A study of anti-fuzzy quasi-ideals in ordered semigroups

Anwar Zeb and Asghar Khan

Abstract. In this paper, we introduce the concept of anti-fuzzy quasi-ideals in ordered semigroups and investigate the quasi-ideals of ordered semigroups in terms of anti-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of anti-fuzzy quasi-ideals and semiprime anti-fuzzy quasi-ideals.

1. Introduction


In this paper, we introduce the concept of anti-fuzzy quasi-ideals in ordered semigroups and investigate the basic properties of quasi-ideals of ordered semigroups in terms of anti-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of anti-fuzzy quasi-ideals. We define semiprime anti-fuzzy quasi-ideals and characterize completely regular ordered semigroups in terms of semiprime anti-fuzzy quasi-ideals.

2010 MSC: 06F05, 06D72, 08A72

Keywords: Fuzzy subset, anti-fuzzy quasi-ideal, simple semigroup, completely regular ordered semigroup.
2. Some basic definitions and results

By an ordered semigroup \((p\text{-}semigroup}\) we mean a structure \((S, \cdot, \leq)\) in which

\begin{enumerate}[(OS1)]
  \item \((S, \cdot)\) is a semigroup,
  \item \((S, \leq)\) is a poset,
  \item \((\forall a, b, x \in S)(a \leq b \implies ax \leq bx\) and \(xa \leq xb)\).
\end{enumerate}

Throughout this paper \(S\) will denote an ordered semigroup unless otherwise specified.

For \(A, B \subseteq S\), we denote \((A] := \{t \in S| t \leq h \text{ for some } h \in A\}\) and \(AB := \{ab | a \in A, b \in B\}\). Then \(A \subseteq (A], \ (A)[B] \subseteq (AB], \ ((A]) = (A]\) and \(((A)[B]) \subseteq (AB]\).

A non-empty subset \(A\) of \(S\) is called a right (resp. left) ideal of \(S\) if:

\begin{enumerate}[(1)]
  \item \(A \subseteq (A]\) (resp. \(SA \subseteq A]\),
  \item \(a \in A\) and \(S \ni b \leq a\) imply \(b \in A\).
\end{enumerate}

If \(A\) is both a right and a left ideal of \(S\), then it is called an ideal of \(S\).

A non-empty subset \(Q\) of \(S\) is called a quasi-ideal of \(S\) if:

\begin{enumerate}[(1)]
  \item \((QS) \cap (SQ) \subseteq Q\),
  \item \(a \in Q\) and \(S \ni b \leq a\) imply \(b \in Q\).
\end{enumerate}

A subsemigroup \(B\) of \(S\) is called a bi-ideal of \(S\) if:

\begin{enumerate}[(1)]
  \item \(BSB \subseteq B\),
  \item \(a \in B\) and \(S \ni b \leq a\) imply \(b \in B\).
\end{enumerate}

A fuzzy subset \(f\) of \(S\) is called a fuzzy left (resp. right) ideal of \(S\) if:

\begin{enumerate}[(1)]
  \item \(x \leq y \implies f(x) \geq f(y)\),
  \item \(f(xy) \geq f(y)\) (resp. \(f(xy) \geq f(x)\)) for all \(x, y \in S\).
\end{enumerate}

A fuzzy subsemigroup \(f\) of \(S\) is called a fuzzy bi-ideal of \(S\) if:

\begin{enumerate}[(1)]
  \item \(x \leq y \implies f(x) \geq f(y)\),
  \item \(f(xyz) \geq \min\{f(x), f(z)\}\) for all \(x, y \in S\).
\end{enumerate}

For a non-empty family of fuzzy subsets \(\{f_i\}_{i \in I}\) of \(S\), the fuzzy subsets \(\bigwedge_{i \in I} f_i\) and \(\bigvee_{i \in I} f_i\) of \(S\) are defined as follows:

\[
\left(\bigwedge_{i \in I} f_i\right)(x) := \inf_{i \in I}\{f_i(x)\}, \quad \left(\bigvee_{i \in I} f_i\right)(x) := \sup_{i \in I}\{f_i(x)\}.
\]
For any two fuzzy subsets $f$ and $g$ of $S$ we put

$$(f \circ g)(x) := \begin{cases} \bigvee_{(y,z) \in A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x := \{(y, z) \in S \times S \mid x \leq yz\}$.

