, a_j$ with brackets associated to the left. Formally, $\lambda(a_i) = a_i$ $(1 \leqslant i \leqslant n)$ and $\lambda(a_i, \dots, a_{j-1}, a_j) = \lambda(a_i, \dots, a_{j-1}) \cdot a_j$ $(1 \leqslant i < j \leqslant n)$.

Further, we define $\lambda(\pm a_1, \pm a_n) = a_1$ if $n = 1$, $\lambda(\pm a_1, \pm a_n) = a_1 a_n$ if $n > 1$ and

$$\lambda(\pm a_1, a_i, \dots, a_j, \pm a_n) = \begin{cases} \lambda(a_i, \dots, a_j); & \text{if } 1 = i \leqslant j = n \\ \lambda(a_i, \dots, a_j, a_n); & \text{if } 1 = i \leqslant j < n \\ \lambda(a_1, a_i, \dots, a_j); & \text{if } 1 < i \leqslant j = n \\ \lambda(a_1, a_i, \dots, a_j, a_n); & \text{if } 1 < i \leqslant j < n. \end{cases}$$

Therefore, both a_1 and a_n should appear in the product $\lambda(\pm a_1, a_i, \dots, a_j, \pm a_n)$, but just once each. If $n = 1$ then $a_1 = a_n$ should also appear just once.

Of course, there is an analogue of Theorem 3.18.

Theorem 3.19. *Let $a_1, \dots, a_n$ $(n > 0)$ be a sequence of elements of the right product right loop S, such that $a_{p_1}, \dots, a_{p_m}$ $(1 \leqslant p_1 < \dots < p_m \leqslant n;$ $0 \leqslant m \leqslant n)$ and only them among $a_1, \dots, a_n$ are nonidempotents. Then $\lambda(a_1, \dots, a_n) = \lambda(\pm a_1, a_{p_1}, \dots, a_{p_m}, \pm a_n)$.*

Proof. The proof is by induction on n.
(1) $n = 1$.
If $m = 0$ then $\lambda(a_1) = a_1 = \lambda(\pm a_1, \pm a_1)$.
If $m = 1$ then $\lambda(a_1) = a_1 = \lambda(\pm a_1, a_1, \pm a_1)$.
(2) $n > 1$.
(2a) Let $1 = p_1, p_{m-1} = n-1, p_m = n$ (i.e., $a_1, a_{n-1}, a_n \notin E_S$). Then, using induction argument, $\lambda(a_1, \dots, a_n) = \lambda(a_1, \dots, a_{n-1}) \cdot a_n = \lambda(\pm a_1, a_{p_1}, \dots,$

$a_{p_{m-1}}, \pm a_{n-1}) \cdot a_{p_m} = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}) \cdot a_{p_m} = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}, a_{p_m}) = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_n)$.
(2b) Let $1 = p_1, p_m = n-1$ (i.e. $a_1, a_{n-1} \notin E_S; a_n \in E_S$). Then, using induction argument again, we get $\lambda(a_1, \ldots, a_n) = \lambda(a_1, \ldots, a_{n-1}) \cdot a_n = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_{n-1}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}, a_n) = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_n)$.
(2c) Let $1 = p_1, p_{m-1} < n-1, p_m = n$ (i.e., $a_1, a_n \notin E_S; a_{n-1} \in E_S$). Then, by the induction argument and (RL), $\lambda(a_1, \ldots, a_n) = \lambda(a_1, \ldots, a_{n-1}) \cdot a_n = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_{m-1}}, \pm a_{n-1}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}, a_{n-1}) \cdot a_{p_m} = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}) a_{n-1} \cdot a_{p_m} = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}) \cdot a_{p_m} = \lambda(a_{p_1}, \ldots, a_{p_{m-1}}, a_{p_m}) = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_n)$.
(2d) Let $1 = p_1, p_{m-1} < n$ (i.e., $a_1 \notin E_S; a_{n-1}, a_n \in E_S$). It follows that $\lambda(a_1, \ldots, a_n) = \lambda(a_1, \ldots, a_{n-1}) \cdot a_n = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_{n-1}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}, a_{n-1}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}) a_{n-1} \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}) \cdot a_n = \lambda(a_{p_1}, \ldots, a_{p_m}, a_n) = \lambda(\pm a_1, a_{p_1}, \ldots, a_{p_m}, \pm a_n)$.

