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Orthodox ternary semigroups

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Abstract. Using the notion of idempotent pairs we define orthodox ternary semigroups and generalize various properties of binary orthodox semigroups to the case of ternary semigroups. Also right strongly regular ternary semigroups are characterized.

1. Introduction

The study of algebras with one n-ary operation was initiated in 1904 by Kasner (see [4]). An n-ary analog of groups was studied by Dörnte [2], Post [7] and many others. A special case of such algebras are *ternary semigroups*, i.e., algebras with one *ternary operation* $T \times T \times T \longrightarrow T : (x, y, z) \longrightarrow [xyz]$ satisfying the associative law

$$[xy[uvw]] = [x[yuv]w] = [[xyu]vw].$$

Ternary semigroups have been studied by many authors (see for example [6, 5, 11]). The study of ideals and radicals in ternary semigroups was initiated in [11]. Ternary groups are studied in [1] and [5]. The concept of regular ternary semigroups was introduced in [10]. In [3] regular ternary semigroups was characterized by ideals. Santiago and Sri Bala [9] have investigated regular ternary semigroups.

In this paper we generalize the concept of orthodox semigroups to ternary case and characterize such ternary semigroups.

2. Preliminaries

For simplicity a ternary semigroup $(T, [\ \ ])$ will be denoted by T and the symbol of an inner ternary operation $[\ \ ]$ will be deleted, i.e., instead of $[[xyz]uw]$ or $[x[yzu]w]$ we will write $[xyzuw]$.

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An element x of a ternary semigroup T is called *regular* if there exists $y \in T$ such that $[xyx] = x$. A ternary semigroup in which each element is regular is called *regular*. An element $x \in T$ is *inverse* to $y \in T$ if $[xyx] = x$ and $[yxy] = y$. Clearly, if x is inverse to y, then y is inverse to x. Thus every regular element has an inverse. The set of all inverses of x in T is denoted by $I(x)$.

Definition 2.1. A pair (a, b) of elements of T is an *idempotent pair* if $[ab[abt]] = [abt]$ and $[[tab]ab] = [tab]$ for all $t \in T$. An idempotent pair (a, b) in which an element a is inverse to b is called a *natural idempotent pair*.

Recall that according to Post [7] two pairs (a, b) and (c, d) are *equivalent* if $[abt] = [cdt]$ and $[tab] = [tcd]$ for all $t \in T$. Equivalent pairs are denoted by $(a, b) \sim (c, d)$. If (a, b) is an idempotent pair, then $([aba], [bab])$ is a natural idempotent pair and $(a, b) \sim ([aba], [bab])$. The equivalence class containing (a, b) will be denoted by $\langle a, b\rangle$. By E_T we denote the set of all equivalence classes of idempotent pairs in T.

Definition 2.2. Two idempotent pairs (a, b) and (c, d) *commute* if $[abcdt] = [cdabt]$ and $[tabcd] = [tcdab]$ for all $t \in T$. A ternary semigroup in which any two idempotent pairs commute is called *strongly regular*.

Proposition 2.3. *In a strongly regular ternary semigroup every element has a unique inverse.*

Proof. Indeed, if x_1 and x_2 are two inverses of x, then $x_1 = [x_1xx_1] = [x_1xx_2xx_1] = [x_1xx_1xx_2] = [x_1xx_2] = [x_1xx_2xx_2] = [x_2xx_1xx_2] = [x_2xx_2] = x_2$ which completes the proof. □

The unique inverse of an element a is denoted by a^{-1}. In the case of ternary groups it coincides with the skew element (see [1] or [2]).

Definition 2.4. A non-empty subset A of a ternary semigroup T is said to be its *left ideal* if $[TTA] \subseteq A$, and a *right ideal* if $[ATT] \subseteq A$. The *left ideal generated by* A has the form $A_l = A \cup [TTA]$, the *right ideal generated by* A has the form $A_r = A \cup [ATT]$.

