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Characterizations of hemirings by interval valued fuzzy ideals

Muhammad Shabir and *Tahir Mahmood*

Abstract. In this paper we define interval valued fuzzy h-quasi-ideals and interval valued fuzzy h-bi-ideals. We characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy h-ideals, interval valued fuzzy h-quasi-ideals, and interval valued fuzzy h-bi-ideals.

1. Introduction

Semirings, which are the generalization of associative rings, introduced by H. S. Vandiver in 1934 [12], are very useful for solving problems in different areas of applied mathematics and information sciences, like as, optimization theory, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, automata theory, formal language theory, coding theory, analysis of computer programs, and so on. Hemirings, which are semirings with commutative addition and zero element appears in a natural manner in some applications to the theory of automata and formal languages (see [4]).

Like in rings theory, ideals play important role in the study of hemirings and are useful for many purposes. But they do not coincide with ring ideals. Thus many results of ring theory have no analogues in semirings using only ideals. In order to overcome this deficiency, Henriksen [5] defined a class of ideals in semirings, called k-ideals. These ideals have the property that if the semiring R is a ring then a subset of R is a k-ideal if and only if it is a ring ideal. A more restricted class of ideals in hemirings is defined by Iizuka [6], called h-ideals. La Torre [8] thoroughly studied h-ideals and k-ideals and established some analogues ring results for semirings.

Zadeh [14], in 1965, introduced the concept of fuzzy set. Which proved

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a very useful tool to describe situation in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The concept of fuzzy set was further carried out by many researchers to generalize some notions of algebra. In [2] J. Ahsan initiated the study of fuzzy semirings (see also [1]). Fuzzy k-ideals in semirings are studied in [3] by Ghosh and fuzzy h-ideals are studied in [7, 10, 13, 15].

In [9] Ma and Zhan introduced the concept of interval valued fuzzy h-ideals in hemirings and develop some results associated with it. In [11] Sun et al characterized h-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy left and right h-ideals. In this paper we extend this idea and define interval valued fuzzy h-quasi-ideals and interval valued fuzzy h-bi-ideals. We characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy h-ideals, interval valued fuzzy h-quasi-ideals, and interval valued fuzzy h-bi-ideals.

2. Preliminaries

For basic definitions of ideals see [4]. A left (right) ideal I of a hemiring R is called a *left* (*right*) *h-ideal* if for all $x, z \in R$ and for any $a_1, a_2 \in I$ from $x + a_1 + z = a_2 + z$, it follows $x \in I$ (cf. [8]). A bi-ideal B of a hemiring R is called an *h-bi-ideal* of R if for all $x, z \in R$ and $a_1, a_2 \in B$ from $x + a_1 + z = a_2 + z$, it follows $x \in B$ (cf. [13]).

The *h-closure* $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \quad \text{for some } a, b \in A, z \in R\}.$$

A quasi-ideal Q of a hemiring R is called an *h-quasi-ideal* of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x + a_1 + z = a_2 + z$ implies $x \in Q$, for all $x, z \in R$ and $a_1, a_2 \in Q$ (cf. [13]). Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general (cf. [13]).

Definition 2.1. A hemiring R is said to be *h-hemiregular* if for each $x \in R$, there exist $a, b, z \in R$ such that $x + xax + z = xbx + z$.

Lemma 2.2. [15] *A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.* □

Lemma 2.3. [13] *For a hemiring R the following conditions are equivalent.*

(i) *R is h-hemiregular.*

(ii) *$B = \overline{BRB}$ for every h-bi-ideal B of R.*

(iii) *$Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.* □

Lemma 2.4. [13] *A hemiring R is h-hemiregular if and only if the right and left h-ideals of R are idempotent and for any right h-ideal I and left h-ideal L of R, $\overline{IL}$ is an h-quasi- ideal of R.* □

Definition 2.5. [13] A hemiring R is said to be *h-intra-hemiregular* if for each $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$.

Lemma 2.6. [13] *A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.* □

Lemma 2.7. [13] *The following conditions are equivalent for a hemiring R.*

(1) *R is both h-hemiregular and h-intra-hemiregular.*

(2) *$B = \overline{B^2}$ for every h-bi-ideal B of R.*

(3) *$Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.* □

3. Interval valued fuzzy sets

A fuzzy subset f is a function $f : X \to [0, 1]$. Now let $\mathcal{L}$ be the family of all closed subintervals of $[0, 1]$ with minimal element $\tilde{O} = [0, 0]$ and maximal element $\tilde{I} = [1, 1]$ according to the partial order $[\alpha, \alpha'] \leqslant [\beta, \beta']$ if and only if $\alpha \leqslant \beta, \alpha' \leqslant \beta'$ defined on $\mathcal{L}$ for all $[\alpha, \alpha'], [\beta, \beta'] \in \mathcal{L}$. An interval valued fuzzy subset λ of a hemiring R is a function $\lambda : R \to \mathcal{L}$.

We write $\lambda(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0, 1]$, for all $x \in R$, where λ^-, λ^+ are fuzzy subsets of R such that for each $x \in R$, $0 \leqslant \lambda^-(x) \leqslant \lambda^+(x) \leqslant 1$. For simplicity we write $\lambda = [\lambda^-, \lambda^+]$.

