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# Roughness in ternary semigroups

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**Abstract.** In this paper we introduced the notions of rough left (right, lateral) ideal and rough prime ideals in ternary semigroup, and studied some properties of these ideals.

## 1. Introduction and preliminaries

The notion of rough set introduced by Z. Pawlak in his pioneering paper [7] and that of ternary semigroup by D. H. Lehmer in 1932 [6]. In this paper, we introduce lower and upper approximations with respect to the congruences on a ternary semigroup.

A *ternary semigroup* is an algebraic structure $(S, f)$ such that $S$ is a non-empty set and $f : S^3 \to S$ is a ternary operation satisfying the following associative law:

$$f(f(a,b,c),d,e) = f(a,f(b,c,d),e) = f(a,b,f(c,d,e)).$$

For simplicity we write $f(a,b,c)$ as $abc$. A non-empty subset $T$ of a ternary semigroup $S$ is said to be a *ternary subsemigroup* of $S$ if $TTT = T^3 \subseteq T$, that is $abc \in T$ for all $a,b,c \in T$.

By a *left* (*right, lateral* (*middle*)) *ideal* of a ternary semigroup $S$ we mean a non-empty subset $A$ of $S$ such that $SSA \subseteq A$ ($ASS \subseteq A$, $SAS \subseteq A$). By a two sided ideal, we mean a subset of $S$ which is both a left and a right ideal of $S$. If a non-empty subset of $S$ is a left, right and lateral ideal of $S$, then it is called an *ideal* of $S$.

A non-empty subset $A$ of a ternary semigroup $S$ is called a *bi-ideal* of $S$ if $AAA \subseteq A$ and $ASASA \subseteq A$ (cf. [2]).

For an equivalence relation $\rho$ on $S$ and a subset $A$ of $S$ we define two subsets

$$\rho_-(A) = \{x \in S : [x]_\rho \subseteq A\}, \qquad \rho^-(A) = \{x \in S : [x]_\rho \cap A \neq \emptyset\}$$

called $\rho$-*lower* and $\rho$-*upper approximations* of $A$, respectively.

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**Theorem 1.1.** *Let $\rho$ and $\lambda$ be equivalence relations on a certain set $S$. If $A$ and $B$ are non-empty subsets of $S$, then*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,
(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,
(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,
(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$,
(5) $A \subseteq B$ *implies* $\rho^-(A) \subseteq \rho^-(B)$,
(6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$,
(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,
(8) $\rho \subseteq \lambda$ *implies* $\rho_-(A) \supseteq \lambda_-(A)$,
(9) $\rho \subseteq \lambda$ *implies* $\rho^-(A) \subseteq \lambda^-(A)$.

The proof is analogous to the proof presented in [7].

## 2. Rough ideals in a ternary semigroup

**Definition 2.1.** A congruence $\rho$ on a ternary semigroup $S$ is called *stable* or *compatible* with the operation if $[a]_\rho [b]_\rho [c]_\rho = [abc]_\rho$for all $a, b, c \in S$.

The following example shows that there are congruences which are not stable.

**Example 2.2.** Let $S = \{a, b, c, d, e\}$ be a semigroup with respect to $*$ and $xyz = (x * y) * z$ for all $x, y, z \in S$. Where $*$ is defined by the table:

| $*$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $b$ | $d$ | $d$ | $d$ |
| $b$ | $b$ | $b$ | $d$ | $d$ | $d$ |
| $c$ | $d$ | $d$ | $c$ | $d$ | $c$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $e$ | $d$ | $d$ | $c$ | $d$ | $c$ |

Then $S$ is a ternary semigroup. Let $\rho$ be a congruence on $S$ such that the $\rho-$congruence classes are the subsets $\{a, b\}, \{c, d\}, \{e\}$.

Then clearly $\rho$ is not stable, since $[a]_\rho [c]_\rho [a]_\rho \neq [aca]_\rho$.

**Definition 2.3.** Let $\rho$ be a congruence on a ternary semigroup $S$. Then a non-empty subset $A$ of $S$ is called an upper (resp. lower) rough ternary subsemigroup of $S$ if $\rho^-(A)$ (resp. $\rho_-(A)$) is a ternary subsemigroup of $S$. $A$ is called an *upper* (resp. *lower*) *rough left* (*right, lateral, two sided*) *ideal* of $S$ if $\rho^-(A)$ (resp. $\rho_-(A)$) is a left (right, lateral, two sided) ideal of $S$.