A fuzzy subset $f$ of $S$ is called a fuzzy quasi-ideal of $S$ if:

1. $x \leq y \implies f(x) \geq f(y)$,
2. $(f \circ 1) \land (1 \circ f) \preceq f$,

where $f \preceq g$ means that $f(x) \leq g(x)$ for all $x \in S$.

A fuzzy subset $f$ of $S$ is called an anti-fuzzy subsemigroup of $S$ if

$$f(xy) \leq \max\{f(x), f(y)\}$$

for all $x, y \in S$.

An anti-fuzzy subsemigroup $f$ of $S$ is called an anti-fuzzy bi-ideal of $S$ if:

1. $x \leq y$ implies $f(x) \leq f(y)$,
2. $f(xay) \leq \max\{f(x), f(y)\}$

for all $x, a, y \in S$.

For fuzzy subsets $f$ and $g$ of $S$ the product $f * g$ is defined as follows:

$$(f * g)(a) := \begin{cases} \bigwedge_{(y,z) \in A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset, \\ 1 & \text{if } A_x = \emptyset, \end{cases}$$

for all $a \in S$.

The fuzzy subsets “$S$” and “$O$” of $S$ are defined as

$$S(x) = 1, \quad O(x) = 0$$

for all $x \in S$.

**Proposition 2.1.** Let $A, B \subseteq S$. Then

(i) $A \subseteq B$ if and only if $f_{B^c} \preceq f_{A^c}$.

(ii) $f_{A^c} \lor f_{B^c} = f_{A^c \cup B^c} = f_{(A \cap B)^c}$.

(iii) $f_{A^c} \land f_{B^c} = f_{(A \cap B)^c}$.

**An ordered semigroup $S$ is called regular (see [6]) if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$ or equivalently, (1) $(\forall a \in S)(a \in (aSa))$ and (2) $(\forall A \subseteq S)(A \subseteq (ASA))$, and $S$ is called left (resp. right) simple (see [7]) if it has no proper left (resp. right) ideals.**
Lemma 2.2. (cf. [7]). S is left (resp. right) simple if and only if \((Sa) = S\) (resp. \((aS) = S\)) for every \(a \in S\). □

An ordered semigroup \(S\) is called left (resp. right) regular (see [7]) if for every \(a \in S\), there exists \(x \in S\) such that \(a \leq xa^2\) (resp. \(a \leq a^2x\)) or equivalently, \((\forall a \in S)(a \in (Sa^2))\) and \((\forall A \subseteq S)(A \subseteq (SA^2))\). \(S\) is called completely regular if it is regular, left regular and right regular [7].

If \(\emptyset \neq A \subseteq S\), then the set \((A \cup (AS \cap SA))\) is the quasi-ideal of \(S\) generated by \(A\).

Lemma 2.3. (cf. [6]). \(S\) is completely regular if and only if \(A \subseteq (A^2 SA^2)\) for every \(A \subseteq S\). Equivalently, if \(a \in (a^2Sa^2)\) for every \(a \in S\). □

3. Anti-fuzzy quasi-ideals

Definition 3.1. A fuzzy subset \(f\) of \(S\) is called an anti-fuzzy quasi-ideal if

1. \((f \ast \mathcal{O}) \lor (\mathcal{O} \ast f) \geq f\),
2. \(x \leq y\) implies \(f(x) \leq f(y)\) for all \(x, y \in S\).

As a consequence of the transfer principle for fuzzy sets (cf. [9]) we obtain the following two theorems.

Theorem 3.2. Let \(\emptyset \neq A \subseteq S\). Then \(A\) is a quasi-ideal of \(S\) if and only if the characteristic function \(f_A^c\) of the complement of \(A\) is an anti-fuzzy quasi-ideal of \(S\).

Theorem 3.3. Let \(f\) be a fuzzy subset of \(S\). Then each non-empty level \(L(f; t)\) is a quasi-ideal if and only if \(f\) is an anti-fuzzy quasi-ideal.