Definition 2.5. We say that a left (resp. right) ideal L (resp.R) of a ternary semigroup T has an *idempotent representation* if there exists an idempotent pair (a, b) in T such that $L = [Tab]$ (resp.$R = [abT]$). This representation is called *unique* if all idempotent pairs representing L (resp.R) are equivalent.

The following two facts are proved in [8] (see also [9]).

Lemma 2.6. *An element $a \in T$ is regular if and only if the principal left (resp. right) ideal of T generated by a has an idempotent representation.* □

Proposition 2.7. *In a strongly regular ternary semigroup every principal left (resp. right) ideal has a unique idempotent representation.* □

For $a, b \in T$, let $\mathrm{L}_{a,b}, \mathrm{R}_{a,b}$ denote the maps $\mathrm{L}_{a,b} : T \longrightarrow T : x \longrightarrow [abx]$ and $\mathrm{R}_{a,b} : x \longrightarrow [xab]$, $\forall x \in T$. On the set

$$M = \{m(a,b) \,|\, m(a,b) = (\mathrm{L}_{a,b}, \mathrm{R}_{a,b}), a, b \in T\}$$

we introduce a binary product by putting

$$m(a,b)m(c,d) = m([abc], d) = m(a, [bcd]).$$

Then M is a semigroup. This semigroup can be extended to the semigroup $S_T = T \cup M$ as follows. For $A, B \in S_T$ we define

$$AB = \begin{cases} m(a,b) & \text{if} \quad A = a,\ B = b \in T, \\ [abx] & \text{if} \quad A = m(a,b) \in S_T,\ B = x \in T, \\ [xab] & \text{if} \quad A = x \in T,\ B = m(a,b) \in S_T, \\ m([abc], d) & \text{if} \quad A = m(a,b),\ B = m(c,d) \in S_T. \end{cases}$$

The semigroup S_T is a covering semigroup in the sense of Post [7] (see also [1]). The product $[abc]$ in T is equal to (abc) in S_T. The element $m(a,b)$ in S_T is usually denoted by ab. This is called the *standard embedding of the ternary semigroup* T into S_T.

It is shown in [9] that T is a regular (strongly regular) ternary semigroup if and only if S_T is a regular (inverse) semigroup. There is a bijective correspondence between E_T and the set E_{S_T} of idempotents of S_T.

3. Orthodox ternary semigroups

Definition 3.1. An *orthodox ternary semigroup* T is a regular ternary semigroup in which for any two idempotents pairs (a,b) and (c,d) the pair $([abc], d)$ is also an idempotent pair.

For $a, b \in T$ denote by $W(a,b)$ the set of all equivalence classes $\langle u, v\rangle$ such that $(u,v) \in T\times T$ and $[abuvabt] = [abt]$, $[tabuvab] = [tab]$, $[uvabuvt] = [uvt]$, $[tuvabuv] = [tuv]$.

Clearly, $\langle u,v\rangle \in W(a,b)$ if and only if $\langle a,b\rangle \in W(u,v)$. If (a,b) is an idempotent pair, then $\langle a,b\rangle \in W(a,b)$. Moreover, $\langle u,v\rangle \in W(a,b)$ implies $\langle a,b\rangle \in W(u,v)$, but $\langle u,v\rangle \in W(a,b)$ may not imply $W(a,b)=W(u,v)$, in general (see Lemma 4.5).

In the sequel let T denote an orthodox ternary semigroup, unless otherwise specified.

Lemma 3.2. $\{\langle b',a'\rangle \mid a' \in I(a), b' \in I(b)\} \subseteq W(a,b)$ *for any* $a,b \in T$.

Proof. $[abb'a'abt] = [[aa'a]bb'a'abb'bt] = [a[a'abb'a'abb'b]t] = [aa'abb'bt] = [abt]$. Similarly can be proved other equalities. □

Lemma 3.3. *For any* $a,b \in T$, *if* $\langle x,y\rangle \in W(a,b)$, *then* $([abx],y)$ *and* $([xya],b)$ *are idempotent pairs.*

Proof. If $\langle x,y\rangle \in W(a,b)$, then $[abxyabxyt] = [ab[xyabxyt]] = [abxyt]$ for all $t \in T$. Also, $[tabxyabxy] = [tabxy]$. Therefore $([abx],y)$ is an idempotent pair. Similarly $([xya],b)$ is an idempotent pair. □

Corollary 3.4. *If* $a' \in I(a), b' \in I(b)$ *for some* $a,b \in T$, *then* $([abb'],a')$ *and* $([b'a'a],b)$ *are idempotent pairs.* □

Lemma 3.5. $[I(c)I(b)I(a)] \subset I([abc])$, *for all* $a,b,c \in T$.