**Lemma 2.4.** *Let $\rho$ be a congruence relation on a ternary semigroup $S$. If $A, B$ and $C$ are non-empty subsets of $S$, then $\rho^{-}(A)\rho^{-}(B)\rho^{-}(C) \subseteq \rho^{-}(ABC)$.*

*Proof.* If $d \in \rho^{-}(t)\rho^{-}(B)\rho^{-}(C)$, then $d = abc$ for some $a \in \rho^{-}(A)$, $b \in \rho^{-}(B)$, $c \in \rho^{-}(C)$. Thus there exist elements $x, y, z \in S$ such that $x \in [a]_\rho \cap A$, $y \in [b]_\rho \cap B$, $z \in [c]_\rho \cap C$. This implies that $x \in [a]_\rho$, $y \in [b]_\rho$, $z \in [c]_\rho$ and $x \in A$, $y \in B$, $z \in C$.

Since $\rho$ is a congruence on $S$, so $xyz \in [abc]_\rho$. Also $xyz \in ABC$. Thus we have $xyz \in [abc]_\rho \cap ABC$. This implies that $abc \in \rho^{-}(ABC)$. □

The following example shows that the equality in Lemma 2.4 does not hold in general.

**Example 2.5.** Consider the ternary semigroup $S$ and congruence relation $\rho$ as given in Example 2.2. Let $A = \{a, b\}$, $B = \{b, c\}$, $C = \{d\}$. Then $\rho^{-}(A) = \{a, b\}$, $\rho^{-}(B) = \{a, b, c, d\}$, $\rho^{-}(C) = \{c, d\}$. Thus $\rho^{-}(A)\rho^{-}(B)\rho^{-}(C) = \{d\}$ and $\rho^{-}(ABC) = \{x \in S : [x]_\rho \cap (ABC) = \{c, d\}$. Therefore, $\rho^{-}(ABC) \nsubseteq \rho^{-}(A)\rho^{-}(B)\rho^{-}(C)$. □

**Lemma 2.6.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$. If $A, B, C$ are non-empty subsets of $S$, then $\rho_{-}(A)\,\rho_{-}(B)\,\rho_{-}(C) \subseteq \rho_{-}(ABC)$.*

*Proof.* The proof is similar to the proof of Lemma 2.4. □

The following example shows that if $\rho$ is not a stable congruence, then Lemma 2.6 does not hold.

**Example 2.7.** Consider the ternary semigroup $S$ and congruence relation $\rho$ from Example 2.2. For $A = \{a, b\}$, $B = \{c, d\}$, $C = \{e\}$ we have $\rho_{-}(A) = \{a, b\}$, $\rho_{-}(B) = \{c, d\}$, $\rho_{-}(C) = \{e\}$, $\rho_{-}(A)\,\rho_{-}(B)\,\rho_{-}(C) = \{d\}$ and $\rho_{-}(ABC) = \rho_{-}(\{d\}) = \emptyset$. So, $\rho_{-}(A)\,\rho_{-}(B)\,\rho_{-}(C) \nsubseteq \rho_{-}(ABC)$. □

**Theorem 2.8.** *Let $\rho$ be a congruence on a ternary semigroup $S$.*

(1) *If $A$ is a ternary subsemigroup of $S$, then $A$ is an upper rough ternary subsemigroup of $S$.*

(2) *If $A$ is left (right, lateral, two sided) ideal of $S$, then $A$ is an upper rough left (right, lateral, two sided) ideal of $S$.*

*Proof.* (1) Let $A$ be a ternary subsemigroup of $S$. Then $A \subseteq \rho^{-}(A)$ and $\rho^{-}(A)\,\rho^{-}(A)\,\rho^{-}(A) \subseteq \rho^{-}(AAA) \subseteq \rho^{-}(A)$, by Lemma 2.4.

(2) Analogously as (1). □

The following examples show that $\rho^{-}(A)$ is a ternary subsemigroup (ideal) of $S$ even if $A$ is not a ternary subsemigroup (ideal) of $S$.