The remaining cases of (2) in which $p_1 \neq 1$, *i.e.*, $a_1 \in E_S$ can be proved analogously. □

In right product {left, right} loops, Theorems 3.18 and 3.19 give us the means to reduce {right, left} products. The result is much stronger in right product loops.

Definition 3.20. Let $(S; \cdot, \backslash, /)$ be a right product quasigroup and $1 \notin S$. By S^1 we denote a triple magma with operations extending $\cdot, \backslash, /$ to $S \cup \{1\}$ in the following way: $x \circ y \quad (\circ \in \{\cdot, \backslash, /\})$ remains as before if $x, y \in S$. If $x = 1$ then $x \circ y = y$ and if $y = 1$ then $x \circ y = x$.

Note that the new, extended operations $\cdot, \backslash, /$ are well defined and that 1 is the neutral element for all three.

Lemma 3.21. *Let* $a_1, \ldots, a_n \quad (n > 0)$ *be a sequence of elements of a right product loop* S *such that* a_n *is an idempotent and* $p(a_1, \ldots, a_n)$ *some product of* $a_1, \ldots, a_n$ (*in that order*) *with an arbitrary* (*albeit fixed*) *distribution of brackets. Then* $p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{n-1}, 1) \cdot a_n$.

Proof. First, note that if e is an idempotent then $x \cdot ye = xy \cdot e$ for all $x, y \in S$. Namely, if $e \in E_S$ then there is a $z \in S$ such that $e = z/z$ (for example $z = e$ is one). The identity $x \cdot y(z/z) = xy \cdot (z/z)$ is true in all right product loops as it is true in all loops and all right zero semigroups.

The proof of the lemma is by induction on n.
(1) $n = 1$.

$a_1 = a_n$ is an idempotent, so $p(a_1) = a_1 = 1 \cdot a_1 = p(1) \cdot a_1$.
(2) $n > 1$.
Let $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_k) \cdot r(a_{k+1}, \ldots, a_n)$ for some $k \quad (1 \leqslant k \leqslant n)$. By the induction hypothesis $r(a_{k+1}, \ldots, a_n) = r(a_{k+1}, \ldots, a_{n-1}, 1) \cdot a_n$. So $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_k) \cdot (r(a_{k+1}, \ldots, a_{n-1}, 1) \cdot a_n) = (q(a_1, \ldots, a_k) \cdot r(a_{k+1}, \ldots, a_{n-1}, 1)) \cdot a_n = p(a_1, \ldots, a_{n-1}, 1) \cdot a_n$. □

The following result is an improvement of Theorems 3.18 and 3.19.

Theorem 3.22. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \quad (n > 0)$ be two sequences of elements of the right product loop S (with some of b_k possibly being 1) such that*

$$b_k = \begin{cases} 1; & \text{if } k < n \text{ and } a_k \in E_S \\ a_k; & \text{if } k = n \text{ or } a_k \notin E_S \end{cases}$$

and let $p(a_1, \ldots, a_n)$ be as in Lemma 3.21. Then $p(a_1, \ldots, a_n) = p(b_1, \ldots, b_n)$.

Proof. The proof of the Theorem is by induction on n.
(1) $n = 1$.
There is only one product $p(a_1) = a_1$ and, irrespectively of whether a_1 is idempotent or not, $b_1 = a_1$. Therefore $p(a_1) = p(b_1)$.
(2) $n > 1$.
Let $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_k) \cdot r(a_{k+1}, \ldots, a_n)$ for some k $(1 \leqslant k \leqslant n)$. By the induction hypothesis we have $q(a_1, \ldots, a_k) = q(b_1, \ldots, b_{k-1}, a_k)$ and $r(a_{k+1}, \ldots, a_n) = r(b_{k+1}, \ldots, b_n)$.