Proof. By Corollary 3.4 $([b'a'a],b)$ is an idempotent pair. Since T is orthodox $([[b'a'a]bc],c')$ is an idempotent pair. Using Lemma 3.2 we obtain $[abc] = [abb'a'abc]$. Thus $[[abc][c'b'a'][abc]] = [[abb'a'abc][c'b'a'ab][cc'c]] = [ab[b'a'abcc'b'a'abcc'c]] = [[abb'a'abc]c'c] = [abc]$. Similarly $[c'b'a'abcc'b'a'] = [c'b'a']$. □

Theorem 3.6. *For a regular ternary semigroup* T *the following statements are equivalent.*

(1) T *is orthodox;*

(2) *For any* a,b,c,d *in* T, *if* $\langle x,y\rangle \in W(a,b)$ *and* $\langle u,v\rangle \in W(c,d)$, *then* $(\langle [uvx],y\rangle \in W([abc],d)$*;*

(3) *If* (a,b) *is an idempotent pair and* $\langle x,y\rangle \in W(a,b)$, *then* (x,y) *is an idempotent pair.*

Proof. (1) $\Rightarrow$ (2): Let $\langle x,y\rangle \in W(a,b)$ and $\langle u,v\rangle \in W(c,d)$. Then by Lemma 3.3 $([xya],b)$ and $([cdu],v)$ are idempotent pairs. So, $([xyabc],[duv])$ and $([cduvx],[yab])$ are idempotent pairs. Hence for all $t \in T$ we obtain $[abcduvxyabcdt] = [abxyabcduvxyabcduvcdt] = [ab[xyabcduvxyabcduvc]dt]$

$= [abxyabcduvcdt] = [abcdt]$. Analogously $[tabcduvxyabcd] = [tabcd]$ and $[uvxyabcduvxyt] = [uvxyt]$, $[tuvxyabcduvxy] = [tuvxy]$ for all $t \in T$. Hence $\langle [uvx], y\rangle \in W([abc], d)$.

(2) $\Rightarrow$ (3): Let (a, b) be an idempotent pair and let $\langle x, y\rangle \in W(a, b)$. By Lemma 3.3 $([xya], b)$ and $([abx], y)$ are idempotent pairs. Consequently $\langle [xya], b\rangle \in W([xya], b)$ and $\langle [abx], y\rangle \in W([abx], y)$. Then by hypothesis,

$$\langle [[abx]y[xya]], b\rangle \in W([[xya]b[abx]], y) = W([xyabx], y). \tag{1}$$

Since $\langle x, y\rangle \in W(a, b)$ we see that for all $t \in T$ $[xyt] = [xyabxyt] = [xyabxyabxyxyabxyabxyt] = [xyabxyxyabxyt] = [xyxyabxyt] = [xyxyt]$ because $([abx], y)$ is an idempotent pair. Similarly, $[txy] = [txyabxy] = [txyabxyabxyxyabxyabxy] = [txyabxyxyabxyabxy] = [txyxyabxyabxy] = [txyxyabxy] = [txyxy]$ and so (x, y) is an idempotent pair.

