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Bipolar fuzzy Lie superalgebras

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Abstract. We introduce the notion of bipolar fuzzy Lie sub-superalgebras (resp. bipolar fuzzy ideals) and present some of their properties. First we investigate the properties of bipolar fuzzy Lie sub-superalgebras and bipolar fuzzy ideals under homomorphisms of Lie superalgebras. Next we study bipolar fuzzy bracket product, solvable bipolar fuzzy ideals and nilpotent bipolar fuzzy ideals of Lie superalgebras.

1. Introduction

The concept of fuzzy set was first initiated by Zadeh [16] in 1965 and since then, fuzzy set has become an important tool in studying scientific subjects, in particular, it can be applied in a wide variety of disciplines such as computer science, medical science, management science, social science, engineering and so on. There are a number of generalizations of Zadeh's fuzzy set theory so far reported in the literature viz., interval-valued fuzzy theory, intuitionistic fuzzy theory, L-fuzzy theory, probabilistic fuzzy theory and so on. In 1994, Zhang [17, 18] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets [15]. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [12], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places.

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As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

The theory of Lie superalgebras was constructed by V.G. Kac [14] in 1977 as a generalization of the theory of Lie algebras. This theory had played an important role in both mathematics and physics. In particular, Lie superalgebras are important in theoretical physics where they are used to describe the mathematics of supersymmetry [11]. Furthermore, Lie superalgebras had found many applications in computer science such as unimodal polynomials [13].

Recently, Chen [7, 8, 9, 10] have considered Lie superalgebras in fuzzy settings, intuitionistic fuzzy settings, interval-valued fuzzy settings and investigated their several properties. Akram introduced the notion of cofuzzy Lie superalgebras over a cofuzzy field in [4]. Now, it is natural to consider Lie superalgebras in bipolar fuzzy settings. In this paper, we introduce the notion of bipolar fuzzy Lie sub-superalgebras (resp. bipolar fuzzy ideals) and investigate the properties of bipolar fuzzy Lie sub-superalgebras and bipolar fuzzy ideals under homomorphisms of Lie superalgebras. We also introduce the concept of bipolar fuzzy bracket product and study solvable bipolar fuzzy ideals and nilpotent bipolar fuzzy ideals of Lie superalgebras and present the corresponding theorems parallel to Lie superalgebras. We have used standard definitions and terminologies in this paper. For notations, terminologies and applications not mentioned in the paper, the readers are referred to [2–6, 15, 17].

2. Preliminaries

In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1. [14] Suppose that V is a vector space and $V_{\bar{0}}, V_{\bar{1}}$ are its (vector) subspaces. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be the direct sum of the subspaces. Then V (with this decomposition) is called a $\mathbb{Z}_2$-*graded vector space* if each element v of a $\mathbb{Z}_2$-graded vector space has a unique expression of the form $v = v_{\bar{0}} + v_{\bar{1}}$ $(v_{\bar{0}} \in V_{\bar{0}},\, v_{\bar{1}} \in V_{\bar{1}})$. The subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are called the even part and odd part of V, respectively. In particular, if v is an element of either $V_{\bar{0}}$ or $V_{\bar{1}}$, v is said to be homogeneous.

Definition 2.2. [14] A $\mathbb{Z}_2$-graded vector space $\mathbb{L} = \mathbb{L}_{\bar{0}} \oplus \mathbb{L}_{\bar{1}}$ with a Lie bracket

$$[\ ,\ ] : \mathbb{L} \times \mathbb{L} \xrightarrow{\text{bilinear}} \mathbb{L}$$

is called a *Lie superalgebra*, if it satisfies the following conditions:

(1) $[\mathbb{L}_i, \mathbb{L}_j] \subseteq \mathbb{L}_{i+j}$ for $i,\, j \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$,

(2) $[x, y] = -(-1)^{|x||y|}[y, x]$ (antisymmetry),

(3) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[[x, z], y]$(Jacobi identity),

where for any homogeneous element $a \in \mathbb{L}_i$, $i = \bar{0}, \bar{1}$. The subspaces $\mathbb{L}_{\bar{0}}$ and $\mathbb{L}_{\bar{1}}$ are called the even and odd parts of $\mathbb{L}$, respectively. Therefore, a Lie algebra is a Lie superalgebra with trivial odd part.

Definition 2.3. [14] If $\varphi : \mathbb{L}_1 \to \mathbb{L}_2$ is a linear map between Lie superalgebras $\mathbb{L}_1 = \mathbb{L}_{1\bar{0}} \oplus \mathbb{L}_{1\bar{1}}$ and $\mathbb{L}_2 = \mathbb{L}_{2\bar{0}} \oplus \mathbb{L}_{2\bar{1}}$ such that

(4) $\varphi(\mathbb{L}_{1i}) \subseteq \mathbb{L}_{2i}$ $(i \in \mathbb{Z}_2)$ (preserving the grading),

(5) $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (preserving the Lie bracket).

Then φ is called a *homomorphism* of Lie superalgebras.

Throughout this paper, we denote V a vector space, $\mathbb{L}$ a Lie superalgebra over field F.

Let μ be a *fuzzy subset* on V, i.e., a map $\mu : V \to [0, 1]$. In this paper, the notations $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

Definition 2.4. [17] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu_B^P(x),\, \mu_B^N(x)) \,|\, x \in X\}$$

where $\mu_B^P : X \to [0,\, 1]$ and $\mu_B^N : X \to [-1,\, 0]$ are mappings. For the sake of simplicity, we shall use the symbol $B = (\mu_B^P,\, \mu_B^N)$ for the bipolar fuzzy set $B = \{(x,\, \mu_B^P(x),\, \mu_B^N(x)) \,|\, x \in X\}$.

Definition 2.5. [15] For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define

- $(A \bigcap B)(x) = (\min(\mu_A^P(x), \mu_B^P(x)), \max(\mu_A^N(x), \mu_B^N(x)))$,
- $(A \bigcup B)(x) = (\max(\mu_A^P(x), \mu_B^P(x)), \min(\mu_A^N(x), \mu_B^N(x)))$.

In order to point out the differences between intuitionistic fuzzy Lie sub-superalgebras and bipolar fuzzy Lie sub-superalgebras, we omit the similar proofs in this paper.

Lemma 2.6. *$A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subspace of V if and only if μ_A^P and μ_A^N are fuzzy subspaces of V.* □

Lemma 2.7. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy subspaces of V. Then $A + B$ is also a bipolar fuzzy subspace of V.* □

Lemma 2.8. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy subspaces of V. Then $A \cap B$ is also a bipolar fuzzy subspace of V.* $\square$

Lemma 2.9. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subspace of V' and ϕ be a mapping from vector space V to V'. Then the inverse image $\phi^{-1}(A)$ is also a bipolar fuzzy subspace of V.* $\square$

Lemma 2.10. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subspace of V and f be a mapping from V to V'. Then the image $\phi(A)$ is also a bipolar fuzzy subspace of V'.* $\square$

3. Bipolar fuzzy Lie sub-superalgebras

Definition 3.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Suppose that $A_{\bar{0}} = (\mu_{A_{\bar{0}}}^P, \mu_{A_{\bar{0}}}^N)$ and $A_{\bar{1}} = (\mu_{A_{\bar{1}}}^P, \mu_{A_{\bar{1}}}^N)$ are bipolar fuzzy vector subspaces of $V_{\bar{0}}$, $V_{\bar{1}}$, respectively. We define $A'_{\bar{0}} = (\mu_{A'_{\bar{0}}}^P, \mu_{A'_{\bar{0}}}^N)$ where

$$\mu_{A'_{\bar{0}}}^P(x) = \begin{cases} \mu_{A_{\bar{0}}}^P(x) & x \in V_{\bar{0}} \\ 0 & x \notin V_{\bar{0}} \end{cases}, \qquad \mu_{A'_{\bar{0}}}^N(x) = \begin{cases} \mu_{A_{\bar{0}}}^N(x) & x \in V_{\bar{0}} \\ 0 & x \notin V_{\bar{0}} \end{cases}$$

and define $A'_{\bar{1}} = (\mu_{A'_{\bar{1}}}^P, \mu_{A'_{\bar{1}}}^N)$ where

$$\mu_{A'_{\bar{1}}}^P(x) = \begin{cases} \mu_{A_{\bar{1}}}^P(x) & x \in V_{\bar{1}} \\ 0 & x \notin V_{\bar{1}} \end{cases}, \qquad \mu_{A'_{\bar{1}}}^N(x) = \begin{cases} \mu_{A_{\bar{1}}}^N(x) & x \in V_{\bar{1}} \\ 0 & x \notin V_{\bar{1}} \end{cases}$$

Then $A'_{\bar{0}} = (\mu_{A'_{\bar{0}}}^P, \mu_{A'_{\bar{0}}}^N)$ and $A'_{\bar{1}} = (\mu_{A'_{\bar{1}}}^P, \mu_{A'_{\bar{1}}}^N)$ are the bipolar fuzzy vector subspaces of V. Moreover, we have $A'_{\bar{0}} \cap A'_{\bar{1}} = (\mu_{A'_{\bar{0}} \cap A'_{\bar{1}}}^P, \mu_{A'_{\bar{0}} \cap A'_{\bar{1}}}^N)$, where

$$\mu_{A'_{\bar{0}} \cap A'_{\bar{1}}}^P(x) = \mu_{A'_{\bar{0}}}^P(x) \wedge \mu_{A'_{\bar{1}}}^P(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases},$$

$$\mu_{A'_{\bar{0}} \cap A'_{\bar{1}}}^N(x) = \mu_{A'_{\bar{0}}}^N(x) \vee \mu_{A'_{\bar{1}}}^N(x) = \begin{cases} -1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

So $A'_{\bar{0}} + A'_{\bar{1}}$ is the direct sum and denoted by $A_{\bar{0}} \oplus A_{\bar{1}}$. If $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy vector subspace of V and $A = A_{\bar{0}} \oplus A_{\bar{1}}$, then $A = (\mu_A^P, \mu_A^N)$ is called a $\mathbb{Z}_2$-*graded bipolar fuzzy vector subspace* of V.