Example 3.4. The set \(S = \{a, b, c, d, f\}\) with the multiplication

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>d</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>f</td>
<td>c</td>
<td>c</td>
<td>f</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>b</td>
<td>d</td>
<td>b</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>a</td>
<td>f</td>
<td>a</td>
<td>c</td>
<td>a</td>
</tr>
</tbody>
</table>

and the order \(\leq:=\{ (a, a), (a, b), (a, c), (a, d), (a, f), (b, b), (c, c), (d, d), (f, f)\}\)

is an ordered semigroup with the following quasi-ideals:

\(\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, f\}, \{a, b, d\}, \{a, c, d\}, \{a, b, f\}, \{a, c, f\}, S\).
For a fuzzy set \( f \) defined by \( f(a) = 0.3, f(b) = 0.5, f(c) = f(f) = 0.8, f(d) = 0.6 \) we have

\[
L(f; t) := \begin{cases} 
S & \text{if } t \in [0.8, 1), \\
\{a, b, d\} & \text{if } t \in [0.6, 0.8), \\
\{a, b\} & \text{if } t \in [0.5, 0.6), \\
\{a\} & \text{if } t \in [0.3, 0.5), \\
\emptyset & \text{if } t \in [0, 0.3).
\end{cases}
\]

\( L(f; t) \) is a quasi-ideal. By Theorem 3.3, \( f \) is an anti-fuzzy quasi-ideal. \( \square \)

**Lemma 3.5.** Every anti-fuzzy quasi-ideal of \( S \) is its anti-fuzzy bi-ideal.

**Proof.** Let \( x, y, z \in S \). Then \( xyz = x(yz) = (xy)z \). Hence \((x, yz) \in A_{xyz}\) and \((xy, z) \in A_{xyz}\). Since \( A_{xyz} \neq \emptyset \), we have

\[
f(xyz) \leq [(f * O) \vee (O * f)](xyz) \\
= \max \left[ \bigwedge_{(p,q) \in A_{xyz}} \max\{f(p), O(q)\}, \bigwedge_{(p,q) \in A_{xyz}} \max\{O(p_1), f(q_1)\} \right] \\
\leq \max[\max\{f(x), O(yz)\}, \max\{O(xy), f(z)\}] \\
= \max[\max\{f(x), 0\}, \max\{0, f(z)\}] = \max[0, f(z)].
\]

Let \( x, y \in S \), then \( xy = x(y) \) and hence \((x, y) \in A_{xy}\). Since \( A_{xy} \neq \emptyset \), we have

\[
f(xy) \leq [(f * O) \vee (O * f)](xy) \\
= \max \left[ \bigwedge_{(p,q) \in A_{xy}} \max\{f(p), O(q)\}, \bigwedge_{(p,q) \in A_{xy}} \max\{O(p), f(q)\} \right] \\
\leq \max[\max\{f(x), O(y)\}, \max\{O(x), f(y)\}] \\
= \max[\max\{f(x), 0\}, \max\{0, f(y)\}] = \max[0, f(y)].
\]

Let \( x, y \in S \) be such that \( x \leq y \). Then \( f(x) \leq f(y) \), because \( f \) is an anti-fuzzy quasi-ideal of \( S \). Thus \( f \) is an anti-fuzzy bi-ideal of \( S \). \( \square \)

The converse of above Lemma is not true, in general.
Example 3.6. The set $S = \{a, b, c, d\}$ with the multiplication table
\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & a & b & a \\
d & a & a & b & b \\
\end{array}
\]
and the order $\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$ is an ordered semigroup.

\{a, d\} is its bi-ideal but not a quasi-ideal.

For a fuzzy set $f(a) = f(d) = 0.7, f(b) = f(c) = 0.3$ we have
\[
L(f; t) := \begin{cases}
S & \text{if } t \in [0.7, 1), \\
\{a, d\} & \text{if } t \in [0.3, 0.7), \\
\emptyset & \text{if } t \in [0, 0.3).
\end{cases}
\]
$L(f; t)$ is a bi-ideal for every $t$, but for $t \in [0.3, 0.7)$ it is not a quasi-ideal of $S$. By Theorem 3.3, $f$ is an anti-fuzzy bi-ideal of $S$ but not an anti-fuzzy quasi-ideal of $S$. □

4. Completely regular ordered semigroups

Theorem 4.1. The following are equivalent:

(i) $S$ is regular, left and right simple,
(ii) every anti-fuzzy quasi-ideal of $S$ is a constant function.

Proof. (i) $\implies$ (ii). Let $S$ be a fixed regular, left and right simple ordered semigroup. Let $f$ be an anti-fuzzy quasi-ideal of $S$. We consider the set $E_\Omega = \{e \in S | e^2 \geq e\}$. $E_\Omega$ is non-empty, because for $a \in S$ there exists $x \in S$ such that $a \leq ax$, hence $(ax)^2 = (ax)x \geq ax$, which means that $ax \in E_\Omega$.