(3) $\Rightarrow$ (1): Let (a, b) and (c, d) be two idempotent pairs. Then for $\langle x, y\rangle \in W([abc], d)$ we have $[cdxyabcdxyabt] = [cd[xyabcdxya]bt] = [cdxyabt]$ and $[tcdxyabcdxyab] = [tcdxyab]$ for all $t \in T$. Thus $([cdx], [yab])$ is an idempotent pair. Next $[abcdcdxyababcdt] = [abcdxyabcdt] = [abcdt]$ and $[tabcdcdxyababcd] = [tabcdxyabcd] = [tabcd]$ for all $t \in T$. Also $[cdxyababcdcdxyabt] = [cdxyabcdxyabt] = [cdxyabt]$ and $[tcdxyababcdcdxyab] = [tcdxyabcdxyab] = [tcdxyab]$ for all $t \in T$. This means that $\langle [abc], d\rangle$ belongs to $W([cdx], [yab])$. Hence by our hypothesis $([abc], d)$ is an idempotent pair which proves that T is orthodox. □

Proposition 3.7. *Let (x, y) be an idempotent pair in T. Then $([a'xy], a)$ and $(a, [xya'])$ are idempotent pairs for all $a \in T$ and $a' \in I(a)$.*

Proof. Since T is orthodox $([aa'x], y)$ is an idempotent pair, so $[a'xyaa'xyat] = [[a'aa']xyaa'xyat] = [a'[aa'xyaa'xya]t] = [a'aa'xyat] = [a'xyat]$ $\forall t \in T$. Also $[ta'xyaa'xya] = [ta'xya]$ for all $t \in T$. Thus $([a'xy], a)$ is an idempotent pair. Similarly $(a, [xya'])$ is an idempotent pair. □

Theorem 3.8. *$I(a) = \{[xya'uv] \mid \langle x, y\rangle \in W(a', a), \langle u, v\rangle \in W(a, a')\}$ for all $a \in T$ and $a' \in I(a)$.*

Proof. Let $W(a) = \{[xya'uv] \mid \langle x, y\rangle \in W(a', a),\ \langle u, v\rangle \in W(a, a')\}$. Then for all $\langle x, y\rangle \in W(a', a)$ and $\langle u, v\rangle \in W(a, a')$ we obtain $[a[xya'uv]a] = [a[a'axya'aa'][aa'uvaa'a]] = [a[a'aa'][aa'a]] = a$. Also $[[xya'uv]a[xya'uv]] = [[xya'][aa'a][[a'aa']uv]] = [xya'uv]$. Thus $W(a) \subseteq I(a)$. Conversely, if $x \in I(a)$, then $\langle x, a\rangle \in W(a', a)$ and $\langle a, x\rangle \in W(a, a')$, by Lemma 3.2. Also $x = [xax] = [xaa'ax]$. Therefore $I(a) \subseteq W(a)$. Hence $I(a) = W(a)$. □

It is clear that if a ternary semigroup T is orthodox, then E_T is a band where the product is defined by $\langle a,b\rangle\ \langle c,d\rangle = \langle [abc],d\rangle = \langle a,[bcd]\rangle$. Thus E_T is a semilattice of rectangular bands, $E_T = \cup_{\alpha\in\Gamma} E_\alpha$. Then $\langle a,a'\rangle \in E_\alpha$ for any $a\in T$, and $a'\in I(a)$, so $W(a,a') = E_\alpha$, by Theorem3.6 (3). For any other $a^*\in I(a)$, $\langle a,a^*\rangle \in W(a,a') = E_\alpha$. Thus $W(a,a') = W(a,a^*)$ for all $a',a^*\in I(a)$. Similarly $W(a',a) = W(a^*,a)$. Hence we have the following lemma.

Lemma 3.9. *For any $a\in T$, if $a',a^*\in I(a)$, then $W(a,a^*) = W(a,a')$ and $W(a^*,a) = W(a',a)$.* □

Theorem 3.10. *A regular ternary semigroup T is orthodox if and only if for all $a,b\in T$ $I(a)\cap I(b)\neq\emptyset$ implies $I(a) = I(b)$.*

Proof. Let T be orthodox and $I(a)\cap I(b)\neq\emptyset$. If $x\in I(a)\cap I(b)$, then $a,b\in I(x)$. Hence, by Lemma 3.9, we have $W(x,a) = W(x,b)$ and $W(a,x) = W(b,x)$. Recalling the definition of $W(a)$ in the proof of Theorem 3.8, we see that $W(a) = \{[uvxpq] \mid \langle u,v\rangle \in W(x,a), \langle p,q\rangle \in W(a,x)\}$ and $W(b) = \{[stxwz] \mid \langle s,t\rangle \in W(x,b), \langle w,z\rangle \in W(b,x)\}$. Hence $W(a) = W(b)$. By Theorem 3.8, we have $I(a) = W(a) = W(b) = I(b)$.

