Affine–regular hexagons in the parallelogram space

Vladimir Volenec, Zdenka Kolar–Begović and Ružica Kolar–Šuper

Abstract The concept of the affine–regular hexagon, by means of six parallelograms, is defined and investigated in any parallelogram space and geometrical interpretation in the affine plane is also given.

1. Introduction

Let Q be a given set whose elements are said to be *points*. Let a quaternary relation $\mathrm{Par} \subset Q^4$ is defined on the set Q. We shall say that the points a, b, c, d form a *parallelogram* and we shall write $\mathrm{Par}(a, b, c, d)$ in the case when $(a, b, c, d) \in Par$.

The pair (Q,Par) is called a *parallelogram space* if the quaternary relation $\mathrm{Par} \subset Q^4$ has the following properties:

(P1) For any three points a, b, c there is one and only one point d so that $\mathrm{Par}(a, b, c, d)$.

(P2) If (e, f, g, h) is any cyclic permutation of (a, b, c, d) or of (d, c, b, a) then $\mathrm{Par}(a, b, c, d)$ implies $\mathrm{Par}(e, f, g, h)$.

(P3) From $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(c, d, e, f)$ it follows $\mathrm{Par}(a, b, f, e,)$.

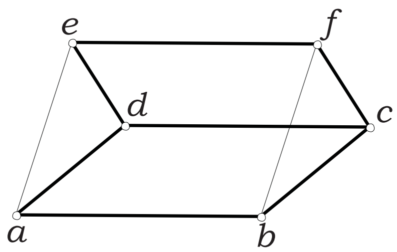

Figure 1.

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The parallelograms (including degenerated parallelograms) form a parallelogram space in the affine space of any dimension. The property (P3) in the affine plane (space) is illustrated in the Figure 1. Some other properties will be illustrated in the same plane.

2. Midpoint in the parallelogram space

In the parallelogram space the midpoint of the pair of points can be defined. We shall say that b is the *midpoint* of the pair $\{a, c\}$ and we shall write $M(a, b, c)$ if the statement $\mathrm{Par}(a, b, c, b)$ (Figure 2) is valid. From $M(a, b, c)$ obviously follows $M(c, b, a)$ and for any two points a and b there is the unique point c so that the statement $M(a, b, c)$ is valid. There are the examples of the parallelogram space, in which every pair of points does not have to have the unique midpoint.

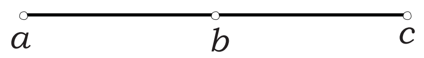

Figure 2.

Theorem 2.1. *Each statement of the three statements $Par(o, a_1, a_2, a_3)$, $Par(o, a_2, a_3, a_4)$ and $M(a_1, o, a_4)$ is the consequence of the remaining two statements (Figure 3).*

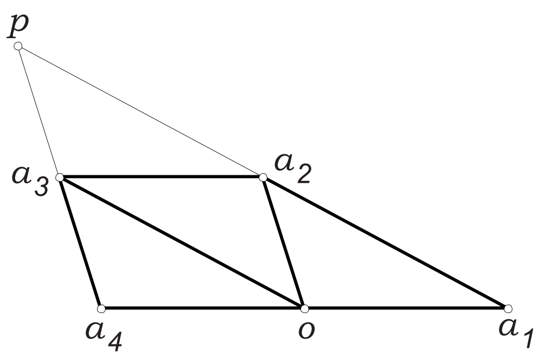

Figure 3.

Proof. The property (P3) by means of the property (P2) gives the implications

$$Par(o, a_1, a_2, a_3), Par(a_2, a_3, a_4, o) \Rightarrow Par(o, a_1, o, a_4)$$
$$Par(a_2, a_3, o, a_1), Par(o, a_1, o, a_4) \Rightarrow Par(a_2, a_3, a_4, o)$$
$$Par(a_2, a_3, a_4, o), Par(a_4, o, a_1, o) \Rightarrow Par(a_2, a_3, o, a_1).$$

But, the statements Par(o, a_1, o, a_4) and Par(a_4, o, a_1, o) are equivalent to the statement $M(a_1, o, a_4)$. □

3. Affine regular hexagon in the parallelogram space

We shall say that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is the *affine–regular hexagon* with the vertices $a_1, a_2, a_3, a_4, a_5, a_6$ and the center o and we write ARH$_o(a_1, a_2, a_3, a_4, a_5, a_6)$ if the statements

$$Par(o, a_{i-1}, a_i, a_{i+1}),\ (i = 1, 2, 3, 4, 5, 6)$$

are valid, where the indexes are taken modulo 6 from the set $\{1, 2, 3, 4, 5, 6\}$ (Figure 4).