Definition 3.2. Let $A = (\mu_A^P, \mu_A^N)$ be an bipolar fuzzy set of $\mathbb{L}$. Then $A = (\mu_A^P, \mu_A^N)$ is called a *bipolar fuzzy Lie sub-superalgebra* of $\mathbb{L}$, if it satisfies the following conditions:

(1) $A = (\mu_A^P, \mu_A^N)$ is a $\mathbb{Z}_2$-graded bipolar fuzzy vector subspace,

(2) $\mu_A^P([x, y]) \geq \mu_A^P(x) \wedge \mu_A^P(y)$ and $\mu_A^N([x, y]) \leq \mu_A^N(x) \vee \mu_A^N(y)$.

If the condition (2) is replaced by

(3) $\mu_A^P([x,y]) \geq \mu_A^P(x) \vee \mu_A^P(y)$ and $\mu_A^N([x,y]) \leq \mu_A^N(x) \wedge \mu_A^N(y)$,

then $A = (\mu_A^P, \mu_A^N)$ is called a *bipolar fuzzy ideal* of $\mathbb{L}$.

Example 3.3. Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$, where $N_{\bar{0}} = \langle e \rangle, N_{\bar{1}} = \langle a_1, \cdots, a_n, b_1, \cdots, b_n \rangle$ and $[a_i, b_i] = e, i = 1, 2, \cdots n$, the remaining brackets being zero. Then N is Lie superalgebra. Define $A_{\bar{0}} = (\mu_{A_{\bar{0}}}^P, \mu_{A_{\bar{0}}}^N)$ where

$$\mu_{A_{\bar{0}}}^P(x) = \begin{cases} 0.7 & x \in N_{\bar{0}} \setminus \{0\} \\ 1 & x = 0 \end{cases}, \quad \mu_{A_{\bar{0}}}^N(x) = \begin{cases} -0.2 & x \in N_{\bar{0}} \setminus \{0\} \\ -1 & x = 0 \end{cases}.$$

Define $A_{\bar{1}} = (\mu_{A_{\bar{1}}}^P, \mu_{A_{\bar{1}}}^N)$ where

$$\mu_{A_{\bar{1}}}^P(x) = \begin{cases} 0.5 & x \in N_{\bar{1}} \setminus \{0\} \\ 1 & x = 0 \end{cases}, \quad \mu_{A_{\bar{1}}}^N(x) = \begin{cases} -0.4 & x \in N_{\bar{1}} \setminus \{0\} \\ -1 & x = 0 \end{cases}.$$

Define $A = (\mu_A^P, \mu_A^N)$ by $A = A_{\bar{0}} \oplus A_{\bar{1}}$. Then $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy ideal of N. □

Definition 3.4. For any $t \in [0, 1]$ and fuzzy subset μ^P of $\mathbb{L}$, the set $U(\mu^P, t) = \{x \in \mathbb{L} | \mu^P(x) \geqslant t\}$ (resp. $L(\mu^P, t) = \{x \in \mathbb{L} | \mu^P(x) \leqslant t\}$) is called an *upper* (resp. *lower*) *t-level cut* of μ^P.

The proofs of the following theorems are omitted.

Theorem 3.5. *If $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy Lie sub-superalgebra (resp. bipolar fuzzy ideal) of $\mathbb{L}$, then the sets $U(\mu_A^P, t)$ and $L(\mu_A^N, t)$ are Lie sub-superalgebras (resp. ideals) of $\mathbb{L}$ for every $t \in Im\mu_A^P \cap Im\mu_A^N$.* □

Theorem 3.6. *If $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy set of $\mathbb{L}$ such that all non-empty level sets $U(\mu_A^P, t)$ and $L(\mu_A^N, t)$ are Lie sub-superalgebras (resp. ideals) of $\mathbb{L}$, then $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy Lie sub-superalgebra (resp. bipolar fuzzy ideal) of $\mathbb{L}$.* □

Theorem 3.7. *If $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ are bipolar fuzzy Lie sub-superalgebras (resp. bipolar fuzzy ideals) of $\mathbb{L}$, then so is $A + B = (\mu_{A+B}^P, \mu_{A+B}^N)$.* □

Theorem 3.8. *If $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ are bipolar fuzzy Lie sub-superalgebras (resp. bipolar fuzzy ideals) of $\mathbb{L}$, then so is $A \cap B = (\mu_{A\cap B}^P, \mu_{A\cap B}^N)$.* □

Proposition 3.9. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a Lie homomorphism. If $A = (\mu_A, \mu_A^N)$ is a bipolar fuzzy Lie sub-superalgebra (resp. bipolar fuzzy ideal) of $\mathbb{L}'$, then the bipolar fuzzy set $\varphi^{-1}(A)$ of $\mathbb{L}$ is also a bipolar fuzzy Lie sub-superalgebra (resp. bipolar fuzzy ideal).*

Proof. Since φ preserves the grading, we have $\varphi(x) = \varphi(x_{\bar{0}}+x_{\bar{1}}) = \varphi(x_{\bar{0}})+\varphi(x_{\bar{1}}) \in \mathbb{L}'_{\bar{0}} \oplus \mathbb{L}'_{\bar{1}}$, for $x = x_{\bar{0}} + x_{\bar{1}} \in \mathbb{L}$. We define $\varphi^{-1}(A)_{\bar{0}} = (\mu^P_{\varphi^{-1}(A)_{\bar{0}}}, \mu^N_{\varphi^{-1}(A)_{\bar{0}}})$ where $\mu^P_{\varphi^{-1}(A)_{\bar{0}}} = \varphi^{-1}(\mu^P_{A_{\bar{0}}})$, $\mu^N_{\varphi^{-1}(A)_{\bar{0}}} = \varphi^{-1}(\mu^N_{A_{\bar{0}}})$ and $\varphi^{-1}(A)_{\bar{1}} = (\mu^P_{\varphi^{-1}(A)_{\bar{1}}}, \mu^N_{\varphi^{-1}(A)_{\bar{1}}})$ where $\mu^P_{\varphi^{-1}(A)_{\bar{1}}} = \varphi^{-1}(\mu^P_{A_{\bar{1}}})$, $\mu^N_{\varphi^{-1}(A)_{\bar{1}}} = \varphi^{-1}(\mu^N_{A_{\bar{1}}})$. By Lemma 2.9, we have that they are bipolar fuzzy subspaces of $\mathbb{L}_{\bar{0}}, \mathbb{L}_{\bar{1}}$, respectively.

Then we define $\varphi^{-1}(A)'_{\bar{0}} = (\mu^P_{\varphi^{-1}(A)'_{\bar{0}}}, \mu^N_{\varphi^{-1}(A)'_{\bar{0}}})$, where $\mu^P_{\varphi^{-1}(A)'_{\bar{0}}} = \varphi^{-1}(\mu^P_{A'_{\bar{0}}})$, $\mu^N_{\varphi^{-1}(A)'_{\bar{0}}} = \varphi^{-1}(\mu^N_{A'_{\bar{0}}})$, and $\varphi^{-1}(A)'_{\bar{1}} = (\mu^P_{\varphi^{-1}(A)'_{\bar{1}}}, \mu^N_{\varphi^{-1}(A)'_{\bar{1}}})$, where $\mu^P_{\varphi^{-1}(A)'_{\bar{1}}} = \varphi^{-1}(\mu^P_{A'_{\bar{1}}})$, $\mu^N_{\varphi^{-1}(A)'_{\bar{1}}} = \varphi^{-1}(\mu^N_{A'_{\bar{1}}})$.

Clearly,

$$\mu^P_{\varphi^{-1}(A)'_{\bar{0}}}(x) = \begin{cases} \mu^P_{\varphi^{-1}(A)_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 0 & x \notin \mathbb{L}_{\bar{0}} \end{cases}, \quad \mu^N_{\varphi^{-1}(A)'_{\bar{0}}}(x) = \begin{cases} \mu^N_{\varphi^{-1}(A)'_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 0 & x \notin \mathbb{L}_{\bar{0}} \end{cases}$$

and

$$\mu^P_{\varphi^{-1}(A)'_{\bar{1}}}(x) = \begin{cases} \mu^P_{\varphi^{-1}(A)'_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 0 & x \notin \mathbb{L}_{\bar{1}} \end{cases}, \quad \mu^N_{\varphi^{-1}(A)'_{\bar{1}}}(x) = \begin{cases} \mu^N_{\varphi^{-1}(A)'_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 0 & x \notin \mathbb{L}_{\bar{1}} \end{cases}$$

These show that $\varphi^{-1}(A)'_{\bar{0}}$ and $\varphi^{-1}(A)'_{\bar{1}}$ are the extensions of $\varphi^{-1}(A)_{\bar{0}}$ and $\varphi^{-1}(A)_{\bar{1}}$.

For $0 \neq x \in \mathbb{L}$, we have

$$\begin{aligned} \mu^P_{\varphi^{-1}(A)'_{\bar{0}}}(x) \wedge \mu^P_{\varphi^{-1}(A)'_{\bar{1}}}(x) &= \varphi^{-1}(\mu^P_{A'_{\bar{0}}})(x) \wedge \varphi^{-1}(\mu^P_{A'_{\bar{1}}})(x) \\ &= \mu^P_{A'_{\bar{0}}}(\varphi(x)) \wedge \mu^P_{A'_{\bar{1}}}(\varphi(x)) = 0 \end{aligned}$$

and

$$\begin{aligned} \mu^N_{\varphi^{-1}(A)'_{\bar{0}}}(x) \vee \mu^N_{\varphi^{-1}(A)'_{\bar{1}}}(x) &= \varphi^{-1}(\mu^N_{A'_{\bar{0}}})(x) \vee \varphi^{-1}(\mu^N_{A'_{\bar{1}}})(x) \\ &= \mu^N_{A'_{\bar{0}}}(\varphi(x)) \vee \mu^N_{A'_{\bar{1}}}(\varphi(x)) = 0. \end{aligned}$$