Let $A \subseteq R$. Then the interval valued characteristic function χ_A of A is defined to be a function $\chi_A : R \to \mathcal{L}$ such that for all $x \in R$

$$\chi_A(x) = \begin{cases} \tilde{I} = [1, 1] & \text{if } x \in A, \\ \tilde{O} = [0, 0] & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of R is also an interval valued fuzzy subset of R. Note that $\chi_R(x) = \tilde{I}$ for all $x \in R$.

For any two interval valued fuzzy subsets λ and μ of a hemiring R we define

$$(\lambda \vee \mu)(x) = [\lambda^-(x) \vee \mu^-(x), \lambda^+(x) \vee \mu^+(x)],$$
$$(\lambda \wedge \mu)(x) = [\lambda^-(x) \wedge \mu^-(x), \lambda^+(x) \wedge \mu^+(x)],$$

where
$\lambda^-(x) \vee \mu^-(x) = \sup\{\lambda^-(x), \mu^-(x)\}$, $\lambda^-(x) \wedge \mu^-(x) = \inf\{\lambda^-(x), \mu^-(x)\}$,
$\lambda^+(x) \vee \mu^+(x) = \sup\{\lambda^+(x), \mu^+(x)\}$, $\lambda^+(x) \wedge \mu^+(x) = \inf\{\lambda^+(x), \mu^+(x)\}$.

For any two interval valued fuzzy subsets λ and μ of a hemiring R, $\lambda \leqslant \mu$ if and only if $\lambda(x) \leqslant \mu(x)$, that is $\lambda^-(x) \leqslant \mu^-(x)$ and $\lambda^+(x) \leqslant \mu^+(x)$, for all $x \in R$.

Definition 3.1. [11] Let λ and μ be two interval valued fuzzy subsets in a hemiring R. Then the *h-intrinsic product* of λ and μ is defined by

$$(\lambda \odot \mu)(x) = \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \left(\lambda^-(a_i) \wedge \mu^-(b_i)\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^-(a'_j) \wedge \mu^-(b'_j)\right), \\ \bigwedge\limits_{i=1}^{m} \left(\lambda^+(a_i) \wedge \mu^+(b_i)\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^+(a'_j) \wedge \mu^+(b'_j)\right), \end{array} \right\}$$

for all $x \in R$, if x can be expressed as $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$, and $\tilde{O}$ if x cannot be expressed as $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$.

An interval valued fuzzy subset λ of a hemiring R is said to be *idempotent* if $\lambda \odot \lambda = \lambda$.

Lemma 3.2. [11] *Let R be a hemiring and $A, B \subseteq R$. Then we have*
(1) $A \subseteq B \longleftrightarrow \chi_A \leqslant \chi_B.$,
(2) $\chi_A \wedge \chi_B = \chi_{A \cap B}$,
(3) $\chi_A \odot \chi_B = \chi_{\overline{AB}}$.

Definition 3.3. Let λ be an interval valued fuzzy subset of a hemiring R. Then λ is said to be an *interval valued fuzzy left* (resp. *right*) *h-ideal* of R if and only if for all $x, y \in R$
(i) $\lambda(x + y) \geqslant \lambda(x) \wedge \lambda(y)$,
(ii) $\lambda(xy) \geqslant \lambda(y)$ (resp. $\lambda(xy) \geqslant \lambda(x)$)
(iii) $x + a + y = b + y \longrightarrow \lambda(x) \geqslant \lambda(a) \wedge \lambda(b)$, for all $a, b, x, y \in R$.

An interval valued fuzzy subset $\lambda : R \to \mathcal{L}$ is called an *interval valued fuzzy h-ideal* of hemiring R if it is both, interval valued fuzzy left and right h-ideal of R.

Definition 3.4. An interval valued fuzzy subset λ of a hemiring R is called an *interval valued fuzzy h-bi-ideal* of R if it satisfies (i), (iii) and

(*iv*) $\lambda(xy) \geqslant \min\{\lambda(x), \lambda(y)\}$,
(*v*) $\lambda(xyz) \geqslant \min\{\lambda(x), \lambda(z)\}$
for all $x, y, z \in R$.

An interval valued fuzzy subset λ of a hemiring R is called an *interval valued fuzzy h-quasi-ideal* of R if it satisfies (*i*), (*iii*) and
(*vi*) $(\lambda \odot \mathcal{R}) \wedge (\mathcal{R} \odot \lambda) \leqslant \lambda$.

Note that if λ is any interval valued fuzzy left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal), then $\lambda(0) \geqslant \lambda(x)$ for all $x \in R$.

Lemma 3.5. *A subset A of a hemiring R is an h-ideal (resp., h-bi-ideal, h-quasi-ideal) of R if and only if χ_A is an interval valued fuzzy h-ideal (resp., h-bi-ideal, h-quasi-ideal) of R.* □

Theorem 3.6. *Every interval valued fuzzy right(left) h-ideal is interval valued fuzzy h-quasi-ideal.* □

Converse of the Theorem 3.6 is not true in general.