The vertices a_i and a_{i+3} are said to be *opposite vertices* of the considered affine-regular hexagon, and the vertices a_i, a_{i+1}, a_{i+2} are said to be *adjacent vertices.*

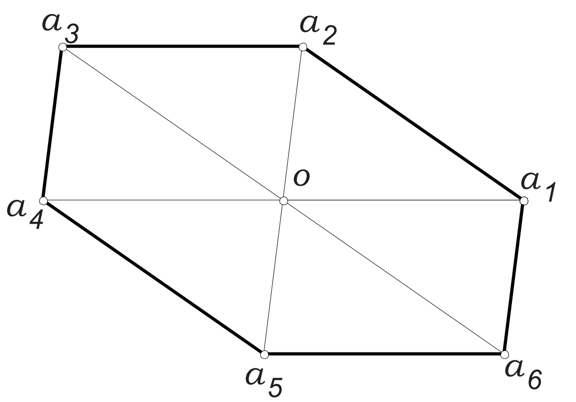

Figure 4.

Corollary 3.1. *The statement* $ARH_o(a_1, a_2, a_3, a_4, a_5, a_6)$ *imply the statement* $ARH_o(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6})$ *where* $(i_1, i_2, i_3, i_4, i_5, i_6)$ *is any cyclic permutation of* $(1, 2, 3, 4, 5, 6)$ *or of* $(6, 5, 4, 3, 2, 1)$. □

Theorem 3.2. *If the statement* $ARH_o(a_1, a_2, a_3, a_4, a_5, a_6)$ *is valid then for any* $i \in \{1, 2, 3, 4, 5, 6\}$ *the statements* $Par(a_i, a_{i+1}, a_{i+3}, a_{i+4})$ *and* $M(a_i,\ o, a_{i+3})$ *are valid where indexes are taken modulo* 6.

Proof. The second statement for the case $i = 1$ is proved by Theorem 2.1 The first statement follows according to (P3) by the implication

$$Par(a_1, a_2, a_3, o), Par(a_3, o, a_5, a_4) \Rightarrow Par(a_1, a_2, a_4, a_5).$$

For the other indexes it is enough to apply cyclic permutations and Corollary 3.1. □

Theorem 3.3. *Affine–regular hexagon is uniquely determined by its center and by any two of its vertices which are not opposite or by any of its three adjacent vertices.*

Proof. Firstly let us prove the last statement. Let the vertices a_1, a_2, a_3 be given. Then according to (P1) there are the points o, a_4, a_5, a_6 so that the following statements

$$Par(a_1, a_2, a_3, o), Par(o, a_2, a_3, a_4), Par(o, a_3, a_4, a_5), Par(o, a_4, a_5, a_6) \quad (1)$$

are valid. From the two first statements (1) according to Theorem 2.1 it follows $M(a_1, o, a_4)$, which together with the fourth statement (1), again according to Theorem 2.1, gives $\mathrm{Par}(o, a_5, a_6, a_1)$. Analogously (increasing all indexes for one) the statement $Par(o, a_6,\ a_1, a_2)$ could be proved, so we get the statement $\mathrm{ARH}_o(a_1, a_2, a_3, a_4,\ a_5, a_6)$. Yet, it is necessary to prove that this affine–regular hexagon is uniquely determined by its center and for example by the vertices a_1, a_2 or by the vertices a_1, a_3. These proofs are reduced to the considered case if the vertex a_3 respectively a_2 is determined so that the statement $\mathrm{Par}(o, a_1, a_2, a_3)$ is valid and then we can conclude as in the previous case. □

The points o, a_1, a_2, a_3, a_4 with the properties from Theorem 2.1 determine figure, which will be denoted by the symbol $\mathrm{HARH}_o(a_1, a_2, a_3, a_4)$, "half" of the affine regular hexagon with the center o (Figure 3).

Corollary 3.4. *If the statement* $HARH_o(a_1, a_2, a_3, a_4)$ *is valid, then there are the uniquely determined points* a_5 *and* a_6 *so that the statement* $ARH_o(a_1, a_2, a_3, a_4, a_5, a_6)$ *holds.*

Theorem 3.5. *If the statement*

$$HARH_o(a_1, a_2, a_3, a_4) \tag{2}$$

is valid, then there is a point p such that the statements

$$M(a_1, a_2, p),\ M(a_4, a_3, p) \tag{3}$$

are valid. Conversely, if the statements (3) *are valid, then there is a point o such that the statement* (2) *is valid* (*Figure* 3).