For $x \in \mathbb{L}$ we have

$$\begin{aligned} \mu^P_{\varphi^{-1}(A)'_{\bar{0}}+\varphi^{-1}(A)'_{\bar{1}}}(x) &= \sup_{x=a+b} \{\mu^P_{\varphi^{-1}(A)'_{\bar{0}}}(a) \wedge \mu^P_{\varphi^{-1}(A)'_{\bar{1}}}(b)\} \\ &= \sup_{x=a+b} \{\varphi^{-1}(\mu^P_{A'_{\bar{0}}})(a) \wedge \varphi^{-1}(\mu^P_{A'_{\bar{1}}})(b)\} \\ &= \sup_{x=a+b} \{\mu^P_{A'_{\bar{0}}}(\varphi(a)) \wedge \mu^P_{A'_{\bar{1}}}(\varphi(b))\} \\ &= \sup_{\varphi(x)=\varphi(a)+\varphi(b)} \{\mu^P_{A'_{\bar{0}}}(\varphi(a)) \wedge \mu^P_{A'_{\bar{1}}}(\varphi(b))\} \\ &= \mu^P_{A'_{\bar{0}}+A'_{\bar{1}}}(\varphi(x)) = \mu^P_A(\varphi(x)) = \mu^P_{\varphi^{-1}(A)}(x) \end{aligned}$$

and

$$\mu^N_{\varphi^{-1}(A)'_{\bar{0}}+\varphi^{-1}(A)'_{\bar{1}}}(x) = \inf_{x=a+b} \{\mu^N_{\varphi^{-1}(A)'_{\bar{0}}}(a) \vee \mu^N_{\varphi^{-1}(A)'_{\bar{1}}}(b)\}$$

$$
\begin{aligned}
&= \inf_{x=a+b}\{\varphi^{-1}(\mu^N_{A'_{\bar 0}})(a) \vee \varphi^{-1}(\mu^N_{A'_{\bar 1}})(b)\} \\
&= \inf_{x=a+b}\{\mu^N_{A'_{\bar 0}}(\varphi(a)) \vee \mu^N_{A'_{\bar 1}}(\varphi(b))\} \\
&= \inf_{\varphi(x)=\varphi(a)+\varphi(b)}\{\mu^N_{A'_{\bar 0}}(\varphi(a)) \vee \mu^N_{A'_{\bar 1}}(\varphi(b))\} \\
&= \mu^N_{A'_{\bar 0}+A'_{\bar 1}}(\varphi(x)) = \mu^N_A(\varphi(x)) = \mu^N_{\varphi^{-1}(A)}(x).
\end{aligned}
$$

So, $\varphi^{-1}(A) = \varphi^{-1}(A)_{\bar 0} \oplus \varphi^{-1}(A)_{\bar 1}$ is a $\mathbb{Z}_2$-graded bipolar fuzzy vector subspace of $\mathbb{L}$.

Let $x, y \in \mathbb{L}$. Then

(1) $\mu^P_{\varphi^{-1}(A)}([x,y]) = \mu^P_A(\varphi([x,y])) = \mu^P_A([\varphi(x),\varphi(y)]) \geq \mu^P_A(\varphi(x)) \wedge \mu^P_A(\varphi(y)) = \mu^P_{\varphi^{-1}(A)}(x) \wedge \mu^P_{\varphi^{-1}(A)}(y)$, and $\mu^N_{\varphi^{-1}(A)}([x,y]) = \mu^N_A(\varphi([x,y])) = \mu^N_A([\varphi(x),\varphi(y)]) \leq \mu^N_A(\varphi(x)) \vee \mu^N_A(\varphi(y)) = \mu^N_{\varphi^{-1}(A)}(x) \vee \mu^N_{\varphi^{-1}(A)}(y)$, thus $\varphi^{-1}(A)$ is a bipolar fuzzy Lie sub-superalgebra.

(2) $\mu^P_{\varphi^{-1}(A)}([x,y]) = \mu^P_A(\varphi([x,y])) = \mu^P_A([\varphi(x),\varphi(y)]) \geq \mu^P_A(\varphi(x)) \vee \mu^P_A(\varphi(y)) = \mu^P_{\varphi^{-1}(A)}(x) \vee \mu^P_{\varphi^{-1}(A)}(y)$, and $\mu^N_{\varphi^{-1}(A)}([x,y]) = \mu^N_A(\varphi([x,y])) = \mu^N_A([\varphi(x),\varphi(y)]) \leq \mu^N_A(\varphi(x)) \wedge \mu^N_A(\varphi(y)) = \mu^N_{\varphi^{-1}(A)}(x) \wedge \mu^N_{\varphi^{-1}(A)}(y)$, thus $\varphi^{-1}(A)$ is a bipolar fuzzy ideal. □

Proposition 3.10. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a Lie homomorphism. If $A = (\mu^P_A, \mu^N_A)$ is a bipolar fuzzy Lie sub-superalgebra of $\mathbb{L}$, then the bipolar fuzzy set $\varphi(A)$ is also a bipolar fuzzy Lie sub-superalgebra of $\mathbb{L}'$.*

Proof. Since $A = (\mu^P_A, \mu^N_A)$ is a bipolar fuzzy Lie sub-superalgebra of $\mathbb{L}$, we have $A = A_{\bar 0} \oplus A_{\bar 1}$ where $A_{\bar 0} = (\mu_{A_{\bar 0}}, \mu^N_{A_{\bar 0}}), A_{\bar 1} = (\mu_{A_{\bar 1}}, \mu^N_{A_{\bar 1}})$ are bipolar fuzzy vector subspaces of $\mathbb{L}_{\bar 0}$ and $\mathbb{L}_{\bar 1}$, respectively. We define $\varphi(A)_{\bar 0} = (\mu^P_{\varphi(A)_{\bar 0}}, \mu^N_{\varphi(A)_{\bar 0}})$, where $\mu^P_{\varphi(A)_{\bar 0}} = \varphi(\mu^P_{A_{\bar 0}})$, $\mu^N_{\varphi(A)_{\bar 0}} = \varphi(\mu^N_{A_{\bar 0}})$, $\varphi(A)_{\bar 1} = (\mu^P_{\varphi(A)_{\bar 1}}, \mu^N_{\varphi(A)_{\bar 1}})$, where $\mu^P_{\varphi(A)_{\bar 1}} = \varphi(\mu^P_{A_{\bar 1}})$, $\mu^N_{\varphi(A)_{\bar 1}} = \varphi(\mu^N_{A_{\bar 1}})$. By Lemma 2.10, $\varphi(A)_{\bar 0}$ and $\varphi(A)_{\bar 1}$ are bipolar fuzzy subspaces of $\mathbb{L}_{\bar 0}$, $\mathbb{L}_{\bar 1}$, respectively. And extend them to $\varphi(A)'_{\bar 0}, \varphi(A)'_{\bar 1}$, we define $\varphi(A)'_{\bar 0} = (\mu^P_{\varphi(A)'_{\bar 0}}, \mu^N_{\varphi(A)'_{\bar 0}})$ where $\mu^P_{\varphi(A)'_{\bar 0}} = \varphi(\mu^P_{A'_{\bar 0}}), \mu^N_{\varphi(A)'_{\bar 0}} = \varphi(\mu^N_{A'_{\bar 0}})$ and $\varphi(A)'_{\bar 1} = (\mu^P_{\varphi(A)'_{\bar 1}}, \mu^N_{\varphi(A)'_{\bar 1}})$ where $\mu^P_{\varphi(A)'_{\bar 1}} = \varphi(\mu^P_{A'_{\bar 1}}), \mu^N_{\varphi(A)'_{\bar 1}} = \varphi(\mu^N_{A'_{\bar 1}})$. Clearly,

$$
\mu^P_{\varphi(A)'_{\bar 0}}(x) = \begin{cases} \mu^P_{\varphi(A)_{\bar 0}}(x) & x \in \mathbb{L}_{\bar 0} \\ 0 & x \notin \mathbb{L}_{\bar 0} \end{cases}, \quad \mu^N_{\varphi(A)'_{\bar 0}}(x) = \begin{cases} \mu^N_{\varphi(A)_{\bar 0}}(x) & x \in \mathbb{L}_{\bar 0} \\ 0 & x \notin \mathbb{L}_{\bar 0} \end{cases},
$$

$$
\mu^P_{\varphi(A)'_{\bar 1}}(x) = \begin{cases} \mu^P_{\varphi(A)_{\bar 1}}(x) & x \in \mathbb{L}_{\bar 1} \\ 0 & x \notin \mathbb{L}_{\bar 1} \end{cases}, \quad \mu^N_{\varphi(A)'_{\bar 1}}(x) = \begin{cases} \mu^N_{\varphi(A)_{\bar 1}}(x) & x \in \mathbb{L}_{\bar 1} \\ 0 & x \notin \mathbb{L}_{\bar 1} \end{cases}.
$$

For $0 \neq x \in \mathbb{L}'$ we have

$$
\begin{aligned}
\mu^P_{\varphi(A)'_{\bar 0}}(x) \wedge \mu^P_{\varphi(A)'_{\bar 1}}(x) &= \varphi(\mu^P_{A'_{\bar 0}})(x) \wedge \varphi(\mu^P_{A'_{\bar 1}})(x) = \sup_{x=\varphi(a)}\{\mu^P_{A'_{\bar 0}}(a)\} \wedge \sup_{x=\varphi(a)}\{\mu^P_{A'_{\bar 1}}(a)\} \\
&= \sup_{x=\varphi(a)}\{\mu^P_{A'_{\bar 0}}(a) \wedge \mu^P_{A'_{\bar 1}}(a)\} = 0,
\end{aligned}
$$

$$\mu^N_{\varphi(A)'_{\bar{0}}}(x) \vee \mu^N_{\varphi(A)'_{\bar{1}}}(x) = \varphi(\mu^N_{A'_{\bar{0}}})(x) \vee \varphi(\mu^N_{A'_{\bar{1}}})(x) = \inf_{x=\varphi(a)}\{\mu^N_{A'_{\bar{0}}}(a)\} \vee \inf_{x=\varphi(a)}\{\mu^N_{A'_{\bar{1}}}(a)\}$$
$$= \inf_{x=\varphi(a)}\{\mu^N_{A'_{\bar{0}}}(a) \vee \mu^N_{A'_{\bar{1}}}(a)\} = 0.$$