Example 3.7. Let $\mathbb{Z}_0 = \mathbb{Z}^+ \cup \{0\}$, $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_0 \right\}$ and $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0 \right\}$. Then R is a hemiring under the usual binary operations of addition and multiplication of matrices, and Q is h–quasi-ideal of R but Q is not left (right) h-ideal of R. Then by Lemma 3.5, the characteristic function χ_Q is an interval valued fuzzy h-quasi-ideal of R and by Lemma 3.5, χ_Q is not interval valued fuzzy left (right) h-ideal of R. □

Theorem 3.8. *If λ is an interval valued fuzzy right h-ideal and μ an interval valued fuzzy left h-ideal of a hemiring R, then $\lambda \wedge \mu$ is an interval valued fuzzy h-quasi-ideal of R.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}
(\lambda \wedge \mu)(x+y) &= [\lambda^-(x+y) \wedge \mu^-(x+y), \lambda^+(x+y) \wedge \mu^+(x+y)] \\
&\geqslant [\lambda^-(x) \wedge \lambda^-(y) \wedge \mu^-(x) \wedge \mu^-(y), \lambda^+(x) \wedge \lambda^+(y) \wedge \mu^+(x) \wedge \mu^+(y)] \\
&= [\lambda^-(x) \wedge \mu^-(x),\ \lambda^+(x) \wedge \mu^+(x)] \wedge [\lambda^-(y) \wedge \mu^-(y),\ \lambda^+(y) \wedge \mu^+(y)] \\
&= (\lambda \wedge \mu)(x) \wedge (\lambda \wedge \mu)(y).
\end{aligned}$$

Now $((\lambda \wedge \mu) \odot \mathcal{R}) \wedge (\mathcal{R} \odot (\lambda \wedge \mu)) \leqslant (\lambda \odot \mathcal{R}) \wedge (\mathcal{R} \odot \mu) \leqslant \lambda \wedge \mu$.

Next let $a, b, x, z \in R$ such that $x + a + z = b + z$. Then

$(\lambda \wedge \mu)(x) = [\lambda^-(x) \wedge \mu^-(x), \lambda^+(x) \wedge \mu^+(x)]$

$\geqslant [\lambda^-(a) \wedge \lambda^-(b) \wedge \mu^-(a) \wedge \mu^-(b), \lambda^+(a) \wedge \lambda^+(b) \wedge \mu^+(a) \wedge \mu^+(b)]$
$= [\lambda^-(a) \wedge \mu^-(a),\ \lambda^+(a) \wedge \mu^+(a)] \wedge [\lambda^-(b) \wedge \mu^-(b),\ \lambda^+(b) \wedge \mu^+(b)]$
$= (\lambda \wedge \mu)(a) \wedge (\lambda \wedge \mu)(b).$

Hence $\lambda \wedge \mu$ is an interval valued fuzzy h-quasi-ideal of R. □

Theorem 3.9. *Any interval valued fuzzy h-quasi-ideal of a hemiring R is an interval valued fuzzy h-bi-ideal of R.*

Proof. Let λ be any interval valued fuzzy h-quasi-ideal of R. Then for all $x, y, z \in R$, for all expressions $xyz + \Sigma_{i=1}^m a_i b_i + z' = \Sigma_{j=1}^n a'_j b'_j + z'$, we have

$$\lambda(xyz) \geqslant ((\lambda \odot \mathcal{R}) \wedge (\mathcal{R} \odot \lambda))(xyz) = (\lambda \odot \mathcal{R})(xyz) \wedge (\mathcal{R} \odot \lambda)(xyz)$$

$$= \begin{bmatrix} \sup \left\{ \Big(\bigwedge_{i=1}^m \lambda^-(a_i) \Big) \wedge \Big(\bigwedge_{j=1}^n \lambda^-(a'_j) \Big), \Big(\bigwedge_{i=1}^m \lambda^+(a_i) \Big) \wedge \Big(\bigwedge_{j=1}^n \lambda^+(a'_j) \Big) \right\} \\ \wedge\ \sup \left\{ \Big(\bigwedge_{i=1}^m \lambda^-(b_i) \Big) \wedge \Big(\bigwedge_{j=1}^n \lambda^-(b'_j) \Big), \Big(\bigwedge_{i=1}^m \lambda^+(b_i) \Big) \wedge \Big(\bigwedge_{j=1}^n \lambda^+(b'_j) \Big) \right\} \end{bmatrix}$$

$$\geqslant \{\lambda^-(0) \wedge \lambda^-(x), \lambda^+(0) \wedge \lambda^+(x)\} \wedge \{\lambda^-(0) \wedge \lambda^-(z), \lambda^+(0) \wedge \lambda^+(z)\}$$

(because $xyz+00+0 = x(yz)+0$ and $xyz+00+0 = (xy)z+0$)

$$= \lambda(x) \wedge \lambda(z)$$

Similarly we can show that $\lambda(xy) \geqslant \lambda(x) \wedge \lambda(y)$ for all $x, y \in R$. Hence λ is an interval valued fuzzy h-bi-ideal of R. □

Converse of the Theorem 3.9 is not true in general.