Conversely assume that for all $a,b\in T$ from $I(a)\cap I(b)\neq\phi$ it follows $I(a) = I(b)$. Let $(a,a'),(b,b')$ be two idempotent pairs of T and let x be an inverse of $[aa'b]$ in T. Then for $y = [bxaa'b]$, $z = [b'bxaa']$ and $w = [xaa'bb']$ we have $[yzy] = [bxaa'bb'bxaa'bxaa'b] = [bxaa'b] = y$, $[zyz] = [b'bxaa'bxaa'bb'bxaa'] = [b'bxaa'] = z$, $[ywy] = [bxaa'bxaa'bb'bxaa'b] = [bxaa'b] = y$ and $[wyw] = [xaa'bb'bxaa'bxaa'bb'] = [xaa'bb'] = w$. Therefore $y\in I(z)\cap I(w)$ and so by hypothesis $I(z) = I(w)$. Thus $[aa'b]$ is in $I(z)$ because, $[[aa'b]z[aa'b]] = [aa'bb'bxaa'aa'b] = [aa'bxaa'b] = [aa'b]$ and $[z[aa'b]z] = [b'bxaa'aa'bb'bxaa'] = [b'b[xaa'bx]aa'] = [b'bxaa'] = z$. Hence by our hypothesis $[aa'b]\in I(w)$. Hence $[aa'bb'aa'bb't] = [[aa'bxaa'b]b'aa'bb't]$ $= [aa'b[xaa'bb']aa'bb't] = [[aa'bwaa'b]b't] = [aa'bb't]$ for all $t\in T$. Similarly $[taa'bb'aa'bb'] = [taa'bb']$. Therefore $([aa'b],b')$ is an idempotent pair which proves that T is orthodox. □

4. Right strongly regular ternary semigroups

Binary right inverse semigroups are characterized in [12]. Below we present similar characterizations for ternary semigroups.

Definition 4.1. A regular ternary semigroup T with zero is called a *right* (resp. *left*) *strongly regular ternary semigroup* if every principal left (resp. right) ideal of T has a unique idempotent representation.

Since T is regular, by Lemma 2.6 every principal left ideal has an idempotent representation $(a)_l = [TTa] = [Ta'a]$. Suppose there are two idempotent pairs (x, y) and (u, v) such that $(a)_l = [TTa] = [Txy] = [Tuv]$. If T is right strongly regular, then (x, y) and (u, v) are equivalent. In other words, $[xyt] = [uvt]$ and $[txy] = [tuv]$ $\forall t \in T$. It can easily be seen that if (a, b) is an idempotent pair in a ternary semigroup T, then $[Tab] = [TT[bab]]$ is a principal left ideal. Thus every principal left ideal of T can be taken to be $[Tab]$ for some idempotent pair (a, b).

Definition 4.2. A ternary semigroup T is called a *right zero ternary semigroup* if $[abc] = c$ for all $a, b, c \in T$.

Theorem 4.3. *The following statements on a regular ternary semigroup T are equivalent:*

(1) *If (a, b), (u, v) are two idempotent pairs of T, then $[abT] \cap [uvT] \neq \emptyset$ and $[abT] \cap [uvT] = [abuvT] = [uvabT]$.*
(2) *If $(a, b), (u, v)$ are idempotent pairs in T, then $[abuvabt] = [uvabt]$ and $[tabuvab] = [tuvab]$ for all $t \in T$.*
(3) *If $\langle u, v\rangle$ and $\langle u_1, v_1\rangle$ belong to $W(a, b)$, then $([uva], b)$ and $([u_1v_1a], b)$ are equivalent idempotent pairs.*
(4) *For any idempotent pair (a, b), $W(a, b)$ is a right zero semigroup.*
(5) *Let (a, b) be an idempotent pair and $x' \in I(x)$. Then $x \in [Tab]$ implies $x' \in [abT]$.*
(6) *T is a right strongly regular ternary semigroup.*