If a_k is nonidempotent then $a_k = b_k$ and $p(a_1, \ldots, a_n) = q(b_1, \ldots, b_k) \cdot r(b_{k+1}, \ldots, b_n) = p(b_1, \ldots, b_n)$.

If a_k is idempotent then $b_k = 1$ and by the Lemma 3.21 $p(a_1, \ldots, a_n) = q(b_1, \ldots, b_{k-1}, a_k) \cdot r(b_{k+1}, \ldots, b_n) = (q(b_1, \ldots, a_{k-1}, 1) \cdot a_k) \cdot r(b_{k+1}, \ldots, b_n) = q(b_1, \ldots, b_k) \cdot r(b_{k+1}, \ldots, b_n) = p(b_1, \ldots, b_n)$. □

The following corollary is an analogue of ([7], Theorem 2.4).

Corollary 3.23. *Let $a_1, \ldots, a_n$ be a sequence of elements of the right product loop S, such that at most two of them are nonidempotents. Then all products of $a_1, \ldots, a_n$, in that order, are equal to the following product of at most three of them: First – nonidempotents of $a_1, \ldots, a_{n-1}$ if any (the one with the smaller index first) and then a_n if it is not used already.* □

In right product pointed loops we need not use 1.

Theorem 3.24. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ $(n > 0)$ be two sequences of elements of the right product pointed loop S with the distinguished element e such that*

$$b_k = \begin{cases} e; & \text{if } k < n \text{ and } a_k \in E_S \\ a_k; & \text{if } k = n \text{ or } a_k \notin E_S \end{cases}$$

and let $p(a_1, \dots, a_n)$ be some product of $a_1, \dots, a_n$. Then $p(a_1, \dots, a_n) = p(b_1, \dots, b_n)$. □

4. Independence of axioms

Finally, we consider the independence of the axioms (A) for right product quasigroups.

It is well-known that the quasigroup axioms (Q1)–(Q4) are independent. It follows that axioms (A1) and (A2) are independent. To give just one concrete example, here is a model in which (Q2) = (A2) fails.

Example 4.1. The model $(\mathbb{Z}; \cdot, \backslash, /)$ where $x \cdot y = x + y$, $x/y = x - y$ and $x \backslash y = \max\{y - x, 0\}$ is a left quasigroup satisfying (Q1) but not (Q2), and hence satisfies (A1), (A3), (A4) and (A5), but not (A2). □

As it turns out, the independence of the remaining axioms can be easily shown by models of size 2. These were found using Mace4 [12].

Example 4.2. Table 3 is a model satisfying (A1), (A2), (A4), (A5), but not (A3).

$\cdot$	0	1
0	0	1
1	0	1

$\backslash$	0	1
0	0	1
1	0	1

$/$	0	1
0	1	0
1	1	0

Table 3: (A1), (A2), (A4), (A5), but not (A3).

Example 4.3. Table 4 is a model satisfying (A1), (A2), (A3), (A5), but not (A4). □

Example 4.4. Table 5 is a model satisfying (A1), (A2), (A3), (A4), but not (A5).

$\cdot$	0	1
0	0	1
1	1	0

$\backslash$	0	1
0	0	1
1	1	0

$/$	0	1
0	1	0
1	0	1

Table 4: (A1), (A2), (A3), (A5), but not (A4).

$\cdot$	0	1
0	1	0
1	1	0

$\backslash$	0	1
0	1	0
1	1	0

$/$	0	1
0	1	0
1	1	0

Table 5: (A1), (A2), (A3), (A4), but not (A5).

Acknowledgment. Our investigations were aided by the automated deduction program Prover9 and the finite model builder Mace4, both developed by McCune [12]

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Received February 15, 2011

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