Example 3.10. Let $\mathbb{Z}^+$ and $\mathbb{R}^+$ be the sets of all positive integers and positive real numbers, respectively. And

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in \mathbb{R}^+, c \in \mathbb{Z}^+ \right\},$$
$$I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in \mathbb{R}^+, c \in \mathbb{Z}^+, a < b \right\},$$
$$J = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in \mathbb{R}^+, c \in \mathbb{Z}^+, b > 3 \right\}.$$

Then R is a hemiring under the usual binary operations of addition and multiplication of matrices, and I is right h-ideal and J is left h-ideal of R. Now the product IJ is an h-bi-ideal of R and it is not an h-quasi-ideal of R. Then by Lemma 3.5, the function χ_{IJ} is an interval valued fuzzy h-bi-ideal of R and it is not interval valued fuzzy h-quasi-ideal of R. □

Theorem 3.11. *Let $\lambda = [\lambda^-, \lambda^+]$ be an interval valued fuzzy subset of a hemiring R. Then λ is an interval valued fuzzy h-ideal (h-bi-ideal, h-quasi-ideal) of R if and only if λ^- and λ^+ are fuzzy h-ideals (h-bi-ideals, h-quasi-ideals) of R.* □

Theorem 3.12. [11] *A hemiring R is h-hemiregular if and only if for any interval valued fuzzy right h-ideal λ and interval valued fuzzy left h-ideal μ of R we have $\lambda \odot \mu = \lambda \wedge \mu$.* □

Theorem 3.13. *For a hemiring R the following conditions are equivalent.*

(i) *R is h-hemiregular.*

(ii) *$\lambda \leqslant \lambda \odot \mathcal{R} \odot \lambda$, for every interval valued fuzzy h-bi-ideal of R.*

(iii) *$\lambda \leqslant \lambda \odot \mathcal{R} \odot \lambda$, for every interval valued fuzzy h-quasi-ideal of R.*

Proof. (i) $\Rightarrow$ (ii) Let λ be an interval valued fuzzy h-bi-ideal of R and $x \in R$. Then there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Then for all expressions $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$, we have

$$
\begin{aligned}
&(\lambda \odot \mathcal{R} \odot \lambda)(x) \\
&= \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \Big((\lambda \odot \mathcal{R})^-(a_i) \wedge \lambda^-(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big((\lambda \odot \mathcal{R})^-(a'_j) \wedge \lambda^-(b'_j) \Big), \\ \bigwedge\limits_{i=1}^{m} \Big((\lambda \odot \mathcal{R})^+(a_i) \wedge \lambda^+(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big((\lambda \odot \mathcal{R})^+(a'_j) \wedge \lambda^+(b'_j) \Big) \end{array} \right\} \\
&\geqslant \left\{ \begin{array}{l} (\lambda \odot \mathcal{R})^- (xa) \wedge \lambda^- (x) \wedge (\lambda \odot \mathcal{R})^- (xa') \wedge \lambda^- (x), \\ (\lambda \odot \mathcal{R})^+ (xa) \wedge \lambda^+ (x) \wedge (\lambda \odot \mathcal{R})^+ (xa') \wedge \lambda^+ (x) \end{array} \right\} \\
&= [(\lambda \odot \mathcal{R})^-(xa), (\lambda \odot \mathcal{R})^+(xa)] \wedge [(\lambda \odot \mathcal{R})^-(xa'), (\lambda \odot \mathcal{R})^+(xa')] \wedge [\lambda^-(x), \lambda^+(x)] \\
&= (\lambda \odot \mathcal{R})(xa) \wedge (\lambda \odot \mathcal{R})(xa') \wedge \lambda(x) \\
&= \sup \left\{ \bigwedge_{i=1}^{m} \lambda^-(a_i) \wedge \bigwedge_{j=1}^{n} \lambda^-(a'_j), \bigwedge_{i=1}^{m} \lambda^+(a_i) \wedge \bigwedge_{j=1}^{n} \lambda^+(a'_j) \right\} \wedge \\
&\qquad \sup \left\{ \bigwedge_{i=1}^{m} \lambda^-(c_i) \wedge \bigwedge_{j=1}^{n} \lambda^-(c'_j), \bigwedge_{i=1}^{m} \lambda^+(c_i) \wedge \bigwedge_{j=1}^{n} \lambda^+(c'_j) \right\} \wedge \lambda(x)
\end{aligned}
$$

(for all $xa + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$ and $xa' + \Sigma_{i=1}^{m} c_i d_i + z = \Sigma_{j=1}^{n} c'_j d'_j + z$)

$$
\geqslant \left\{ \begin{array}{l} [\lambda^- (xax) \wedge \lambda^- (xa'x), \lambda^+ (xax) \wedge \lambda^+ (xa'x)] \wedge \\ [\lambda^- (xax) \wedge \lambda^- (xa'x), \lambda^+ (xax) \wedge \lambda^+ (xa'x)] \wedge \lambda(x) \end{array} \right\}
$$

(because $xa + xaxa + za = xa'xa + za$ and $xa' + xaxa' + za' = xa'xa' + za'$)

$\geqslant [\lambda^- (xax), \lambda^+ (xax)] \wedge [\lambda^- (xa'x), \lambda^+ (xa'x)] \wedge \lambda(x) = \lambda(x)$.

Thus $\lambda \leqslant \lambda \odot \mathcal{R} \odot \lambda$.

$(ii) \Rightarrow (iii)$ is straightforward because each interval valued fuzzy h-quasi-ideal is an interval valued fuzzy h-bi-ideal.