Proof. (1) $\Rightarrow$ (2): Let (a, b), (u, v) be two idempotent pairs of T. Suppose $[abT] \cap [uvT] = [abuvT] = [uvabT]$. Then $[abuvT] \subseteq [uvT]$ and so for any $t \in T$, $[abuvt] = [uvt_1]$ for some $t_1 \in T$. Multiplying on the left by u and v we get $[uvabuvt] = [uvuvt_1] = [uvt_1] = [abuvt]$. So, $[tuvabuv] = [tuvabuvuv] = [tabuvuv] = [tabuv]$. Therefore $[tuvabuv] = [tabuv]$. Since $[uvabT] \subseteq [abT]$. Similarly we get $[abuvabt] = [uvabt]$ and $[tabuvab] = [tuvab]$ for every $t \in T$.

(2) $\Rightarrow$ (3): Suppose (2) holds for all idempotent pairs. If $\langle u, v\rangle \in W(a, b)$ and $\langle u_1, v_1\rangle \in W(a, b)$, then $([uva], b)$ and $([u_1v_1a], b)$ are idempotent pairs. So, $[u_1v_1abt] = [u_1v_1abu_1v_1abt] = [u_1v_1abuvabu_1v_1abt] = [uvabu_1v_1abt] = [uvabt]$ for all $t \in T$. Therefore $[u_1v_1abt] = [uvabt]$ and $[tu_1v_1ab] = [tu_1v_1abu_1v_1ab] = [tu_1v_1abuvabu_1v_1ab] = [tu_1v_1abuvabu_1v_1abab]$

$= [tuvabu_1v_1abab] = [tuvab]$. Hence $[tu_1v_1ab] = [tuvab]$. Thus $([uva], b)$ and $([u_1v_1a], b)$ are equivalent idempotent pairs. This proves (3).

(3) $\Rightarrow$ (4): Let (a, b) be an idempotent pair and $\langle u, v\rangle \in W(a, b)$. Since $\langle a, b\rangle \in W(a, b)$ by hypothesis, $([uva], b) \sim ([aba], b) \sim (a, b)$. Therefore $[uvt] = [uvabuvt] = [abuvt]$ and so, $[uvuvt] = [abuvabuvt] = [abuvt] = [uvt]$. Thus $[uvuvt] = [uvt]$ and $[tuvuv] = [tabuvabuv] = [tabuv] = [tuv]$ for every $t \in T$. Hence (u, v) is an idempotent pair. Let $\langle u_1, v_1\rangle \in W(a, b)$. By hypothesis, $([uva], b)$ and $([u_1v_1a], b)$ are equivalent idempotent pairs. Thus for all $t \in T$ we have $[uvu_1v_1abuvu_1v_1t] = [uvuvabuvu_1v_1t] = [uvabuvu_1v_1t]$ $= [uvu_1v_1t]$. Similarly, $[tuvu_1v_1abuvu_1v_1] = [tuvu_1v_1]$. Also for all $t \in T$, $[abuvu_1v_1abt] = [abuvuvabt] = [abuvabt] = [abt]$. Similarly, $[tabuvu_1v_1ab] = [tab]$. Therefore $\langle [uvu_1], v_1\rangle \in W(a, b)$. Hence for all $t \in T$ will be $[uvu_1v_1t]$ $= [uvu_1v_1abu_1v_1t] = [uvu_1v_1abuvu_1v_1abu_1v_1t] = [uvuvabuvuvabu_1v_1t] = [uvabu_1v_1t] = [u_1v_1abu_1v_1t] = [u_1v_1t]$. Similarly, $[tuvu_1v_1] = [tu_1v_1]$. Therefore $([uvu_1], v_1) \sim (u_1, v_1)$. Hence $W(a, b)$ is a right zero semigroup.