(A) We first prove that $f$ is a constant function on $E_\Omega$. That is, $f(e) = f(t)$ for every $t \in E_\Omega$. In fact: since $S$ is left and right simple, we have $(St) = S$ and $(tS) = S$. But $e \in S$. Then $e \in (St)$ and $e \in (tS)$. Thus $e \leq xt$ and $e \leq ty$ for some $x, y \in S$. If $e \leq xt$ then $e^2 = ee \leq (xt)(xt) = (tx)t$ and $(tx, t) \in A_{\epsilon^2}$. If $e \leq ty$ then $e^2 = ee \leq (ty)(ty) = t(yty)$ and $(t, yty) \in A_{\epsilon^2}$.
Since \( A_{1,2} \neq \emptyset \) we have
\[
f(e^2) \leq ((f \circ \Omega) \land (\Omega \circ f))(e^2) = \max[(f \circ \Omega)(e^2), (\Omega \circ f)(e^2)]
\]
\[
= \max \left( \bigwedge_{(y_1, z_1) \in A_{1,2}} \max\{f(y_1), \Omega(z_1)\}, \bigwedge_{(y_2, z_2) \in A_{1,2}} \max\{\Omega(y_2), f(z_2)\} \right)
\]
\[
\leq \max[\max\{f(t), \Omega(yty)\}, \max\{\Omega(\Omega^{\circ}(xtx), f(t)\}]
\]
\[
= \max[\max\{f(t), 0\}, \max\{0, f(t)\}] = \max[f(t), f(t)] = f(t).
\]

Since \( e \in E_{\Omega} \), we have \( e^2 \geq e \) and \( f(e^2) \geq f(e) \). Thus \( f(e) \leq f(t) \). On the other hand since \( S \) is left and right simple and \( e \in S \), we have \( S = (Se) \) and \( S = (eS) \). Since \( t \in S \) we have \( t \in (Se) \) and \( t \in (eS) \). Then \( t \leq ze \) and \( t \leq es \) for some \( z, s \in S \). If \( t \leq ze \) then \( t^2 = tt \leq (ze)(ze) = (ze)z \) and \( (ze, e) \in A_{1,2} \). If \( t \leq es \) then \( t^2 = tt \leq (es)(es) = e(ses) \) and \( (e, ses) \in A_{1,2} \).

Since \( A_{1,2} \neq \emptyset \) we have
\[
f(t^2) \leq ((f \circ \Omega) \land (\Omega \circ f))(t^2) = \max[(f \circ \Omega)(t^2), (\Omega \circ f)(t^2)]
\]
\[
= \max \left( \bigwedge_{(y_1, z_1) \in A_{1,2}} \max\{f(y_1), \Omega(z_1)\}, \bigwedge_{(y_2, z_2) \in A_{1,2}} \max\{\Omega(y_2), f(z_2)\} \right)
\]
\[
\leq \max[\max\{f(e), \Omega(\Omega^{\circ}(es))\}, \max\{\Omega(\Omega^{\circ}(es)), f(e)\}]
\]
\[
= \max[\max\{f(e), 0\}, \max\{0, f(e)\}] = \max[\max\{f(e), f(e)\}] = f(e).
\]

Since \( t \in E_{\Omega} \) then \( t^2 \geq t \) and \( f(t^2) \geq f(t) \). Thus \( f(t) \leq f(e) \). Consequently, \( f(t) = f(e) \).