$(iii) \Rightarrow (i)$ Let Q be any h-quasi-ideal of a hemiring R. Then by Lemma 3.5, χ_Q is an interval valued fuzzy h-quasi-ideal of R. Then by Lemma 3.2 and the hypothesis $\chi_Q \subseteq \chi_Q \odot \mathcal{R} \odot \chi_Q = \chi_{\overline{QRQ}}$, which implies $Q \subseteq \overline{QRQ}$. Also, as Q is an h-quasi-ideal, so $\overline{QRQ} \subseteq \overline{RQ} \cap \overline{QR} \subseteq Q$. Hence $\overline{QRQ} = Q$. Thus, by Lemma 2.3, R is h-hemiregular hemiring. □

Theorem 3.14. *For a hemiring R, the following conditions are equivalent.*

(i) *R is h-hemiregular.*

(ii) $\lambda \wedge \mu \leqslant \lambda \odot \mu$ *for every interval valued fuzzy h-bi-ideal* λ *and inter-val valued fuzzy left h-ideal* μ *of R.*

(iii) $\lambda \wedge \mu \leqslant \lambda \odot \mu$ *for every interval valued fuzzy h-quasi-ideal* λ *and interval valued fuzzy left h-ideal* μ *of R.*

(iv) $\lambda \wedge \mu \leqslant \lambda \odot \mu$*for every interval valued fuzzy right h-ideal* λ *and interval valued fuzzy h-bi-ideal* μ *of R.*

(v) $\lambda \wedge \mu \leqslant \lambda \odot \mu$ *for every interval valued fuzzy right h-ideal* λ *and interval valued fuzzy h-quasi-ideal* μ *of R.*

(vi) $\lambda \wedge \mu \wedge \nu \leqslant \lambda \odot \mu \odot \nu$ *for every interval valued fuzzy right h-ideal* λ*, fuzzy h-bi-ideal* μ *and interval valued fuzzy left h-ideal* ν *of R.*

(vii) $\lambda \wedge \mu \wedge \nu \leqslant \lambda \odot \mu \odot \nu$ *for every interval valued fuzzy right h-ideal* λ*, interval valued fuzzy h-quasi-ideal* μ *and interval valued fuzzy left h-ideal* ν *of R.*

Proof. $(i) \Rightarrow (ii)$ Let λ be any *interval valued* fuzzy h-bi-ideal and μ be any *interval valued* fuzzy left h-ideal of R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Thus for all expressions $a + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$, we have

$$(\lambda \odot \mu)(a) = \sup \left\{ \begin{array}{c} \bigwedge_{i=1}^{m} \left(\lambda^-(a_i) \wedge \mu^-(b_i) \right) \wedge \bigwedge_{j=1}^{n} \left(\lambda^-(a'_j) \wedge \mu^-(b'_j) \right), \\ \bigwedge_{i=1}^{m} \left(\lambda^+(a_i) \wedge \mu^+(b_i) \right) \wedge \bigwedge_{j=1}^{n} \left(\lambda^+(a'_j) \wedge \mu^+(b'_j) \right) \end{array} \right\}$$

$$\geqslant [\lambda^-(a), \lambda^+(a)] \wedge [\mu^-(x_1a), \mu^+(x_1a)] \wedge [\mu^-(x_2a), \mu^+(x_2a)]$$

$$(\text{because } a + ax_1a + z = ax_2a + z)$$

$$\geqslant [\lambda^-(a), \lambda^+(a)] \wedge [\mu^-(a), \mu^+(a)] \geqslant \lambda(a) \wedge \mu(a) = (\lambda \wedge \mu)(a).$$

So $\lambda \odot \mu \geqslant \lambda \wedge \mu$.

$(ii) \Rightarrow (iii)$ By Theorem 3.9.

$(iii) \Rightarrow (i)$ Let λ be an interval valued fuzzy right h-ideal and μ be an interval valued fuzzy left h-ideal of R. Since every interval valued fuzzy right h-ideal is interval valued fuzzy h-quasi-ideal, so by (iii) we have $\lambda \odot \mu \geqslant \lambda \wedge \mu$. But $\lambda \odot \mu \leqslant \lambda \wedge \mu$. Hence $\lambda \odot \mu = \lambda \wedge \mu$ for every interval valued fuzzy right h-ideal λ of R, and for every interval valued fuzzy left h-ideal μ of R. Thus by Theorem 3.12, R is h-hemiregular.

Similarly we can prove $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.

$(i) \Rightarrow (vi)$ Let λ be any interval valued fuzzy right h-ideal, μ be an interval valued fuzzy h-bi-ideal and ν be an interval valued fuzzy left h-ideal of R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Then for all expressions $a + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$, we have

$$(\lambda \odot \mu \odot \nu)(a) = \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \Big((\lambda \odot \mu)^-(a_i) \wedge \nu^-(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big((\lambda \odot \mu)^-(a'_j) \wedge \nu^-(b'_j) \Big), \\ \bigwedge\limits_{i=1}^{m} \Big((\lambda \odot \mu)^+(a_i) \wedge \nu^+(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big((\lambda \odot \mu)^+(a'_j) \wedge \nu^+(b'_j) \Big) \end{array} \right\}$$

$$\geqslant [(\lambda \odot \mu)^-(a) \wedge \nu^-(x_1 at) \wedge \nu^-(x_2 a), (\lambda \odot \mu)^+(a) \wedge \nu^+(x_1 a) \wedge \nu^+(x_2 a)]$$

$$= [(\lambda \odot \mu)^-(a), (\lambda \odot \mu)^+(a)] \wedge [\nu^-(x_1 a), \nu^+(x_1 a)] \wedge [\nu^-(x_2 a), \nu^+(x_2 a)]$$

$$= (\lambda \odot \mu)(a) \wedge \nu(x_1 a) \wedge \nu(x_2 a)$$

$$\geqslant \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \Big(\lambda^-(a_i) \wedge \mu^-(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big(\lambda^-(a'_j) \wedge \mu^-(b'_j) \Big), \\ \bigwedge\limits_{i=1}^{m} \Big(\lambda^+(a_i) \wedge \mu^+(b_i) \Big) \wedge \bigwedge\limits_{j=1}^{n} \Big(\lambda^+(a'_j) \wedge \mu^+(b'_j) \Big) \end{array} \right\} \wedge \nu(a)$$