(4) $\Rightarrow$ (3): Let $\langle u, v\rangle$ and $\langle u_1, v_1\rangle$ be in $W(a, b)$. Since $\langle a, b\rangle \in W(a, b)$ and $W(a, b)$ is a right zero semigroup, $([uva], b) \sim (a, b) \sim ([u_1v_1a], b)$.

(4) $\Rightarrow$ (5): Let (a, b) be an idempotent pair and $x \in [Tab]$. Then $x = [xab]\ \forall x \in T$. Thus for any $x' \in I(x)$, $[x[abx']x] = [[xab]x'x] = [xx'x] = x$ and $[[abx']x[abx']] = [abx'[xab]x'] = [abx'xx'] = [abx']$. Therefore x' and $[abx']$ are inverses of x. Next, we prove that $\langle [abx'], x\rangle \in W(x', x)$. Hence $\forall t \in T$, $[abx'xx'xabx'xt] = [abx'[xab]x'xt] = [abx'xt]$. Similarly $[tabx'xx'xabx'x] = [tabx'x]$ and $[x'xabx'xx'xt] = [x'[xab]x'xt] = [x'xx'xt] = [x'xt]$. Similarly $[tx'xabx'xx'x] = [tx'x]$. Therefore $\langle [abx'], x\rangle \in W(x', x)$. Hence, $([abx'], x)$ is equivalent to (x', x). Therefore $\forall t \in T$, $[abx'xt] = [x'xt]$ and $[tabx'x] = [tx'x]$. For $t = x'$ we have $x' = [x'xx'] = [abx'xx'] = [abx']$, which means that $x' \in [abT]$.

(5) $\Rightarrow$ (6): Since T is regular, by Lemma 2.6 every principal left ideal is of the form $[Tab]$ where (a, b) is an idempotent pair. Suppose (c, d) is an idempotent pair such that $[Tab] = [Tcd]$. Since $(a, b), (c, d)$ are idempotent pairs $([aba], [bab])$ are inverses of one another and $([cdc], [dcd])$ are inverses of one another. Thus $[bab] \in [Tab] = [Tcd]$ and $[dcd] \in [Tcd] = [Tab]$. Therefore $[bab] = [babcd]$ and $[dcd] = [[dcd]ab]$. By hypothesis, $[aba] \in [cdT]$ and $[cdc] \in [abT]$. Therefore $[aba] = [cd[aba]]$ and $[cdc] = [ab[cdc]]$. Thus for all $t \in T$ we have $[abt] = [a[bab]t] = [a[babcd]t] = [abcdt] = [[abcdc]dt] = [cdcdt] = [cdt]$ and $[tab] = [t[aba]b] = [tcdabab] = [tcdab] = [tcd]$. Hence $(a, b) \sim (c, d)$ and T is right inverse.

(6) $\Rightarrow$ (1): Let (a, b) be an idempotent pair and $\langle u, v\rangle \in W(a, b)$. Then