Let $y \in \mathbb{L}'$. Then

$$\begin{aligned}
\mu^P_{\varphi(A)'_{\bar{0}}+\varphi(A)'_{\bar{1}}}(y) &= \sup_{y=a+b}\{\mu^P_{\varphi(A)'_{\bar{0}}}(a) \wedge \mu^P_{\varphi(A)'_{\bar{1}}}(b)\} = \sup_{y=a+b}\{\varphi(\mu^P_{A'_{\bar{0}}})(a) \wedge \varphi(\mu^P_{A'_{\bar{1}}})(b)\} \\
&= \sup_{y=a+b}\{\sup_{a=\varphi(m)}\{\mu^P_{A'_{\bar{0}}}(m)\} \wedge \sup_{b=\varphi(n)}\{\mu^P_{A'_{\bar{1}}}(n)\}\} \\
&= \sup_{y=\varphi(x)}\{\sup_{x=m+n}\{\mu^P_{A'_{\bar{0}}}(m) \wedge \mu^P_{A'_{\bar{1}}}(n)\}\} \\
&= \sup_{y=\varphi(x)}\{(\mu^P_{A'_{\bar{0}}+A'_{\bar{1}}})(x)\} = \sup_{y=\varphi(x)}\{\mu^P_A(x)\} = \mu^P_{\varphi(A)}(y),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{\varphi(A)'_{\bar{0}}+\varphi(A)'_{\bar{1}}}(y) &= \inf_{y=a+b}\{\mu^N_{\varphi(A)'_{\bar{0}}}(a) \vee \mu^N_{\varphi(A)'_{\bar{1}}}(b)\} = \inf_{y=a+b}\{\varphi(\mu^N_{A'_{\bar{0}}})(a) \vee \varphi(\mu^N_{A'_{\bar{1}}})(b)\} \\
&= \inf_{y=a+b}\{\inf_{a=\varphi(m)}\{\mu^N_{A'_{\bar{0}}}(m)\} \vee \inf_{b=\varphi(n)}\{\mu^N_{A'_{\bar{1}}}(n)\}\} \\
&= \inf_{y=\varphi(x)}\{\inf_{x=m+n}\{\mu^N_{A'_{\bar{0}}}(m) \vee \mu^N_{A'_{\bar{1}}}(n)\}\} \\
&= \inf_{y=\varphi(x)}\{(\mu^N_{A'_{\bar{0}}+A'_{\bar{1}}})(x)\} = \inf_{y=\varphi(x)}\{\mu^N_A(x)\} = \mu^N_{\varphi(A)}(y).
\end{aligned}$$

So $\varphi(A) = \varphi(A)_{\bar{0}} \oplus \varphi(A)_{\bar{1}}$ is a $\mathbb{Z}_2$-graded bipolar fuzzy vector subspace.

Let $x, y \in \mathbb{L}'$. It is enough to show $\mu^P_{\varphi(A)}([x,y]) \geq \mu^P_{\varphi(A)}(x) \wedge \mu^P_{\varphi(A)}(y)$ and $\mu^N_{\varphi(A)}([x,y]) \leq \mu^N_{\varphi(A)}(x) \vee \mu^N_{\varphi(A)}(y)$. If $\mu^P_{\varphi(A)}([x,y]) < \mu^P_{\varphi(A)}(x) \wedge \mu^P_{\varphi(A)}(y)$, we have $\mu^P_{\varphi(A)}([x,y]) < \mu^P_{\varphi(A)}(x)$ and $\mu^P_{\varphi(A)}([x,y]) < \mu^P_{\varphi(A)}(y)$. We choose a number $t \in [0,1]$ such that $\mu^P_{\varphi(A)}([x,y]) < t < \mu^P_{\varphi(A)}(x)$ and $\mu^P_{\varphi(A)}([x,y]) < t < \mu^P_{\varphi(A)}(y)$. Then there exist $a \in \varphi^{-1}(x), b \in \varphi^{-1}(y)$ such that $\mu^P_A(a) > t$, $\mu^P_A(b) > t$. Since $\varphi([a,b]) = [\varphi(a), \varphi(b)] = [x,y]$, we have $\mu^P_{\varphi(A)}([x,y]) = \sup\limits_{[x,y]=\varphi([a,b])}\{\mu^P_A([a,b])\} \geq \mu^P_A([a,b]) \geqslant \mu^P_A(a) \wedge \mu^P_A(b) > t > \mu^P_{\varphi(A)}([x,y])$. This is a contradiction.

Suppose that $\mu^N_{\varphi(A)}([x,y]) > \mu^N_{\varphi(A)}(x) \vee \mu^N_{\varphi(A)}(y)$, we have $\mu^N_{\varphi(A)}([x,y]) > \mu^N_{\varphi(A)}(x)$ and $\mu^N_{\varphi(A)}([x,y]) > \mu^N_{\varphi(A)}(y)$. We choose $t \in [-1,0]$ such that $\varphi(\mu^N)([x,y]) > t > \mu^N_{\varphi(A)}(x)$ and $\mu^N_{\varphi(A)}([x,y]) > t > \mu^N_{\varphi(A)}(y)$. Then there exist $a \in \varphi^{-1}(x), b \in \varphi^{-1}(y)$ such that $\mu^N_A(a) < t, \mu^N_A(b) < t$. Since $\varphi([a,b]) = [\varphi(a), \varphi(b)] = [x,y]$, we have $\mu^N_{\varphi(A)}([x,y]) = \inf\limits_{[x,y]=\varphi([a,b])}\{\mu^N_A([a,b])\} \leqslant \mu^N_A([a,b]) \leq \mu^N_A(a) \vee \mu^N_A(b) < t < \mu^N_{\varphi(A)}([x,y])$. This is a contradiction.

Therefore, $\varphi(A)$ is a bipolar fuzzy Lie sub-superalgebra of $\mathbb{L}'$. □

We state the following results without proofs.

Proposition 3.11. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a surjective Lie homomorphism. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy ideal of $\mathbb{L}$, then $\varphi(A)$ is also a bipolar fuzzy ideal of $\mathbb{L}'$.* $\square$

Theorem 3.12. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a surjective Lie homomorphism. Then for any bipolar fuzzy ideals $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B, \mu_B^N)$ of $\mathbb{L}$ we have $\varphi(A + B) = \varphi(A) + \varphi(B)$.* $\square$

4. Bipolar fuzzy bracket product

Definition 4.1. For any bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ of $\mathbb{L}$, we define the *bipolar fuzzy bracket product* $[A, B] = (\mu_{[A,B]}^P, \mu_{[A,B]}^N)$ putting

$$\mu_{[A,B]}^P(x) = \begin{cases} \sup\limits_{x=\sum\limits_{i\in N} \alpha_i[x_i,y_i]} \{\min\limits_{i\in N}\{\mu_A^P(x_i) \wedge \mu_B^P(y_i)\}\} & \text{where } \alpha_i \in F,\ x_i, y_i \in \mathbb{L} \\ 0 & \text{if } x \text{ is not expressed as } x = \sum\limits_{i\in N} \alpha_i[x_i, y_i] \end{cases}$$

and

$$\mu_{[A,B]}^N(x) = \begin{cases} \inf\limits_{x=\sum\limits_{i\in N} \alpha_i[x_i,y_i]} \{\max\limits_{i\in N}\{\mu_A^N(x_i) \vee \mu_B^N(y_i)\}\} & \text{where } \alpha_i \in F,\ x_i, y_i \in \mathbb{L} \\ 0 & \text{if } x \text{ is not expressed as } x = \sum\limits_{i\in N} \alpha_i[x_i, y_i] \end{cases}.$$

Lemma 4.2. *Let $A_1 = (\mu_{A_1}^P, \mu_{A_1}^N), A_2 = (\mu_{A_2}^P, \mu_{A_2}^N), B_1 = (\mu_{B_1}^P, \mu_{B_1}^N)$ and $B_2 = (\mu_{B_2}^P, \mu_{B_2}^N)$ be bipolar fuzzy sets of $\mathbb{L}$ such that $A_1 \subseteq A_2$, $B_1 \subseteq B_2$. Then $[A_1, B_1] \subseteq [A_2, B_2]$. In particular, if $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ are bipolar fuzzy sets of $\mathbb{L}$, then $[A_1, B] \subseteq [A_2, B]$ and $[A, B_1] \subseteq [A, B_2]$.* $\square$

Lemma 4.3. *Let $A_1 = (\mu_{A_1}^P, \mu_{A_1}^N), A_2 = (\mu_{A_2}^P, \mu_{A_2}^N), B_1 = (\mu_{B_1}^P, \mu_{B_1}^N), B_2 = (\mu_{B_2}^P, \mu_{B_2}^N)$ and $A = (\mu_A^P, \mu_A^N), B = (\mu_B^P, \mu_B^N)$ be any bipolar fuzzy vector subspaces of $\mathbb{L}$. Then $[A_1+A_2, B] = [A_1, B]+[A_2, B]$ and $[A, B_1+B_2] = [A, B_1]+[A, B_2]$.* $\square$

Lemma 4.4. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy vector subspaces of $\mathbb{L}$. Then for any $\alpha, \beta \in F$, we have $[\alpha A, B] = \alpha[A, B]$ and $[A, \beta B] = \beta[A, B]$.* $\square$

Theorem 4.5. *Let $A_1 = (\mu_{A_1}^P, \mu_{A_1}^N), A_2 = (\mu_{A_2}^P, \mu_{A_2}^N), B_1 = (\mu_{B_1}^P, \mu_{B_1}^N), B_2 = (\mu_{B_2}^P, \mu_{B_2}^N)$ and $A = (\mu_A^P, \mu_A^N), B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy vector subspaces of $\mathbb{L}$. Then for any $\alpha, \beta \in F$, we have*

$$[\alpha A_1 + \beta A_2, B] = \alpha[A_1, B] + \beta[A_2, B],$$

$$[A, \alpha B_1 + \beta B_2] = \alpha[A, B_1] + \beta[A, B_2].$$

Proof. The results follow from Theorem 4.3 and Lemma 4.4. $\square$

Lemma 4.6. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be any two bipolar fuzzy vector subspaces of $\mathbb{L}$. Then $[A, B]$ is a bipolar fuzzy vector subspace of $\mathbb{L}$.* $\square$

Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be $\mathbb{Z}_2$-graded bipolar fuzzy vector subspaces of $\mathbb{L}$. Then $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $B = B_{\bar{0}} \oplus B_{\bar{1}}$, where $A_{\bar{0}}, B_{\bar{0}}$ are bipolar fuzzy vector subspaces of $\mathbb{L}_{\bar{0}}$ and $A_{\bar{1}}, B_{\bar{1}}$ are bipolar fuzzy vector subspaces of $\mathbb{L}_{\bar{1}}$.