(for all expressions $a + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z$)

$$\geqslant [\lambda^-(ax_1) \wedge \lambda^-(ax_2) \wedge \mu^-(a), \lambda^+(ax_1) \wedge \lambda^+(ax_2) \wedge \mu^+(a)] \wedge \nu(a)$$

$$= \lambda(ax_1) \wedge \lambda(ax_2) \wedge \mu(a) \wedge \nu(a) \geqslant \lambda(a) \wedge \mu(a) \wedge \nu(a) = (\lambda \wedge \mu \wedge \nu)(a).$$

Thus $\lambda \wedge \mu \wedge \nu \leqslant \lambda \odot \mu \odot \nu$.

$(vi) \Rightarrow (vii)$ Obvious.

$(vii) \Rightarrow (i)$ Let λ be any interval valued fuzzy right h-ideal, and ν be an interval valued fuzzy left h-ideal of R. Then

$$\lambda \wedge \nu = \lambda \wedge \mathcal{R} \wedge \nu \leqslant \lambda \odot \mathcal{R} \odot \nu \leqslant \lambda \odot \nu.]$$

But $\lambda \odot \nu \leqslant \lambda \wedge \nu$ always holds. Hence $\lambda \odot \nu = \lambda \wedge \nu$ for every interval valued fuzzy right h-ideal λ and for every interval valued fuzzy left h-ideal ν of R. Thus by Theorem 3.12, R is h-hemiregular. $\square$

Theorem 3.15. *A hemiring R is h-hemiregular if and only if*

(i) *every interval valued fuzzy right and interval valued fuzzy left h-ideal of R are idempotent,*

(ii) *for any interval valued fuzzy right h-ideal λ and for any interval valued fuzzy left h-ideal μ of R, $\lambda \odot \mu$ is an interval valued fuzzy h-quasi-ideal of R.*

Proof. Assume R is h-hemiregular and let λ be an interval valued fuzzy left h-ideal of R. Then $\lambda \odot \lambda \leqslant \mathcal{R} \odot \lambda \leqslant \lambda$.

Also as R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a+ax_1a+z = ax_2a+z$. Then for all expressions $a+\Sigma_{i=1}^{m} a_ib_i+z = \Sigma_{j=1}^{n} a'_jb'_j + z$, we have

$$(\lambda \odot \lambda)(a) = \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \Big(\lambda^-(a_i) \wedge \lambda^-(b_i)\Big) \wedge \bigwedge\limits_{j=1}^{n} \Big(\lambda^-(a'_j) \wedge \lambda^-(b'_j)\Big), \\ \bigwedge\limits_{i=1}^{m} \Big(\lambda^+(a_i) \wedge \lambda^+(b_i)\Big) \wedge \bigwedge\limits_{j=1}^{n} \Big(\lambda^+(a'_j) \wedge \lambda^+(b'_j)\Big) \end{array} \right\}$$

$$\geqslant [\lambda^-(a), \lambda^+(a)] \wedge [\lambda^-(x_1a), \lambda^+(x_1a)] \wedge [\lambda^-(x_2a), \lambda^+(x_2a)]$$

$$\text{(because } a + ax_1a + z = ax_2a + z\text{)}$$

$$\geqslant [\lambda^-(a), \lambda^+(a)] \wedge [\lambda^-(a), \lambda^+(a)] \geqslant \lambda(a) \wedge \lambda(a) = \lambda(a),$$

that is, $\lambda \odot \lambda \geqslant \lambda$. Thus $\lambda \odot \lambda = \lambda$.

Similarly we can prove for interval valued fuzzy right h-ideal of R. Hence (i) holds. Now let λ be any interval valued fuzzy right h-ideal and μ be any interval valued fuzzy left h-ideal of R, then by Theorem 3.12, $\lambda \odot \mu = \lambda \wedge \mu$. Then by Theorem 3.8, $\lambda \odot \mu$ is interval valued fuzzy h-quasi-ideal of R. Hence (ii) holds.

Conversely, assume that (i) and (ii) holds. Let I be any right h-ideal of R. Then by Lemma 3.5, χ_I is interval valued fuzzy right h-ideal of R. Now by using Lemma 3.2, and hypothesis $\chi_I = \chi_I \odot \chi_I = \chi_{\overline{I^2}}$, which implies $I = I^2$. So I is an idempotent.