$([uva], b)$ is an idempotent pair and $[Tab] = [Tabuvab] \subseteq [Tuvab] \subset [Tab]$. Therefore $[Tab] = [Tuvab]$. Let $\langle u_1, v_1\rangle \in W(a, b)$. Then $[Tuvab] = [Tab] = [Tu_1v_1ab]$. Thus $([uva], b)$, $([u_1v_1a], b)$ are equivalent idempotent pairs. Thus (6) implies (3). We now show that (1) holds. Let (a, b), (u, v) be idempotent pairs of T. We claim that $([abu], v)$ is an idempotent pair. Let $\langle x, y\rangle \in W([abu], v)$. Then it can easily be seen that $\langle [uvx], y\rangle$ and $\langle [xya], b\rangle$ are both in $W([abu], v)$. By hypothesis (3) we have $([xyababu], v) \sim ([uvxyabu], v)$. Therefore $[xyt] = [xyabuvxyt] = [uvxyabuvxyt] = [uvxyt]$ and so $[xyt] = [xyab[uvxyt]] = [xyabxyt]$. Consequently, $[xyabxyabt] = [xyabt]$. Similarly $[txyabxyab] = [txyab]$. Hence $([xya], b)$ is an idempotent pair. As $\langle [abu], v\rangle \in W([xya], b)$, hence by (4) $([abu], v)$ is an idempotent pair. Next, $[uvabuvuvabuvt] = [uvabuvabuvt] = [uvabuvt]$. Similarly $[tuvabuvuvabuv] = [tuvabuv]$. Hence $([uva], [buv])$ is an idempotent pair. Thus $[Tabuv] = [Tabuvabuv] \subseteq [Tuvabuv] \subset [Tabuv]$. Therefore $[Tabuv] = [Tuvabuv]$. By hypothesis $([abu], v) \sim ([uva], [buv])$ and so for all $t \in T$,$[abuvt] = [uvabuvt]$ and $[tabuv] = [tuvabuv]$. Hence (2) holds. Thus $[abuvT] \subseteq [uvT]$ and $[abuvT] \subset [abT]$. Therefore $[abuvT] \subseteq [abT] \cap [uvT]$ and so $[abt] \cap [uvT] \neq \emptyset$. Also $[abT] \cap [uvT] \subseteq [abuvT]$. Hence $[abT] \cap [uvT] = [abuvT]$. Similarly $[abT] \cap [uvT] = [uvabT]$. Hence (1). □

As a consequence of Theorem 4.3 (2) we obtain

Corollary 4.4. *A right strongly regular ternary semigroup is orthodox.* □

Lemma 4.5. *In a right strongly regular ternary semigroup for any idempotent pair (p, q) from $\langle u, v\rangle \in W(p, q)$ it follows $W(u, v) = W(p, q)$.*

Proof. By Theorem 4.3, $W(p, q)$ is a right zero semigroup. For any $\langle u, v\rangle$ from $W(p, q)$ we have $[uvuvt] = [uvt] = [pquvt]$, $[uvpqt] = [pqt]$, $[tuvuv] = [tuv] = [tpquv]$ and $[tuvpq] = [tpq]$. Suppose $\langle x, y\rangle \in W(u, v)$. Then $[xyxyt] = [xyt] = [uvxyt]$ and $[xyuvt] = [uvt]$. Similarly $[txyxy] = [txy] = [tuvxy]$ and $[txyuv] = [tuv], \forall t \in T$. Therefore $[xypq[xyt]] = [xypq[uvxyt]] = [xyuvxyt] = [xyt]$, $[txypqxy] = [txypquvxy] = [txyuvxy] = [txy]$ and $[pqxypqt] = [pqxyuvpqt] = [pquvpqt] = [pqt]$. Similarly we obtain $[tpqxypq] = [tp[qxyuvp]pq] = [tpq]$ and so $\langle x, y\rangle \in W(p, q)$. Thus $W(u, v) \subseteq W(p, q)$. Analogously $W(p, q) \subseteq W(u, v)$. Hence $W(u, v) = W(p, q)$. □

Lemma 4.6. *Let T be a right strongly regular ternary semigroup. Then $(b', a') \subseteq W(a, b)$ for any $a, b \in T$ $a' \in I(a)$, $b' \in I(b)$.* □

Lemma 4.7. *Let T be a right strongly regular ternary semigroups. Then $[I(b)I(a)T] = \{[uvt] \mid \langle u, v\rangle \in W(a, b), t \in T\}$ for any $a, b \in T$.*

Proof. By Lemma 4.6, $[I(b)I(a)T] \subseteq \{[uvt] \mid \langle u,v\rangle \in W(a,b), t \in T\}$. If $b' \in I(b)$, $a' \in I(a)$ and $\langle u,v\rangle \in W(a,b)$, then $([b'a'a],b)$ and $([uva],b)$ are equivalent idempotent pairs by Theorem 4.3 (3). Hence for $\langle u,v\rangle \in W(a,b)$ we have $[uvt] = [uvabuvt] = [b'a'abuvt] \in [I(b)I(a)T]$. □

Similar results are valid for left strongly regular ternary semigroups.

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