We define:

- $[A_{\bar{0}}, B_{\bar{0}}] = (\mu^P_{[A_{\bar{0}}, B_{\bar{0}}]}, \mu^N_{[A_{\bar{0}}, B_{\bar{0}}]})$, where

$$\mu^P_{[A_{\bar{0}}, B_{\bar{0}}]}(x) = \sup_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{0}}}(x_i) \wedge \mu^P_{B_{\bar{0}}}(y_i)\}\}$$

and

$$\mu^N_{[A_{\bar{0}}, B_{\bar{0}}]}(x) = \inf_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\max_{i\in N}\{\mu^N_{A_{\bar{0}}}(x_i) \vee \mu^N_{B_{\bar{0}}}(y_i)\}\},$$

for $x_i \in \mathbb{L}_{\bar{0}}$ and $y_i \in \mathbb{L}_{\bar{0}}$,

- $[A_{\bar{0}}, B_{\bar{1}}] = (\mu^P_{[A_{\bar{0}}, B_{\bar{1}}]}, \mu^N_{[A_{\bar{0}}, B_{\bar{1}}]})$, where

$$\mu^P_{[A_{\bar{0}}, B_{\bar{1}}]}(x) = \sup_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{0}}}(x_i) \wedge \mu^P_{B_{\bar{1}}}(y_i)\}\}$$

and

$$\mu^N_{[A_{\bar{0}}, B_{\bar{1}}]}(x) = \inf_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\max_{i\in N}\{\mu^N_{A_{\bar{0}}}(x_i) \vee \mu^N_{B_{\bar{1}}}(y_i)\}\},$$

for $x_i \in \mathbb{L}_{\bar{0}}$ and $y_i \in \mathbb{L}_{\bar{1}}$,

- $[A_{\bar{1}}, B_{\bar{0}}] = (\mu^P_{[A_{\bar{1}}, B_{\bar{0}}]}, \mu^N_{[A_{\bar{1}}, B_{\bar{0}}]})$, where

$$\mu^P_{[A_{\bar{1}}, B_{\bar{0}}]}(x) = \sup_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{1}}}(x_i) \wedge \mu^P_{B_{\bar{0}}}(y_i)\}\}$$

and

$$\mu^N_{[A_{\bar{1}}, B_{\bar{0}}]}(x) = \inf_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\max_{i\in N}\{\mu^N_{A_{\bar{1}}}(x_i) \vee \mu^N_{B_{\bar{0}}}(y_i)\}\},$$

for $x_i \in \mathbb{L}_{\bar{1}}$ and $y_i \in \mathbb{L}_{\bar{0}}$,

- $[A_{\bar{1}}, B_{\bar{1}}] = (\mu^P_{[A_{\bar{1}}, B_{\bar{1}}]}, \mu^N_{[A_{\bar{1}}, B_{\bar{1}}]})$, where

$$\mu^P_{[A_{\bar{1}}, B_{\bar{1}}]}(x) = \sup_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{1}}}(x_i) \wedge \mu^P_{B_{\bar{1}}}(y_i)\}\}$$

and

$$\mu^N_{[A_{\bar{1}}, B_{\bar{1}}]}(x) = \inf_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\max_{i\in N}\{\mu^N_{A_{\bar{1}}}(x_i) \vee \mu^N_{B_{\bar{1}}}(y_i)\}\},$$

for $x_i \in \mathbb{L}_{\bar{1}}$ and $y_i \in \mathbb{L}_{\bar{1}}$.

Note that $[A_{\bar{0}}, B_{\bar{0}}], [A_{\bar{1}}, B_{\bar{1}}]$ are bipolar fuzzy sets of $\mathbb{L}_{\bar{0}}$ and $[A_{\bar{0}}, B_{\bar{1}}], [A_{\bar{1}}, B_{\bar{0}}]$ are bipolar fuzzy sets of $\mathbb{L}_{\bar{1}}$.

Lemma 4.7. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be any two $\mathbb{Z}_2$-graded bipolar fuzzy vector subspaces of $\mathbb{L}$. Then*

$[A, B]_{\bar{0}} := [A_{\bar{0}}, B_{\bar{0}}] + [A_{\bar{1}}, B_{\bar{1}}]$ *is a bipolar fuzzy vector subspace of* $\mathbb{L}_{\bar{0}}$,

$[A, B]_{\bar{1}} := [A_{\bar{0}}, B_{\bar{1}}] + [A_{\bar{1}}, B_{\bar{0}}]$ *is a bipolar fuzzy vector subspace of* $\mathbb{L}_{\bar{1}}$ *and*

$[A, B]$ *is a $\mathbb{Z}_2$-graded bipolar fuzzy vector subspace of* $\mathbb{L}$.

Proof. Since $[A_{\bar{0}}, B_{\bar{0}}]$ and $[A_{\bar{1}}, B_{\bar{1}}]$ are bipolar fuzzy vector subspaces of $\mathbb{L}_{\bar{0}}$ by Lemma 5.5, we can get that $[A, B]_{\bar{0}} := [A_{\bar{0}}, B_{\bar{0}}] + [A_{\bar{1}}, B_{\bar{1}}]$ is a bipolar fuzzy vector subspace of $\mathbb{L}_{\bar{0}}$ by Lemma 2.6. Similarly, $[A, B]_{\bar{1}} := [A_{\bar{0}}, B_{\bar{1}}] + [A_{\bar{1}}, B_{\bar{0}}]$ is a bipolar fuzzy vector subspace of $\mathbb{L}_{\bar{1}}$. We define $[A, B]'_{\bar{0}} := [A'_{\bar{0}}, B'_{\bar{0}}] + [A'_1, B'_{\bar{1}}]$ and $[A, B]'_{\bar{1}} := [A'_{\bar{0}}, B'_{\bar{1}}] + [A'_1, B'_{\bar{0}}]$.

Let $x \in \mathbb{L}_{\bar{0}}$. We have

$$
\begin{aligned}
\mu^P_{[A,B]'_{\bar{0}}}(x) &= (\mu^P_{[A'_{\bar{0}},B'_{\bar{0}}]+[A'_{\bar{1}},B'_{\bar{1}}]})(x) \\
&= \sup_{x=a+b} \{\mu^P_{[A'_{\bar{0}},B'_{\bar{0}}]}(a) \wedge \mu^P_{[A'_{\bar{1}},B'_{\bar{1}}]}(b)\} \\
&= \sup_{x=a+b} \{ \sup_{a=\sum\limits_{i\in N} \alpha_i[k_i,l_i]} \{\min_{i\in N}\{\mu^P_{A'_{\bar{0}}}(k_i) \wedge \mu^P_{B'_{\bar{0}}}(l_i)\}\} \wedge \\
&\qquad\qquad \wedge \sup_{b=\sum\limits_{i\in N} \beta_i[m_i,n_i]} \{\min_{i\in N}\{\mu^P_{A'_{\bar{1}}}(m_i) \wedge \mu^P_{B'_{\bar{1}}}(n_i)\}\} \\
&= \sup_{x=a+b} \{ \sup_{a=\sum\limits_{i\in N} \alpha_i[k_i,l_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{0}}}(k_i) \wedge \mu^P_{B_{\bar{0}}}(l_i)\}\} \wedge \\
&\qquad\qquad \wedge \sup_{b=\sum\limits_{i\in N} \beta_i[m_i,n_i]} \{\min_{i\in N}\{\mu^P_{A_{\bar{1}}}(m_i) \wedge \mu^P_{B_{\bar{1}}}(n_i)\}\} \\
&= \sup_{x=a+b} \{\mu^P_{[A_{\bar{0}},B_{\bar{0}}]}(a) \wedge \mu^P_{[A_{\bar{1}},B_{\bar{1}}]}(b)\} = (\mu^P_{[A_{\bar{0}},B_{\bar{0}}]+[A_{\bar{1}},B_{\bar{1}}]})(x) = \mu^P_{[A,B]_{\bar{0}}}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu^N_{[A,B]'_{\bar{0}}}(x) &= (\mu^N_{[A'_{\bar{0}},B'_{\bar{0}}]+[A'_{\bar{1}},B'_{\bar{1}}]})(x) = \inf_{x=a+b}\{\mu^N_{[A'_{\bar{0}},B'_{\bar{0}}]}(a) \vee \mu^N_{[A'_{\bar{1}},B'_{\bar{1}}]}(b)\} \\
&= \inf_{x=a+b} \{ \inf_{a=\sum\limits_{i\in N} \alpha_i[k_i,l_i]} \{\max_{i\in N}\{\mu^N_{A'_{\bar{0}}}(k_i) \vee \mu^N_{B'_{\bar{0}}}(l_i)\}\} \vee \\
&\qquad\qquad \vee \inf_{b=\sum\limits_{i\in N} \beta_i[m_i,n_i]} \{\max_{i\in N}\{\mu^N_{A'_{\bar{1}}}(m_i) \vee \mu^N_{B'_{\bar{1}}}(n_i)\}\}
\end{aligned}
$$

$$= \inf_{x=a+b}\{\inf_{a=\sum\limits_{i\in N}\alpha_i[k_i,l_i]}\{\max_{i\in N}\{\mu^N_{A_{\bar 0}}(k_i)\vee\mu^N_{B_{\bar 0}}(l_i)\}\}\vee$$

$$\vee\inf_{b=\sum\limits_{i\in N}\beta_i[m_i,n_i]}\{\max_{i\in N}\{\mu^N_{A_{\bar 1}}(m_i)\vee\mu^N_{B_{\bar 1}}(n_i)\}\}$$

$$= \inf_{x=a+b}\{\mu^N_{[A_{\bar 0},B_{\bar 0}]}(a)\vee\mu^N_{[A_{\bar 1},B_{\bar 1}]}(b)\} = (\mu^N_{[A_{\bar 0},B_{\bar 0}]+[A_{\bar 1},B_{\bar 1}]})(x) = \mu^N_{[A,B]_{\bar 0}}(x).$$

Now let $x\notin\mathbb{L}_{\bar 0}$. Then $\mu^P_{[A,B]'_{\bar 0}}(x) = 0$ and $\mu^N_{[A,B]'_{\bar 0}}(x) = -1$. Similarly, for $x\in\mathbb{L}_{\bar 1}$, we have $\mu^P_{[A,B]'_{\bar 1}}(x) = \mu^P_{[A,B]_{\bar 1}}(x)$ and $\mu^N_{[A,B]'_{\bar 1}}(x) = \mu^N_{[A,B]_{\bar 1}}(x)$. For $x\notin\mathbb{L}_{\bar 1}$, we have $\mu^P_{[A,B]'_{\bar 1}}(x) = 0$ and $\mu^N_{[A,B]'_{\bar 1}}(x) = -1$. Thus $[A,B]'_{\bar 0}$ and $[A,B]'_{\bar 1}$ are the extensions of $[A,B]_{\bar 0}$ and $[A,B]_{\bar 1}$.