Now let I be any right h-ideal and L be any left h-ideal of R. Then by using Lemma 3.2, and hypothesis $\chi_{\overline{IL}} = \chi_I \odot \chi_L$ is an h-quasi-ideal of R. Thus by Lemma 3.5, $\overline{IL}$ is an h-quasi-ideal of R. Hence by Lemma 2.4, R is h-hemiregular. □

Theorem 3.16. *A hemiring R is h-intra-hemiregular if and only if for any interval valued fuzzy left h-ideal λ and any interval valued fuzzy right h-ideal μ of R, $\lambda \wedge \mu \leqslant \lambda \odot \mu$.*

Proof. Assume that R is an h-intra-hemiregular hemiring, λ is an interval valued fuzzy right h-ideal and μ an interval valued fuzzy left h-ideal of R. Now as R is h-intra-hemiregular, so for any $x \in R$, there exists $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$. Then for all such expressions

$$(\lambda \odot \mu)(x) = \sup \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \left(\lambda^-(a_i) \wedge \mu^-(b_i)\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^-(a_j') \wedge \mu^-(b_j'\right), \\ \bigwedge\limits_{i=1}^{m} \left(\lambda^-(a_i) \wedge \mu^-(b_i)\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^-(a_j') \wedge \mu^-(b_j'\right) \end{array} \right\}$$

$$\geqslant \left\{ \begin{array}{l} \bigwedge\limits_{i=1}^{m} \left(\lambda^-(a_i x) \wedge \mu^-(x a_i')\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^-(b_j x) \wedge \mu^-(x b_j')\right), \\ \bigwedge\limits_{i=1}^{m} \left(\lambda^+(a_i x) \wedge \mu^+(x a_i')\right) \wedge \bigwedge\limits_{j=1}^{n} \left(\lambda^+(b_j x) \wedge \mu^+(x b_j')\right) \end{array} \right\}$$

$$(\text{because } x + \textstyle\sum_{i=1}^{m} (a_i x)(x a_i') + z = \sum_{j=1}^{n} (b_j x)(x b_j') + z)$$

$$\geqslant [(\lambda^-(x)) \wedge \mu^-(x), (\lambda^+(x)) \wedge \mu^+(x)] = (\lambda \wedge \mu)(x),$$

which shows $\lambda \wedge \mu \leqslant \lambda \odot \mu$.

Conversely, let I and J be any left and right h-ideals of R, respectively. Then by Lemma 3.5, the characteristic functions χ_I and χ_J are interval valued fuzzy left h-ideal and interval valued fuzzy right h-ideal of R, respectively. Then by hypothesis and Lemma 3.2, we have

$$\chi_{I \cap J} = \chi_I \wedge \chi_J \subseteq \chi_I \odot \chi_J = \chi_{\overline{IJ}},$$

which implies $I \cap J \subseteq \overline{IJ}$. Lemma 2.6 completes the proof. □

Theorem 3.17. *The following conditions are equivalent for a hemiring R.*

(i) *R is both h-hemiregular and h-intra-hemiregular.*

(ii) $\lambda = \lambda \odot \lambda$ *for every interval valued fuzzy h-bi-ideal λ of R.*

(iii) $\lambda = \lambda \odot \lambda$ *for every interval valued fuzzy h-quasi-ideal λ of R.*

Proof. $(i) \Rightarrow (ii)$ Let λ be an interval valued fuzzy h-bi-ideal of R and $x \in R$. Then as R is both h-hemiregular and h-intra-hemiregular, so there exist elements $a_1, a_2, p_i, p_i', q_j, q_j', z \in R$ such that

$x + \sum_{j=1}^{n} (x a_2 q_j x)(x q_j' a_1 x) + \sum_{j=1}^{n} (x a_1 q_j x)(x q_j' a_2 x) + \sum_{i=1}^{m} (x a_1 p_i x)(x p_i' a_1 x)$ $+ \sum_{i=1}^{m} (x a_2 p_i x)(x p_i' a_2 x) + z = \sum_{i=1}^{m} (x a_2 p_i x)(x p_i' a_1 x) + \sum_{i=1}^{m} (x a_1 p_i x)(x p_i' a_2 x)$ $+ \sum_{j=1}^{n} (x a_1 q_j x)(x q_j' a_1 x) + \sum_{j=1}^{n} (x a_2 q_j x)(x q_j' a_2 x) + z$ (see Lemma 5.6 [13]).

Now for all expressions $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a_j' b_j' + z$, we have

$$(\lambda \odot \lambda)(x) = \sup \left\{ \begin{array}{l} \bigwedge_{i=1}^{m} \Big(\lambda^-(a_i) \wedge \lambda^-(b_i)\Big) \wedge \bigwedge_{j=1}^{n} \Big(\lambda^-(a'_j) \wedge \lambda^-(b'_j)\Big), \\ \bigwedge_{i=1}^{m} \Big(\lambda^+(a_i) \wedge \lambda^+(b_i)\Big) \wedge \bigwedge_{j=1}^{n} \Big(\lambda^+(a'_j) \wedge \lambda^+(b'_j)\Big) \end{array} \right\}$$

$$\geqslant \left\{ \begin{array}{l} \bigwedge_{j=1}^{n} \Big(\lambda^-(xa_2q_jx) \wedge \lambda^-(xq'_ja_1x) \wedge \lambda^-(xa_1q_jx) \wedge \lambda^-(xq'_ja_2x)\Big) \wedge \\ \bigwedge_{i=1}^{m} \Big(\lambda^-(xa_1p_ix) \wedge \lambda^-(xp'_ia_1x) \wedge \lambda^-(xa_2p_ix) \wedge \lambda^-(xp'_ia_2x)\Big), \\ \bigwedge_{j=1}^{n} \Big(\lambda^+(xa_2q_jx) \wedge \lambda^+(xq'_ja_1x) \wedge \lambda^+(xa_1q_jx) \wedge \lambda^+(xq'_ja_2x)\Big) \wedge \\ \bigwedge_{i=1}^{m} \Big(\lambda^+(xa_1p_ix) \wedge \lambda^+(xp'_ia_1x) \wedge \lambda^+(xa_2p_ix) \wedge \lambda^+(xp'_ia_2x)\Big) \end{array} \right\}$$

$$\geqslant [\lambda^-(x), \lambda^+(x)] = \lambda(x).$$

But as $\lambda \odot \lambda \leqslant \lambda$, so $\lambda \odot \lambda = \lambda$.