**Example 2.9.** Let $S = \{-i, 0, i\}$ be a ternary semigroup. Let $\rho$ be a congruence on $S$ such that the $\rho$-congruence classes are the subsets $\{-i, i\}, \{0\}$. Then $A = \{i\}$ is not a ternary subsemigroup of $S$, but $\rho^{-}(A) = \{i, -i\}$ is a ternary subsemigroup. Moreover, $B = \{0, i\}$ is not a left ideal of $S$, but $\rho^{-}(B) = \{-i, 0, i\}$ is a left ideal. □

**Theorem 2.10.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$ and let $A \subset S$ be such that $\rho_{-}(A)$ is non-empty.*

(1) *If $A$ is a ternary subsemigroup of $S$, then $\rho_{-}(A)$ also is a ternary subsemigroup of $S$.*

(2) *If $A$ is a left (right, lateral, two sided) ideal of $S$, then $\rho_{-}(A)$ also is a left (right, lateral, two sided) ideal of $S$.* □

The following example shows that if $\rho$ is a stable congruence on a ternary semigroup $S$, then $\rho_{-}(A)$ is a ternary subsemigroup of $S$ even if $A$ is not a ternary subsemigroup of $S$.

**Example 2.11.** Let $S = \{-i, 0, i\}$ and $\rho$ be as in the previous example. Then $A = \{0, i\}$ is not a ternary subsemigroup of $S$ but $\rho_{-}(A) = \{0\}$ is a ternary subsemigroup of $S$. □

**Definition 2.12.** A non-empty subset $A$ of a ternary semigroup $S$ is called a *$\rho-$upper (resp. $\rho-$lower) rough bi-ideal* of $S$, if $\rho^{-}(A)$ (resp. $\rho_{-}(A)$ is a bi-ideal of $S$.

**Theorem 2.13.** *Let $\rho$ be a congruence relation on a ternary semigroup $S$. If $A$ is a bi-ideal of $S$, then it is $\rho-$upper rough bi-ideal of $S$.*

*Proof.* The proof is similar to the proof of Theorem 2.8. □

**Example 2.14.** Let $S$ be as in Example 2.2 and let $\rho$ be a congruence relation on $S$ such the $\rho-$congruence classes are the subsets $\{a, b, d\}$,$\{c, e\}$. Then $A = \{a, b\}$ is not a bi-ideal of $S$ but $\rho^{-}(A) = \{a, b, d\}$ is a bi-ideal. □

**Theorem 2.15.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$ and $A$ a bi-ideal of $S$. Then $\rho_{-}(A)$ is a bi-ideal of $S$ if it is non-empty.*

*Proof.* The proof is similar to the proof of Theorem 2.8. □

**Theorem 2.16.** *Let $\rho$ be a congruence on a ternary semigroup $S$. If $A$, $B$ and $C$ are right, lateral and left ideal of $S$, respectively, then*

(1) $\rho^{-}(ABC) \subseteq \rho^{-}(A) \cap \rho^{-}(B) \cap \rho^{-}(C)$,

(2) $\rho_{-}(ABC) \subseteq \rho_{-}(A) \cap \rho_{-}(B) \cap \rho_{-}(C)$. □

## 3. Rough prime ideals in ternary semigroups

**Definition 3.1.** A left (right, lateral) bi-ideal $A$ of a ternary semigroup $S$ is a *prime* left (right, lateral) bi-ideal of $S$ if $xyz \in A$ implies $x \in A$ or $y \in A$ or $z \in A$ for all $x, y, z \in S$.

**Theorem 3.2.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$ and $A$ a prime ideal of $S$. Then $\rho_{-}(A) \neq \emptyset$ is a prime ideal of $S$.*

*Proof.* Suppose $A$ is a prime ideal of $S$. Then $\rho_{-}(A)$ is an ideal of $S$.

Suppose that $\rho_{-}(A)$ is not a prime ideal of $S$. Then there exist elements $x, y, z \in S$ such that $xyz \in \rho_{-}(A)$ but $x \notin \rho_{-}(A)$, $y \notin \rho_{-}(A)$, and $z \notin \rho_{-}(A)$. Then $[x]_\rho \nsubseteq A$, $[y]_\rho \nsubseteq A$, $[z]_\rho \nsubseteq A$. Thus there exist $x' \in [x]_\rho$ but $x' \notin A$, $y' \in [y]_\rho$ but $y' \notin A$, $z' \in [z]_\rho$ but $z' \notin A$. This implies that $x'y'z' \in [x]_\rho [y]_\rho [z]_\rho \subseteq A$. Since $A$ is a prime ideal of $S$, we have $x' \in A$ or $y' \in A$ or $z' \in A$. A contradiction. Hence $\rho_{-}(A)$ is a prime ideal of $S$. □

**Theorem 3.3.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$. If $A$ is a prime ideal of $S$, then $A$ is an upper rough prime ideal of $S$.*