(B) Now we prove that \( f \) is a constant function on \( S \). That is, \( f(t) = f(a) \) for every \( a \in S \). In fact: since \( S \) is regular and \( a \in S \), there exists \( x \in S \) such that \( a \leq axa \). We consider the elements \( ax \) and \( xa \) of \( S \). Then by \( (OS3) \), we have \((ax)^2 = (axa)x \geq ax \) and \((xa)^2 = x(axa) \geq xa \), then \( ax, xa \in E_{\Omega} \) and by \( (A) \) we have \( f(ax) = f(t) \) and \( f(xa) = f(t) \). Since \((ax)(xa) \geq axa \geq a \), then \( ax, xa \in A_a \) and \( (axa)(xa) \geq axa \geq a \), then \( (ax, xa) \in A_a \) and hence \( A_a \neq \emptyset \). Since \( f \) is an anti-fuzzy quasi-ideal of \( S \), we have
\[
f(a) \leq ((f \circ \Omega) \land (\Omega \circ f))(a) = \max[(f \circ \Omega)(a), (\Omega \circ f)(a)]
\]
\[
= \max \left( \bigwedge_{(y_1, z_1) \in A_a} \max\{f(y_1), \Omega(z_1)\}, \bigwedge_{(y_2, z_2) \in A_a} \max\{\Omega(y_2), f(z_2)\} \right)
\]
\[
\leq \max[\max\{f(ax), \Omega(axa)\}, \max\{\Omega(axa), f(xa)\}]
\]
\[
= \max[\max\{f(ax), 0\}, \max\{0, f(xa)\}] = \max[f(ax), f(xa)] = f(t).
Since $S$ is left and right simple we have $(Sa) = S$, and $(aS) = S$. Since $t \in S$, we have $t \in (Sa)$ and $t \in (aS)$. Then $t \leq pa$ and $t \leq aq$ for some $p, q \in S$. Then $(p, a) \in A_t$ and $(a, q) \in A_t$. Since $A_t \neq \emptyset$, and $f$ is an anti-fuzzy quasi-ideal of $S$, we have

$$f(t) \leq ((f \ast O) \vee (O \ast f))(t) = \max[(f \ast O)(t), (O \ast f)(t)]$$

$$= \max \left[ \bigwedge_{(y_1, z_1) \in A_t} \max\{f(y_1), O(z_1)\}, \bigwedge_{(y_2, z_2) \in A_t} \max\{O(y_2), f(z_2)\} \right]$$

$$\leq \max \left[ \max\{f(a), O(q)\}, \max\{O(p), f(a)\} \right]$$

$$= \max \left[ \max\{f(a), 0\}, \max\{0, f(a)\} \right] = f(a) \quad \text{for all } t \in S.$$

Thus $f(t) \leq f(a)$ and $f(t) = f(a)$.

(ii) $\implies$ (i). Let $a \in S$. Then the set $(aS)$ is a quasi-ideal of $S$. Indeed: $(aS) \cap (Sa) \subseteq (aS)$, and $x \in (aS)$ and $S \ni y \leq x \in (aS)$ imply $y \in ((aS)) = (aS)$. Since $(aS)$ is quasi-ideal of $S$, by Theorem 3.2, the characteristic function $f_{(aS)}$ of $(aS)$ is an anti-fuzzy quasi-ideal of $S$. By hypothesis, $f_{(aS)}$ is a constant function, that is, there exists $t \in \{0, 1\}$ such that $f_{(aS)}(x) = t$ for every $x \in S$. Let $(aS) \subset S$ and $a$ be an element of $S$ such that $a \notin (aS)$, then $f_{(aS)}(a) = 1$. On the other hand, since $a^2 \in (aS)$, then $f_{(aS)}(a^2) = 0$, a contradiction to the fact that $f_{(aS)}$ is a constant function. Hence $(aS) = S$.

By symmetry we can prove that $(Sa) = S$.

Since $a \in S$ and $S = (aS) = (Sa)$, we have $a \in (aS) = (a(aS)) = (aSa)$, consequently $S$ is regular.

**Theorem 4.2.** $S$ is completely regular if and only if for every anti-fuzzy quasi-ideal $f$ of $S$ we have $f(a) = f(a^2)$ for every $a \in S$.

**Proof.** Let $S$ be completely regular and $f$ be an anti-fuzzy quasi-ideal of $S$. Since $S$ is left and right regular we have $a \in (Sa^2)$ and $a \in (a^2S)$ for every $a \in S$. Then there exists $x, y \in S$ such that $a \leq xa^2$ and $a \leq a^2y$. Hence $(x, a^2), (a^2, y) \in A_a$. Since $A_a \neq \emptyset$, we have

$$f(a) \leq ((f \ast O) \vee (O \ast f))(a) = \max[(f \ast O)(a), (O \ast f)(a)]$$

$$= \max \left[ \bigwedge_{(y_1, z_1) \in A_a} \max\{f(y_1), O(z_1)\}, \bigwedge_{(y_2, z_2) \in A_a} \max\{O(y_2), f(z_2)\} \right]$$

$$\leq \max \left[ \max\{f(a^2), O(y)\}, \max\{O(x), f(a^2)\} \right]$$

$$= \max \left[ \max\{f(a^2), 0\}, \max\{0, f(a^2)\} \right]$$

$$= \max\{f(a^2), f(a^2)\} = f(a^2) = f(aa) \leq \max\{f(a), f(a)\} = f(a).$$
Hence \(f(a) = f(a^2)\).