$(ii) \Rightarrow (iii)$ is straightforward by Theorem 3.9.

$(iii) \Rightarrow (i)$ Let Q be an h-quasi-ideal of R. Then by Lemma 3.5, χ_Q is an interval valued fuzzy h-quasi-ideal of R. Thus by hypothesis and Lemma 3.2, we have $\chi_Q = \chi_Q \odot \chi_Q = \chi_{\overline{Q^2}}$. This implies $Q = \overline{Q^2}$. Hence, by Lemma 2.7, R is both h-hemiregular and h-intra-hemiregular. □

Theorem 3.18. *The following conditions are equivalent for a hemiring R.*

- (i) *R is both h-hemiregular and h-intra-hemiregular.*
- (ii) *$\lambda \wedge \mu \leqslant \lambda \odot \mu$ for all interval valued fuzzy h-bi-ideals λ and μ of R.*
- (iii) *$\lambda \wedge g \leqslant \lambda \odot \mu$ for every interval valued fuzzy h-bi-ideal λ and every interval valued fuzzy h-quasi-ideals μ of R.*
- (iv) *$\lambda \wedge g \leqslant \lambda \odot \mu$ for every interval valued fuzzy h-quasi-ideal λ and every interval valued fuzzy h-bi-ideals μ of R.*
- (v) *$\lambda \wedge g \leqslant \lambda \odot \mu$ for all interval valued fuzzy h-quasi-ideals λ and μ of R.*

Proof. $(i) \Rightarrow (ii)$ Similarly as in the previous proof.

$(ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are straightforward.

$(v) \Rightarrow (i)$ Let λ be an interval valued fuzzy left h-ideals of R and μ be an interval valued fuzzy right h-ideal of R. Then λ and μ are interval valued fuzzy h-quasi-ideals of R. So by hypothesis $\lambda \wedge \mu \leqslant \lambda \odot \mu$. But $\lambda \wedge \mu \geqslant \lambda \odot \mu$ (see [11]). Thus $\lambda \wedge \mu = \lambda \odot \mu$. Hence by Theorem 3.12, R is h-hemiregular. On the other hand by hypothesis we also have $\lambda \wedge \mu \leqslant \lambda \odot \mu$. So by Theorem 3.16, R is h-intra-hemiregular. □

References

[1] **J. Ahsan**, *Semirings characterized by their fuzzy ideals*, J. Fuzzy Math. **6** (1998), 181 – 192.

[2] **J. Ahsan, K. Saifullah and M. F. Khan**, *Fuzzy semirings*, Fuzzy Sets and Syst. **60** (1993), 309 – 320.

[3] **S. Ghosh**, *Fuzzy k-ideals of semirings*, Fuzzy Sets Syst. **95** (1998), 103 – 108.

[4] **J. S. Golan**, *Semirings and their applications*, Kluwer Acad. Publ. 1999.

[5] **M. Henriksen**, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices **6** (1958), 321.

[6] **K. Iizuka**, *On Jacobson radical of a semiring*, Tohoku Math. J. **11** (1959), 409 – 421.

[7] **Y. B. Jun, M. A. Özürk and S. Z. Song**, *On fuzzy h-ideals in hemirings*, Inform. Sci. **162** (2004), 211 – 226.

[8] **D. R. La Torre**, *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen **12** (1965), 219 – 226.

[9] **X. Ma and J. Zhan**, *On fuzzy h-ideals of hemirings*, J. Syst. Sci. Complexity **20** (2007), 470 – 478.

[10] **M. Shabir, and T. Mahmood**, *Hemirings characterized by the properties of their fuzzy ideals with thresholds*, Quasigroups and Related Systems **18** (2010), 195 – 212.

[11] **G. Sun, Y. Yin and Y. Li**, *Interval valued fuzzy h-ideals of hemirings*, Int. Math. Forum **5** (2010), 545 – 556.

[12] **H. S. Vandiver**, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934), 914 – 920.

[13] **Y. Q. Yin and H. Li**, *The charatecrizations of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform. Sci. **178** (2008), 3451 – 3464.

[14] **L.A. Zadeh**, *Fuzzy sets*, Information and Control **8** (1965), 338 – 353.

[15] **J. Zhan and W. A. Dudek**, *Fuzzy h-ideals of hemirings*, Inform. Sci. **177** (2007), 876 – 886.

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M.Shabir
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mail: mshabirbhatti@yahoo.co.uk

T.Mahmood
Department of Mathematics, Faculty of Basic and Applied Sciences, International Islamic University, Islamabad, Pakistan
E-mail: tahirbakhat@yahoo.com