Clearly, $[A,B]'_{\bar 0}\cap[A,B]'_{\bar 1} = (\mu^P_{[A,B]'_{\bar 0}\cap[A,B]'_{\bar 1}},\mu^N_{[A,B]'_{\bar 0}\cap[A,B]'_{\bar 1}})$, where

$$\mu^P_{[A,B]'_{\bar 0}\cap[A,B]'_{\bar 1}}(x) = \mu^P_{[A,B]'_{\bar 0}}(x)\wedge\mu^P_{[A,B]'_{\bar 1}}(x) = \begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases},$$

$$\mu^N_{[A,B]'_{\bar 0}\cap[A,B]'_{\bar 1}}(x) = \mu^N_{[A,B]'_{\bar 0}}(x)\vee\mu^N_{[A,B]'_{\bar 1}}(x) = \begin{cases}-1 & x=0\\ \ \ 0 & x\neq 0\end{cases}.$$

For $x\in\mathbb{L}$ we have

$$\begin{aligned}[A,B](x) &= [A'_{\bar 0}+A'_{\bar 1},B'_{\bar 0}+B'_{\bar 1}](x) = ([A'_{\bar 0},B'_{\bar 0}]+[A'_{\bar 1},B'_{\bar 1}]+[A'_{\bar 0},B'_{\bar 1}]+[A'_{\bar 1},B'_{\bar 0}])(x)\\ &= ([A,B]'_{\bar 0}+[A,B]'_{\bar 1})(x).\end{aligned}$$

Hence $[A,B] = [A,B]_{\bar 0}\oplus[A,B]_{\bar 1}$ is a $\mathbb{Z}_2$-graded bipolar fuzzy vector subspace. □

Lemma 4.8. *Let $A = (\mu^P_A,\mu^N_A)$ and $B = (\mu^P_B,\mu^N_B)$ be any two $\mathbb{Z}_2$-graded bipolar fuzzy vector subspaces of $\mathbb{L}$. Then $[A,B] = [B,A]$.* □

The following theorem is our main theorem in this section. The proof is base on Lemma 4.8. The left is similar to intuitionistic fuzzy ideal of Lie superalgebras. For more details see [10].

Theorem 4.9. *Let $A = (\mu^P_A,\mu^N_A)$ and $B = (\mu^P_B,\mu^N_B)$ be any two bipolar fuzzy ideals of $\mathbb{L}$. Then $[A,B]$ is also a bipolar fuzzy ideal of $\mathbb{L}$.* □

5. Solvable and nilpotent bipolar fuzzy ideals

Definition 5.1. Let $A = (\mu^P_A,\mu^N_A)$ be a bipolar fuzzy ideal of $\mathbb{L}$. Define inductively a sequence of bipolar fuzzy ideals of $\mathbb{L}$ by $A^{(0)} = A$, $A^{(1)} = [A^{(0)},A^{(0)}]$, $A^{(2)} = [A^{(1)},A^{(1)}],\cdots,A^{(n)} = [A^{(n-1)},A^{(n-1)}]$, then $A^{(n)}$ is called the *nth derived bipolar fuzzy ideal* of $\mathbb{L}$. In which, $A^{(i+1)} = (\mu^P_{A^{(i+1)}},\mu^N_{A^{(i+1)}})$, where

$$\mu^P_{A^{(i+1)}}(x) = \begin{cases} \sup\limits_{x=\sum\limits_{j\in N}\alpha_j[x_j,y_j]} \{\min\limits_{j\in N}\{\mu^P_{A^{(i)}}(x_j) \wedge \mu^P_{A^{(i)}}(y_j)\}\} & \text{where } \alpha_j \in F,\ x_j, y_j \in \mathbb{L} \\ 0 & \text{if } x \text{ is not expressed as } x = \sum\limits_{j\in N} \alpha_j[x_j, y_j] \end{cases}$$

and

$$\mu^N_{A^{(i+1)}}(x) = \begin{cases} \inf\limits_{x=\sum\limits_{j\in N}\alpha_j[x_j,y_j]} \{\max\limits_{j\in N}\{\mu^N_{A^{(i)}}(x_j) \vee \mu^N_{A^{(i)}}(y_j)\}\} & \text{where } \alpha_j \in F,\ x_j, y_j \in \mathbb{L} \\ 0 & \text{if } x \text{ is not expressed as } x = \sum\limits_{j\in N} \alpha_j[x_j, y_j]. \end{cases}$$

.

From the definition, we can get $\mu^P_{A^{(0)}} \supseteq \mu^P_{A^{(1)}} \supseteq \mu^P_{A^{(2)}} \supseteq \cdots \supseteq \mu^P_{A^{(n)}} \supseteq \cdots$ and $\mu^N_{A^{(0)}} \subseteq \mu^N_{A^{(1)}} \subseteq \mu^N_{A^{(2)}} \subseteq \cdots \subseteq \mu^N_{A^{(n)}} \subseteq \cdots$.

Definition 5.2. Let $A^{(n)}$ be as above. Define: $\eta^{(n)} = \sup\{\mu^P_{A^{(n)}}(x) : 0 \neq x \in \mathbb{L}\}$ and $\kappa^{(n)} = \inf\{\mu^N_{A^{(n)}}(x) : 0 \neq x \in \mathbb{L}\}$. Then it is clear that $\eta^{(0)} \geqslant \eta^{(1)} \geqslant \eta^{(2)} \geqslant \cdots \geqslant \eta^{(n)} \geqslant \cdots$ and $\kappa^{(0)} \leqslant \kappa^{(1)} \leqslant \kappa^{(1)} \leqslant \cdots \leqslant \kappa^{(n)} \leqslant \cdots$.

Definition 5.3. An bipolar fuzzy ideal $A = (\mu^P_A, \mu^N_A)$ of $\mathbb{L}$ is called *solvable*, if there is a positive integer n such that $\eta^{(n)} = 0$ and $\kappa^{(n)} = 0$. So, it is a solvable bipolar fuzzy ideal, then there is positive integer n such that $\mu^P_{A^{(n)}} = 1_0$ and $\mu^N_{A^{(n)}} = (-1)_0$.

Example 5.4. For the Lie superalgebra L from Example 3.3 we define $A_{\bar{0}} = (\mu^P_{A_{\bar{0}}}, \mu^N_{A_{\bar{0}}})$, where $\mu^P_{A_{\bar{0}}}(x) = 1, \mu^N_{A_{\bar{0}}}(x) = -1$ for all $x \in N_{\bar{0}}$. Then it is a bipolar fuzzy subspace of $N_{\bar{0}}$. Let $x \in N_{\bar{1}}$. Then $x = k_1a_1 + k_2a_2 + k_3b_1 + k_4b_2$, for $k_i \neq 0$ and $i = 1,2,3,4$. We define $A_{\bar{1}} = (\mu^P_{A_{\bar{1}}}, \mu^N_{A_{\bar{1}}})$ where $\mu^P_{A_{\bar{1}}}(x) = \mu^P_{A_{\bar{1}}}(a_1) \wedge \mu^P_{A_{\bar{1}}}(a_2) \wedge \mu^P_{A_{\bar{1}}}(b_1) \wedge \mu^P_{A_{\bar{1}}}(b_2)$, in which $\mu^P_{A_{\bar{1}}}(a_1) = 0.2$, $\mu^P_{A_{\bar{1}}}(a_2) = 1$, $\mu^P_{A_{\bar{1}}}(b_1) = 0.1$, $\mu^P_{A_{\bar{1}}}(b_2) = 1$, $\mu^P_{A_{\bar{1}}}(0) = 1$, and $\mu^N_{A_{\bar{1}}}(x) = \mu^N_{A_{\bar{1}}}(a_1) \vee \mu^N_{A_{\bar{1}}}(a_2) \vee \mu^N_{A_{\bar{1}}}(b_1) \vee \mu^N_{A_{\bar{1}}}(b_2)$, in which $\mu^N_{A_{\bar{1}}}(a_1) = -0.7$, $\mu^N_{A_{\bar{1}}}(a_2) = -1$, $\mu^N_{A_{\bar{1}}}(b_1) = -0.9$, $\mu^N_{A_{\bar{1}}}(b_2) = -1$, $\mu^N_{A_{\bar{1}}}(0) = -1$. Then A is a bipolar fuzzy subspace of $N_{\bar{1}}$.

Let $x \in N$. Then $x = ke + k_1a_1 + k_2a_2 + k_3b_1 + k_4b_2$ for $k, k_i \neq 0$ and $i = 1,2,3,4$. We define $A = (\mu^P_A, \mu^N_A)$ where $\mu^P_A(x) = \mu^P_A(e) \wedge \mu^P_A(a_1) \wedge \mu^P_A(a_2) \wedge \mu^P_A(b_1) \wedge \mu^P_A(b_2)$, in which $\mu^P_A(e) = 1, \mu^P_A(a_1) = 0.2$, $\mu^P_A(a_2) = 1$, $\mu^P_A(b_1) = 0.1$, $\mu^P_A(b_2) = 1$, $\mu^P_A(0) = 1$ and $\mu^N_A(x) = \mu^N_B(e) \vee \mu^N_A(a_1) \vee \mu^N_A(a_2) \vee \mu^N_A(b_1) \vee \mu^N_A(b_2)$, in which $\mu^N_A(e) = -1, \mu^N_A(a_1) = -0.7$, $\mu^N_A(a_2) = -1$, $\mu^N_A(b_1) = -0.9$, $\mu^N_A(b_2) = -1$, $\mu^N_A(0) = -1$. Then $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a bipolar fuzzy ideal of N.