Conversely, let \(a \in S\) and let \(Q(a^2)\) be the quasi-ideal generated by \(a^2\). Then \(Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))\). By Theorem 3.2, the characteristic function \(f_{Q(a^2)}\) is an anti-fuzzy quasi-ideal of \(S\). By hypothesis \(f_{Q(a^2)}(a) = f_{Q(a^2)}(a^2)\). Since \(a^2 \in Q(a^2)\), we have \(f_{Q(a^2)}(a^2) = 0\), then \(f_{Q(a^2)}(a) = 0\) and \(a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))\). Then \(a \leq a^2\) or \(a \leq a^2x\) for some \(x \in S\). If \(a \leq a^2\) then \(a \leq a^2 = aa \leq a^2a^2 = aaaa \leq a^2aa^2 \in a^2Sa^2\) and so \(a \in (a^2Sa^2)\). If \(a = a^2x\) and \(a = ya^2\) then \(a \leq (a^2x)(ya^2) = a^2(xy)a^2 \in a^2Sa^2\) and so \(a \in (a^2Sa^2)\).

\(\square\)

A subset \(T\) of \(S\) is called semiprime if for every \(a \in S\) such that \(a^2 \in T\) we have \(a \in T\). An anti-fuzzy quasi-ideal \(f\) of \(S\) is called semiprime if \(f(a) \leq f(a^2)\) all \(a \in S\).

**Theorem 4.3.** \(S\) is completely regular if and only if every its anti-fuzzy quasi-ideal is semiprime.

**Proof.** Let \(S\) be completely regular and \(f\) be its anti-fuzzy quasi-ideal. Then \(f(a) \leq f(a^2)\) for all \(a \in S\). Indeed: since \(S\) is left and right regular, there exist \(x, y \in S\) such that \(a \leq xa^2\) and \(a \leq a^2y\) then \((x, a^2) \in A_a\) and \((a^2, y) \in A_a\). Since \(A_a \neq \emptyset\), and \(f\) is an anti-fuzzy quasi-ideal of \(S\), we have

\[
f(a) \leq \left( (f \circ O) \cup (O \circ f) \right)(a) = \max\{f \circ O)(a), (O \circ f)(a)\}
= \max\left[ \bigwedge_{(y_1, z_1) \in A_a} \max\{f(y_1), O(z_1)\}, \bigwedge_{(y_2, z_2) \in A_a} \max\{O(y_2), f(z_2)\} \right]
= \max\{\max\{f(a^2), O(y)\}, \max\{O(x), f(a^2)\}\}
\leq \max\{\max\{f(a^2), 0\}, \max\{0, f(a^2)\}\} = \max\{f(a^2), f(a^2)\} = f(a^2).
\]

Conversely. Let \(f\) be an anti-fuzzy quasi-ideal of \(S\) such that \(f(a) \leq f(a^2)\) for all \(a \in S\). By Theorem 3.2, the characteristic function \(f_{Q(a^2)}\) of the quasi-ideal \(Q(a^2)\) is an anti-fuzzy quasi-ideal of \(S\). By hypothesis \(f_{Q(a^2)}(a) = f_{Q(a^2)}(a^2)\). Since \(a^2 \in Q(a^2)\), we have \(f_{Q(a^2)}(a^2) = 0\), then \(f_{Q(a^2)}(a) = 0\) and \(a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))\). Thus \(a \leq a^2\) or \(a \leq a^2p\) and \(a \leq qa^2\) for some \(p, q \in S\). If \(a \leq a^2\) then \(a \leq a^2 = aa \leq a^2a^2 = aaaa \leq a^2aa^2 \in a^2Sa^2\) and so \(a \in (a^2Sa^2)\). If \(a \leq a^2p\) and \(a \leq qa^2\) then \(a \leq (a^2p)(qa^2) = a^2(pq)a^2 \in a^2Sa^2\) and so \(a \in (a^2Sa^2)\). Consequently, \(S\) is completely regular. \(\square\)
References


Received September 02, 2010
Revised February 8, 2011

Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan
E-mails: anwar55.ciit@yahoo.com (A. Zeb) azhartset@yahoo.com (A. Khan)