*Proof.* Suppose $A$ is a prime ideal of $S$. Then $\rho^{-}(A)$ is an ideal of $S$. Let $xyz \in \rho^{-}(A)$ for $x, y, z \in S$. Then $[xyz]_\rho \cap A = [x]_\rho [y]_\rho [z]_\rho \cap A \neq \emptyset$. Thus there exist $x' \in [x]_\rho$, $y' \in [y]_\rho$, $z' \in [z]_\rho$ such that $x'y'z' \in A$. Since $A$ is a prime ideal of $S$, so $x' \in A$ or $y' \in A$ or $z' \in A$. This implies that $[x]_\rho \cap A \neq \emptyset$ or $[y]_\rho \cap A \neq \emptyset$ or $[z]_\rho \cap A \neq \emptyset$ and so $x \in \rho^{-}(A)$ or $y \in \rho^{-}(A)$ or $z \in \rho^{-}(A)$. Thus $\rho^{-}(A)$ is a prime ideal of $S$. Hence $A$ is an upper rough prime ideal of $S$. □

**Theorem 3.4.** *Let $\rho$ be a stable congruence on a ternary semigroup $S$.*

(1) *If $A$ a prime left (right, lateral) ideal of $S$, then $\rho_{-}(A)\,(\neq \emptyset)$ and $\rho^{-}(A)$ are prime left (right, lateral) ideal of $S$.*

(2) *If $A$ is a prime bi-ideal of $S$, then it is $\rho-$upper and $\rho-$lower rough prime bi-ideal of $S$.*

*Proof.* Proof is similar to the proofs of Theorem 3.2 and Theorem 3.3. □

## 4. Rough sets and idempotent congruences

A congruence relation $\rho$ on a ternary semigroup $S$ is called an *idempotent congruence* if the quotient ternary semigroup $S/\rho$ is an idempotent ternary semigroup.

**Definition 4.1.** A subset $P$ of a ternary semigroup $S$ is called *semiprime* if $a^3 \in P$ implies $a \in P$ for all $a \in S$.

**Theorem 4.2.** *Let $\rho$ be an idempotent stable congruence on a ternary semigroup $S$. If $A$ is non-empty subset of $S$, then $\rho^{-}(A)$ is semiprime.*

*Proof.* Let $a^3 \in \rho^{-}(A)$, since $\rho$ is idempotent congruence so $[a]_\rho \cap A = [a]_\rho [a]_\rho [a]_\rho \cap A = \left[a^3\right]_\rho \cap A \neq \emptyset$. Therefore $a \in \rho^{-}(A)$. So $\rho^{-}(A)$ is semiprime. □

**Theorem 4.3.** *Let $\rho$ be an idempotent congruence on a ternary semigroup $S$. If $A, B$ and $C$ are non-empty subsets of $S$, then*
(1) $\rho^{-}(A) \cap \rho^{-}(B) \cap \rho^{-}(C) \subseteq \rho^{-}(ABC)$,
(2) $\rho_{-}(A) \cap \rho_{-}(B) \cap \rho_{-}(C) \subseteq \rho_{-}(ABC)$.

*Proof.* (1) Let $d \in \rho^{-}(A) \cap \rho^{-}(B) \cap \rho^{-}(C)$. Then $d \in \rho^{-}(A)$, $d \in \rho^{-}(B)$ and $d \in \rho^{-}(C)$. Then there exist $a, b, c \in S$ such that $a \in [d]_\rho$, $a \in A$, $b \in [d]_\rho$, $b \in B$, $c \in [d]_\rho$, $c \in C$. Since $\rho$ is idempotent, so $abc \in [d]_\rho [d]_\rho [d]_\rho = [d]_\rho$ and since $abc \in ABC$. Therefore $abc \in [d]_\rho \cap ABC$. Thus $d \in \rho^{-}(ABC)$.

The proof of (2) is similar. □

**Theorem 4.4.** *Let $\rho$ be an idempotent congruence on a ternary semigroup $S$. If $A, B, C$ are right, lateral and left ideals of $S$, respectively, then*
(1) $\rho^{-}(A) \cap \rho^{-}(B) \cap \rho^{-}(C) = \rho^{-}(ABC)$,
(2) $\rho_{-}(A) \cap \rho_{-}(B) \cap \rho_{-}(C) = \rho_{-}(ABC)$.