Let $A^{(0)} = A$. Note that $[a_i, b_i] = e$ and the other brackets are zero. Then $\mu^P_{A^{(0)}}(x) = 0.1$, $\mu^N_{A^{(0)}}(x) = -0.7$. We define $A^{(1)} = [A^{(0)}, A^{(0)}]$. If $x \in N_1$, then x can not be expressed as $x = \sum \alpha_i[x_i, y_i]$, $x_i, y_i \in N$, so $\mu^P_{A^{(1)}}(x) = 0, \mu^N_{A^{(1)}}(x) = 0$. If $x \in N_0$, then x can be expressed as $x = \alpha_1[a_1, b_1] + \alpha_2[a_2, b_2]$, $\alpha_1, \alpha_2 \in k$. We calculate

$$\mu^P_{A^{(1)}}(x) = \sup_{x=\sum\limits_{i=1,2}\alpha_i[a_i,b_i]} \{\min_{i=1,2}\{\mu^P_{A^{(0)}}(a_i) \wedge \mu^P_{A^{(0)}}(b_i)\}\} = 0.1,$$
$$\mu^N_{A^{(1)}}(x) = \inf_{x=\sum\limits_{i=1,2}\alpha_i[a_i,b_i]} \{\max_{i=1,2}\{\mu^N_{A^{(0)}}(a_i) \vee \mu^N_{A^{(0)}}(b_i)\}\} = -1.$$

Define $A^{(2)} = [A^{(1)}, A^{(1)}]$, we calculate

$$\mu^P_{A^{(2)}}(x) = \sup_{x=\sum\limits_{i=1,2}\alpha_i[a_i,b_i]} \{\min_{i=1,2}\{\mu^P_{A^{(1)}}(a_i) \wedge \mu^P_{A^{(1)}}(b_i)\}\} = 0,$$
$$\mu^N_{A^{(2)}}(x) = \inf_{x=\sum\limits_{i=1,2}\alpha_i[a_i,b_i]} \{\max_{i=1,2}\{\mu^N_{A^{(1)}}(a_i) \vee \mu^N_{A^{(1)}}(b_i)\}\} = 0.$$

So, $\eta^{(0)} \geqslant \eta^{(1)} \geqslant \eta^{(2)} = 0$ and $\kappa^{(0)} \leqslant \kappa^{(1)} \leqslant \kappa^{(2)} = 0$. These show that A is a solvable bipolar fuzzy ideal of N. □

From the definition of solvable bipolar fuzzy ideals, we can easily get

Lemma 5.5. *Let $A = (\mu^P_A, \mu^N_A)$ be a bipolar fuzzy Lie ideal of $\mathbb{L}$. Then $A = (\mu^P_A, \mu^N_A)$ is a solvable bipolar fuzzy ideal if and only if there is a positive integer n such that $\mu^P_{A^{(m)}} = 1_0, \mu^N_{A^{(m)}} = (-1)_0$ for all $m \geqslant n$.* □

Theorem 5.6. *Homomorphic images of solvable bipolar fuzzy ideals are also solvable bipolar fuzzy Lie ideals.*

Proof. Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a homomorphism of Lie superalgebra and assume that $A = (\mu^P_A, \mu^N_A)$ is a bipolar fuzzy ideal of $\mathbb{L}$. Let $\varphi(A) = B$, i.e, $\mu^P_B = \mu^P_{\varphi(A)}, \mu^N_B = \mu^N_{\varphi(A)}$. We prove $\mu^P_{\varphi(A^{(n)})} = \mu^P_{B^{(n)}}$ and $\mu^N_{\varphi(A^{(n)})} = \mu^N_{B^{(n)}}$ by induction on n, where n is any positive integer. Indeed, let $y \in \mathbb{L}'$. Consider $n = 1$,

$$\begin{aligned}
\mu^P_{\varphi(A^{(1)})}(y) = \mu^P_{\varphi([A,A])}(y) &= \sup_{y=\varphi(x)} \{\mu^P_{[A,A]}(x)\} \\
&= \sup_{y=\varphi(x)} \{ \sup_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]} \{\min_{i\in N}(\mu^P_A(x_i) \wedge \mu^P_A(y_i))\}\} \\
&= \sup_{y=\sum\limits_{i\in N}\alpha_i\varphi[x_i,y_i]} \{\min_{i\in N}(\mu^P_A(x_i) \wedge \mu^P_A(y_i))\} \\
&= \sup_{y=\sum\limits_{i\in N}\alpha_i[a_i,b_i]} \{\min_{i\in N}(\mu^P_A(x_i) \wedge \mu^P_A(y_i)) : \varphi(x_i) = a_i, \varphi(y_i) = b_i\} \\
&= \sup_{\sum\limits_{i\in N}\alpha_i[a_i,b_i]=y} \{\min_{i\in N}(\mu^P_B(a_i) \wedge \mu^P_B(b_i))\} = \mu^P_{[B,B]}(y) = \mu^P_{B^{(1)}}(y),
\end{aligned}$$

and

$$\begin{aligned}\mu^N_{\varphi(A^{(1)})}(y) = \mu^N_{\varphi([A,A])}(y) &= \inf_{y=\varphi(x)}\{\mu^N_{[A,A]}(x)\}\\
&= \inf_{y=\varphi(x)}\{\inf_{x=\sum\limits_{i\in N}\alpha_i[x_i,y_i]}\{\max_{i\in N}(\mu^N_A(x_i)\vee\mu^N_A(y_i))\}\}\\
&= \inf_{y=\sum\limits_{i\in N}\alpha_i\varphi[x_i,y_i]}\{\max_{i\in N}(\mu^N_A(x_i)\vee\mu^N_A(y_i))\}\\
&= \inf_{y=\sum\limits_{i\in N}\alpha_i[a_i,b_i]}\{\max_{i\in N}(\mu^N_A(x_i)\vee\mu^N_A(y_i)):\varphi(x_i)=a_i,\varphi(y_i)=b_i\}\\
&= \inf_{\sum\limits_{i\in N}\alpha_i[a_i,b_i]=y}\{\max_{i\in N}(\mu^N_B(a_i)\vee\mu^N_B(b_i))\} = \mu^N_{[B,B]}(y) = \mu^N_{B^{(1)}}(y).\end{aligned}$$

These prove the case of $n=1$. Suppose that the case of $n-1$ is true, then $\mu^P_{\varphi(A^{(n)})} = \mu^P_{\varphi([A^{(n-1)},A^{(n-1)}])} = \mu^P_{[\varphi(A^{(n-1)}),\varphi(A^{(n-1)})]} = \mu^P_{[B^{(n-1)},B^{(n-1)}]} = \mu^P_{B^{(n)}}$ and $\mu^N_{\varphi(A^{(n)})} = \mu^N_{\varphi([A^{(n-1)},A^{(n-1)}])} = \mu^N_{[\varphi(A^{(n-1)}),\varphi(A^{(n-1)})]} = \mu^N_{[B^{(n-1)},B^{(n-1)}]} = \mu^N_{B^{(n)}}$. Let m be a positive integer such that $\mu^P_{A^{(m)}} = 1_0$ and $\mu^N_{A^{(m)}} = (-1)_0$. Then for any $0\neq y\in\mathbb{L}'$, we get $\mu^P_{B^{(m)}}(y) = \mu^P_{\varphi(A^{(m)})}(y) = \sup\limits_{y=\varphi(x)}\{1_0(x)\} = 0$, $\mu^N_{B^{(m)}}(y) = \varphi(\mu^N_{A^{(m)}})(y) = \inf\limits_{y=\varphi(x)}\{(-1)_0(x)\} = 0$. So $\mu^P_{B^{(m)}} = 1_0$ and $\mu^N_{B^{(m)}} = (-1)_0$. □

Let $A=(\mu^P_A,\mu^N_A)$ be a bipolar fuzzy ideal of $\mathbb{L}$ and I be an ideal of $\mathbb{L}$. We can prove that A/I is a bipolar fuzzy ideal of $\mathbb{L}/I$.

Theorem 5.7. *Let $A=(\mu^P_A,\mu^N_A)$ be an bipolar fuzzy ideal of $\mathbb{L}$ and A/I be a solvable bipolar fuzzy ideal of $\mathbb{L}/I$. If $B=(\mu^P_B,\mu^N_B)$ is a solvable bipolar fuzzy ideal of $\mathbb{L}$ and is also a bipolar fuzzy ideal of $A=(\mu^P_A,\mu^N_A)$ such that $B(I)=A(I)$, then $A=(\mu^P_A,\mu^N_A)$ is solvable.*

Proof. Let φ be the canonical projection from $\mathbb{L}$ to $\mathbb{L}/I$. From the proof of Theorem 5.6, we get $\mu^P_{\varphi(A^{(n)})} = \mu^P_{(A/I)^{(n)}}$ and $\mu^N_{\varphi(A^{(n)})} = \mu^N_{(A/I)^{(n)}}$. Since A/I is solvable, there exists n such that $\mu^P_{(A/I)^{(n)}} = 1_0$ and $\mu^N_{(A/I)^{(n)}} = (-1)_0$.

For $0\neq\bar{y}\in\mathbb{L}/I$, we have $\sup\limits_{m\in\varphi^{-1}(\bar{y})}\{\mu^P_{A^{(n)}}(m)\} = \mu^P_{\varphi(A^{(n)})}(\bar{y}) = \mu^P_{(A/I)^{(n)}}(\bar{y}) = 0$ and $\inf\limits_{m\in\varphi^{-1}(\bar{y})}\{\mu^N_{A^{(n)}}(m)\} = \mu^N_{\varphi(A^{(n)})}(\bar{y}) = \mu^N_{(A/I)^{(n)}}(\bar{y}) = 0$. Notice that $m\in\mathbb{L}$ and $m\neq 0$, we get $\mu^P_{A^{(n)}}(m)=0$ and $\mu^N_{A^{(n)}}(m)=0$.