Proof. Let the statement (2) be valid and let p be the point so that the first statement (3) is valid. Then according to (P3) we have the implications

$$Par(p, a_2, a_1, a_2), Par(a_1, a_2, a_3, o) \Rightarrow Par(p, a_2, o, a_3)$$
$$Par(a_4, a_3, a_2, o), Par(a_2, o, a_3, p) \Rightarrow Par(a_4, a_3, p, a_3),$$

so the second statement (3) is valid too. Conversely, let the statements (3) be valid and let o be such a point that the statement $\mathrm{Par}(a_2, p, a_3, o)$ holds. Then we get the implications

$$Par(o, a_3, p, a_2), Par(p, a_2, a_1, a_2) \Rightarrow Par(o, a_3, a_2, a_1)$$
$$Par(o, a_2, p, a_3), Par(p, a_3, a_4, a_3) \Rightarrow Par(o, a_2, a_3, a_4),$$

so the statement (2) is valid. □

Corollary 3.6. *If the statement* (2) *is valid then from the first statement* (3) *the second statement* (3) *follows.* □

Theorem 3.7. *Let* $n \in \mathbb{N}$, $n \geqslant 3$. *If the statements* $HARH_{c_{12}}(b_1, a_1, a_2, b_2)$, $HARH_{c_{23}}(b_2, a_2, a_3, b_3), \ldots, HARH_{c_{n-1,n}}(b_{n-1}, a_{n-1}, a_n, b_n)$ *are valid then there is a point* c_{n1} *so that the statement* $HARH_{c_{n1}}(b_n, a_n, a_1, b_1)$ *is valid too.* (*The case for* $n = 5$ *is illustrated in the Figure* 5).

Proof. From $\mathrm{HARH}_{c_{12}}(b_1, a_1, a_2, b_2)$ according to Theorem 3.5 it follows that there is a point o so that the statements $M(b_1, a_1, o)$ and $M(b_2, a_2, o)$ are valid, and then from $\mathrm{HARH}_{c_{23}}(b_2, a_2, a_3, b_3)$ follows the statement $M(b_3, a_3, o)$. Let Corollary 3.6 be applied again and after $(n-1)$–th application of this corollary from $\mathrm{HARH}_{c_{n-1,n}}(b_{n-1}, a_{n-1}, a_n, b_n)$ and $M(b_{n-1}, a_{n-1}, o)$ it follows $M(b_n, a_n, o)$. Finally, from the statements $M(b_n, a_n, o)$, $M(b_1, a_1, o)$ owing to Theorem 3.5 it follows that there is a point c_{n1} so that $\mathrm{HARH}_{c_{n1}}(b_n, a_n, a_1, b_1)$ is valid. □

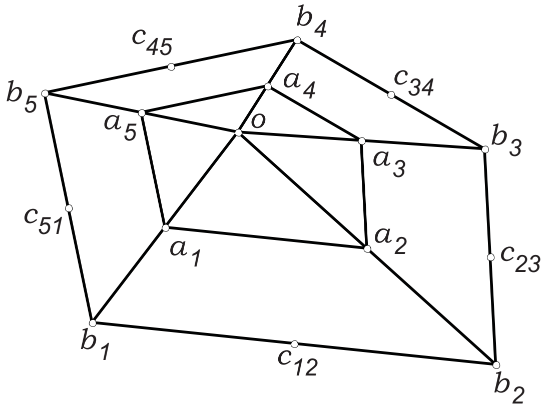

Figure 5.

From Theorem 3.7 by means of Corollary 3.4 we get:

Corollary 3.8. *Let* $n \in \mathbb{N}$, $n \geqslant 3$. *If the statements* $ARH_{c_{12}}(b_1, a_1, a_2, b_2, d_{21}, d_{12})$, $ARH_{c_{23}}(b_2, a_2, a_3, b_3, d_{32}, d_{23}), \ldots,$ $ARH_{c_{n-1,n}}(b_{n-1}, a_{n-1}, a_n,\ b_n, d_{n,n-1}, d_{n-1,n})$ *are valid, then there are the points* c_{n1}, d_{n1}, d_{1n} *so that the statement* $ARH_{c_{n1}}(b_n, a_n, a_1,\ b_1, d_{1n},\ d_{n1})$ *is valid too.* □

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