*Proof.* Follows from Theorem 2.16 and Theorem 4.3. □

## 5. Rough ideals in a quotient ternary semigroup

Let $\rho$ be a congruence relation on a ternary semigroup $S$. The lower and upper approximations can be presented as

$$\underline{\rho}(A) = \left\{[x]_\rho \in S/\rho \colon [x]_\rho \subseteq A\right\}, \qquad \overline{\rho}(A) = \left\{[x]_\rho \in S/\rho \colon [x]_\rho \cap A \neq \emptyset\right\}.$$

**Theorem 5.1.** *Let $\rho$ be a congruence relation on a ternary semigroup $S$. If $A$ is a ternary subsemigroup of $S$, then $\overline{\rho}(A)$ and $\underline{\rho}(A)$ are ternary subsemigroups of $S/\rho$.*

*Proof.* Let $A$ be a ternary subsemigroup. Then $\emptyset \neq A \subseteq \overline{\rho}(A)$. For every $[x]_\rho, [y]_\rho, [z]_\rho \in \overline{\rho}(A)$ we have $[x]_\rho \cap A \neq \emptyset$, $[y]_\rho \cap A \neq \emptyset$, $[z]_\rho \cap A \neq \emptyset$. So, there are $a, b, c \in S$ such that $a \in [x]_\rho \cap A$, $b \in [y]_\rho \cap A$, $c \in [z]_\rho \cap A$. Thus $(a, x) \in \rho$, $(b, y) \in \rho$ $(c, z) \in \rho$. As $\rho$ is a congruence on $S$, we have $(abc, xyz) \in \rho$, this implies that $abc \in [xyz]_\rho$. Then $abc \in [xyz]_\rho \cap A$, so $[xyz]_\rho \in \overline{\rho}(A)$. Hence $\overline{\rho}(A)$ is a ternary subsemigroup of $S/\rho$.

For $\underline{\rho}(A)$ the proof is similar. □

**Theorem 5.2.** *Let $\rho$ be a congruence relation on a ternary semigroup $S$. If $A$ is a left* (*right, lateral, two sided*) *ideal of $S$, then $\overline{\rho}(A)$ is a left* (*right, lateral, two sided*) *ideal of $S/\rho$. Also $\underline{\rho}(A)$ is a left* (*right, lateral, two sided*) *ideal of $S/\rho$, if it is non-empty.*

*Proof.* Let $A$ be a left ideal of $S$. Let $[x]_\rho \in \overline{\rho}(A)$, $[y]_\rho$, $[z]_\rho \in S/\rho$, then $[x]_\rho \cap A \neq \emptyset$. Thus there exists $a \in [x]_\rho \cap A$. As $A$ is a left ideal of $S$, so $zya \in A$. Since $zya \in [z]_\rho [y]_\rho [x]_\rho$, we have $zya \in [z]_\rho [y]_\rho [x]_\rho \cap A$. But $[z]_\rho [y]_\rho [x]_\rho \subseteq [zyx]_\rho$, so $[zyx]_\rho \subseteq \overline{\rho}(A)$. Thus $\overline{\rho}(A)$ is a left ideal of $S/\rho$.

For $\underline{\rho}(A)$ the proof is analogous. □

**Theorem 5.3.** *Let $\rho$ be a congruence relation on a ternary semigroup $S$. If $A$ is a bi-ideal of $S$, then $\overline{\rho}(A)$ is a bi-ideal of $S/\rho$. Also $\underline{\rho}(A)$ is a bi-ideal of $S/\rho$, if it is non-empty.*

*Proof.* Let $A$ be a bi-ideal of $S$, so $\overline{\rho}(A)$ is a ternary subsemigroup of $S$. Let $[x]_\rho, [y]_\rho, [z]_\rho \in \overline{\rho}(A)$ and $[u]_\rho, [v]_\rho \in S/\rho$. Then there exist $a, b, c \in S$ such that $a \in [x]_\rho \cap A$, $b \in [y]_\rho \cap A$, $c \in [z]_\rho \cap A$. Let $u_1 \in [u]_\rho$, $v_1 \in [v]_\rho$. Since $A$ is bi-ideal so $au_1bv_1c \in A$. Now, as $a\rho x$, $u_1\rho u$, $b\rho y$, $v_1\rho v$, $c\rho z$, we have $(au_1bv_1c)\,\rho\,(xuyvz)$. This implies that $au_1bv_1c \in [xuyvz]_\rho$. Thus $[xuyvz]_\rho \cap A \neq \emptyset$. Hence $[xuyvz]_\rho \in \overline{\rho}(A)$. Thus $\overline{\rho}(A)$ is a bi-ideal of $S/\rho$.

For $\underline{\rho}(A)$ the proof is analogous. □

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