For $\bar{y}=0$, we have $\sup\limits_{m\in\varphi^{-1}(0)}\{\mu^P_{A^{(n)}}(m)\} = \mu^P_{\varphi(A^{(n)})}(0) = 1$ and $\inf\limits_{m\in\varphi^{-1}(0)}\{\mu^N_{A^{(n)}}(m)\}$ $= \mu^N_{\varphi(A^{(n)})}(0) = -1$. Since $\varphi^{-1}(0) = I$ and $B(I)=A(I)$, we have $\mu^P_{B^{(n)}}(I) =$

$\mu^P_{A^{(n)}}(I)$ and $\mu^N_{B^{(n)}}(I) = \mu^N_{A^{(n)}}(I)$. For any $x \in I$, B is solvable, then there exists n such that $\mu^P_{B^{(n)}} = 1_0$ and $\mu^N_{B^{(n)}} = (-1)_0$, we have $\mu^P_{A^{(n)}} = 1_0$ and $\mu^N_{A^{(n)}} = (-1)_0$.

Hence for any $x \in \mathbb{L}$, we always have that $\mu^P_{A^{(n)}} = 1_0$ and $\mu^N_{A^{(n)}} = (-1)_0$, which imply that $A = (\mu^P_A, \mu^N_A)$ is solvable. □

Lemma 5.8. *Let $A = (\mu^P_A, \mu^N_A)$ and $B = (\mu^P_B, \mu^N_B)$ be bipolar fuzzy ideals of $\mathbb{L}$. Then $(A \oplus B)^{(n)} = A^{(n)} \oplus B^{(n)}$.*

Proof. Let $0 \neq x \in \mathbb{L}$. Then we have $[A, B] = (\mu^P_{[A,B]}, \mu^P_{[A,B]})$, where

$$\mu^P_{[A,B]}(x) = \sup_{x=\sum\limits_{iBN} \alpha_i[x_i,y_i]} \{\min_{x\in N}(\mu^P_A(x_i) \wedge \mu^P_B(y_i))\} \leq \mu^P_A(x) \wedge \mu^P_B(x) = 0,$$

$$\mu^N_{[A,B]}(x) = \inf_{x=\sum\limits_{iBN} \alpha_i[x_i,y_i]} \{\max_{x\in N}(\mu^N_A(x_i) \vee \mu^N_B(y_i))\} \geq \mu^N_A(x) \vee \mu^N_B(x) = 0.$$

So $\mu^P_{[A,B]} = 1_0$ and $\mu^N_{[A,B]} = (-1)_0$. Consequently, for any positive integer a, b, we have $\mu^P_{[A^{(a)},B^{(b)}]} = 1_0$ and $\mu^N_{[A^{(a)},B^{(b)}]} = (-1)_0$. We prove the lemma by induction on n.

Let $n = 1$. Then

$$(A \oplus B)^{(1)} = [A \oplus B, A \oplus B] = [A, A] \oplus [A, B] \oplus [B, A] \oplus [B, B] = A^{(1)} \oplus B^{(1)}.$$

Suppose that the case of $n - 1$ is true, then

$$\begin{aligned}(A \oplus B)^{(n)} &= [(A \oplus B)^{(n-1)}, (A \oplus B)^{(n-1)}] \\ &= [A^{(n-1)} \oplus B^{(n-1)}, A^{(n-1)} \oplus B^{(n-1)}] = A^{(n)} \oplus B^{(n)}.\end{aligned}$$

So we get $(A \oplus B)^{(n)} = A^{(n)} \oplus B^{(n)}$. □

Theorem 5.9. *Direct sum of any solvable bipolar fuzzy Lie ideals is also a solvable bipolar Lie ideal.*

Proof. Let $A = (\mu^P_A, \mu^N_A)$ and $B = (\mu^P_B, \mu^N_B)$ be solvable bipolar fuzzy ideals. Then there exist positive integers m, n such that $\mu^P_{A^{(m)}} = 1_0, \mu^N_{A^{(m)}} = (-1)_0$ and $\mu^P_{B^{(n)}} = 1_0, \mu^N_{B^{(n)}} = (-1)_0$. Since $(A \oplus B)^{(m+n)} = A^{(m+n)} \oplus B^{(m+n)}$, we have $\mu^P_{(A\oplus B)^{(m+n)}} = \mu^P_{A^{(m+n)}\oplus B^{(m+n)}} = 1_0$ and $\mu^N_{(A\oplus B)^{(m+n)}} = \mu^N_{A^{(m+n)}\oplus B^{(m+n)}} = (-1)_0$. So $A \oplus B$ is a solvable bipolar fuzzy Lie ideal. □

Definition 5.10. Let $A = (\mu^P_A, \mu^N_A)$ be a bipolar fuzzy ideal of $\mathbb{L}$. Define inductively a sequence of bipolar fuzzy ideals of $\mathbb{L}$ by $A^0 = A$, $A^1 = [A, A^0]$, $A^2 = [A, A^1], \cdots, A^n = [A, A^{n-1}] \cdots$, which is called the *descending central series* of a bipolar fuzzy ideal $A = (\mu^P_A, \mu^N_A)$ of $\mathbb{L}$. We get $\mu^P_{A^0} \supseteq \mu^P_{A^1} \supseteq \mu^P_{A^2} \supseteq \cdots \supseteq \mu^P_{A^n} \supseteq \cdots$ and $\mu^N_{A^0} \subseteq \mu^N_{A^1} \subseteq \mu^N_{A^2} \subseteq \cdots \subseteq \mu^N_{A^n} \subseteq \cdots$

Definition 5.11. For any bipolar fuzzy Lie ideal $A = (\mu^P_A, \mu^N_A)$, define $\eta^n = \sup\{\mu^P_{A^n}(x) : 0 \neq x \in \mathbb{L}\}$ and $\kappa^n = \inf\{\mu^N_{A^n}(x) : 0 \neq x \in \mathbb{L}\}$, for any positive integer n. The bipolar fuzzy ideal is called a *nilpotent bipolar fuzzy ideal*, if there is a positive integer m such that $\eta^m = 0$ and $\kappa^m = 1$, or equivalently, $\mu^P_{A^m} = 1_0$ and $\mu^N_{A^m} = (-1)_0$.

Example 5.12. Let us take the basis h, e, f of $\mathfrak{sl}(1|1)$ as follows

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Then h is an even element, and e and f are odd element. Their bracket products are as follows: $[e, f] = [f, e] = h$, the other brackets $= 0$. Then $\mathfrak{sl}(1|1)$ is a three-dimensional Lie superalgebra.

Define $A_{\bar{0}} = (\mu^P_{A_{\bar{0}}}, \mu^N_{A_{\bar{0}}}) : \mathfrak{sl}(1|1)_{\bar{0}} \to [-1, 1]$ where

$$\mu^P_{A_{\bar{0}}}(x) = \begin{cases} 0.6 & x = h \\ 1 & \text{otherwise} \end{cases}, \mu^N_{A_{\bar{0}}}(x) = \begin{cases} -0.4 & x = h \\ -1 & \text{otherwise} \end{cases}$$

Define $A_{\bar{1}} = (\mu^P_{A_{\bar{1}}}, \mu^N_{A_{\bar{1}}}) : \mathfrak{sl}(1|1)_{\bar{1}} \to [-1, 1]$ where

$$\mu^P_{A_{\bar{1}}}(x) = \begin{cases} 0.3 & x = e \\ 0.5 & x = f \\ 1 & \text{otherwise} \end{cases}, \mu^N_{A_{\bar{1}}}(x) = \begin{cases} -0.7 & x = e \\ -0.5 & x = f \\ -1 & \text{otherwise} \end{cases}$$

Define $A = (\mu^P_A, \mu^N_A) : \mathfrak{sl}(1|1) \to [-1, 1]$ where $\mu^P_A(x) = \mu^P_{A_{\bar{0}}}(x_{\bar{0}}) \wedge \mu^P_{A_{\bar{1}}}(x_{\bar{1}})$ and $\mu^N_A(x) = \mu^N_{A_{\bar{0}}}(x_{\bar{0}}) \vee \mu^N_{A_{\bar{1}}}(x_{\bar{1}})$. Then A is a bipolar fuzzy ideal of $\mathfrak{sl}(1|1)$.

Let $A^0 = A$. We define $A^1 = [A, A^0]$, then if $x \in \mathfrak{sl}(1|1)_{\bar{1}}$, x can not be expressed as $x = \sum \alpha_i [x_i, y_i]$, $x_i, y_i \in \mathfrak{sl}(1|1)$ then $\mu^P_{A^1}(x) = 0, \mu^N_{A^1}(x) = 0$. If $x \in \mathfrak{sl}(1|1)_{\bar{0}}$, $x = \alpha[e, f]$, $\alpha \in F$, then $\mu^P_{A^1}(x) = \sup\{\mu^P_A(e) \wedge \mu^P_{A^0}(f)\} = 0.3$ and $\mu^N_{A^1}(x) = \inf\{\mu^N_A(e) \vee \mu^N_{A^0}(f)\} = -0.5$.

Define $A^2 = [A, A^1]$, we calculate if $x \in \mathfrak{sl}(1|1)_{\bar{1}}$, $\mu^P_{A^2}(x) = 0, \mu^N_{A^2}(x) = 0$. If $x \in \mathfrak{sl}(1|1)_{\bar{0}}$, $\mu^P_{A^2}(x) = \sup\{\mu^P_A(e) \wedge \mu^P_{A^1}(f)\} = 0$ and $\mu^N_{A^2}(x) = \inf\{\mu^N_A(e) \vee \mu^N_{A^1}(f)\} = 0$. Then we get $\eta^0 \geqslant \eta^1 \geqslant \eta^2 = 0$ and $\kappa^0 \leq \kappa^1 \leqslant \kappa^2 = 0$. So A is a nilpotent bipolar fuzzy Lie ideal of $\mathfrak{sl}(1|1)$. □

Theorem 5.13. *Homomorphic images of nilpotent bipolar fuzzy ideals are also nilpotent bipolar fuzzy Lie ideals. Direct sum of nilpotent bipolar fuzzy ideals is also a nilpotent bipolar fuzzy ideal.* □

Theorem 5.14. *If $A = (\mu^P_A, \mu^N_A)$ is a nilpotent bipolar fuzzy ideal of $\mathbb{L}$, then it is solvable.* □

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