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A characterization of binary invertible algebras of various types of linearity

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Abstract. In this paper we define the left (right) linear over a group binary invertible algebras and invertible algebras of mixed type of linearity and characterize the classes of such algebras by the second-order formula, namely by the $\forall\exists(\forall)-$ identities.

1. Introduction

Linear quasigroups introduced by V.D. Belousov in 1967 in connection with an investigation of balanced identities in quasigroups [2] play a special role in the study of quasigroups isotopic to groups [5, 4, 6, 10, 11].

A binary algebra $(Q;\Sigma)$ is called *invertible*, if $(Q;A)$ is a quasigroup for any operation $A\in\Sigma$.

Below we introduce the notions of left (right) linear invertible algebras and invertible algebras of mixed type of linearity and characterize the classes of such algebras by the second order formulae, namely by the $\forall\exists(\forall)-$ identities.

For details about $\forall\exists(\forall)-$ identities see [9, 12].

2. Left and right linear invertible algebras

We denote by $L_{A,a}$ and $R_{A,a}$ the left and right translations of the binary algebra $(Q;\Sigma)$: $L_{A,a}: x\mapsto A(a,x)$, $R_{A,a}: x\mapsto A(x,a)$. If the algebra $(Q;\Sigma)$ is an invertible algebra then the translations $L_{A,a}$ and $R_{A,a}$ are bijections for all $a\in Q$ and all $A\in\Sigma$.

It is well known (see [2]) that the quasigroups A^{-1}, ${}^{-1}A$, ${}^{-1}\big(A^{-1}\big)$, $\big({}^{-1}A\big)^{-1}$, A^*, where $A^*(x,y)=A(y,x)$, are associated with the quasigroup A.

Similarly, the invertible algebras:

$$\big(Q;\Sigma^{-1}\big)\,,\ \big(Q;\,{}^{-1}\Sigma\big)\,,\ \big(Q;\,{}^{-1}(\Sigma^{-1})\big)\,,\ \big(Q;({}^{-1}\Sigma)^{-1}\big)\,,\ \big(Q;\Sigma^*\big)\,,$$

where

$$\Sigma^{-1}=\{A^{-1}|\ A\in\Sigma\},\ \ {}^{-1}\Sigma=\{\,{}^{-1}A|\ A\in\Sigma\},\ \ {}^{-1}(\Sigma^{-1})=\{\,{}^{-1}(A^{-1})|\ A\in\Sigma\},$$

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$$({}^{-1}\Sigma)^{-1} = \{({}^{-1}A)^{-1} |\ A \in \Sigma\}, \quad \Sigma^* = \{A^* |\ A \in \Sigma\}$$

are associated with the invertible algebra (Q, Σ). Each of these algebras is called the *parastrophy* of the algebra $(Q; \Sigma)$.

Definition 2.1. An invertible algebra $(Q; \Sigma)$ is called *left* (*right*) *linear over a group* $(Q; +)$, if every operation $A \in \Sigma$ has the form:

$$A(x, y) = \varphi_A x + \beta_A y \quad (A(x, y) = \alpha_A x + \psi_A y)\,,$$

where β_A (respectively α_A) is a permutation of the set Q, and φ_A (respectively ψ_A) is an automorphism of the group $(Q; +)$.

An invertible algebra is called *left* (*right*) *linear* if it is left (right) linear over some group $(Q; +)$.

Theorem 2.2. *A binary invertible algebra* $(Q; \Sigma)$ *is left linear if and only if for all* $X, Y \in \Sigma$ *the following formula*

$$X\left(Y\left(x, Y^{-1}(u, y)\right), z\right) = X\left(Y\left(x, Y^{-1}(u, u)\right), X^{-1}\left(u, X(y, z)\right)\right) \tag{1}$$

is valid in the algebra $(Q; \Sigma \cup \Sigma^{-1})$.

Proof. Let $(Q; \Sigma)$ be an invertible left linear algebra. Then for every $X \in \Sigma$ we have

$$X\,(x, y) = \varphi_X x + \beta_X y, \tag{2}$$

where $\varphi_X \in Aut(Q; +)$ and $\beta_X \in S_Q$. We prove that equality (1) is valid in the algebra $(Q; \Sigma \cup \Sigma^{-1})$ for all $X, Y \in \Sigma$.

Observe that from (2) we obtain

$$X^{-1}(x, y) = \beta_X^{-1}\left(-\varphi_X x + y\right). \tag{3}$$

Thus, according to (2) and (3) we get:

$$\begin{aligned} X(Y(x, Y^{-1}(u, y)), z) &= \varphi_X(\varphi_Y x + \beta_Y Y^{-1}(u, y)) + \beta_X z \\ &= \varphi_X(\varphi_Y x + \beta_Y \beta_Y^{-1}(-\varphi_Y u + y)) + \beta_X z \\ &= \varphi_X \varphi_Y x - \varphi_X \varphi_Y u + \varphi_X y + \beta_X z, \end{aligned}$$

$$\begin{aligned} X(Y(x,Y^{-1}(u, u)),X^{-1}(u,X(y, z))) &= \varphi_X Y(x, Y^{-1}(u, u)) + \beta_X X^{-1}(u, X(y, z)) \\ &= \varphi_X(\varphi_Y x - \varphi_Y u + u) - \varphi_X u + \varphi_X y + \beta_X z \\ &= \varphi_X\varphi_Y x - \varphi_X\varphi_Y u + \varphi_X u - \varphi_X u + \varphi_X y + \beta_X z \\ &= \varphi_X \varphi_Y x - \varphi_X \varphi_Y u + \varphi_X y + \beta_X z. \end{aligned}$$

Hence, the right and left sides of (1) are the same.

Conversely, let (1) holds in $(Q;\Sigma\cup\Sigma^{-1})$ for all $X,Y\in\Sigma$. Then for $u=p$ and $X=A$, $Y=B$, where $A,B\in\Sigma$, we have

$$A(B(x,B^{-1}(p,y)),z)=A(B(x,B^{-1}(p,p)),A^{-1}(p,A(y,z))).$$

From this, by putting $A_1(x,y)=A(x,y)$, $A_2(x,y)=B(x,B^{-1}(p,y))$, $A_3(x,y)=A(B(x,B^{-1}(p,p)),y)$ and $A_4(x,y)=A^{-1}(p,A(x,y))$ we obtain

$$A_1(A_2(x,y),z)=A_3(x,A_4(y,z)),$$

which by Belousov's theorem on four quasigroups (see [3]) shows that operations A_1,A_2,A_3,A_4 are isotopic to the same group $(Q;*)$. Hence, the operations A and B are also isotopic to $(Q;*)$. Since the operations A and B are arbitrary, we obtain that all operations from Σ are isotopic to this group.

For every $X\in\Sigma$, let us define the operation:

$$x\underset{X}{+}y=X(R^{-1}_{X,a}x,L^{-1}_{X,b}y),\tag{4}$$

where a,b are some fixed elements from Q. The operation $\underset{X}{+}$ is a loop operation with the identity element $0_X=X(b,a)$. Obviously, $(Q;\underset{X}{+})$ is a loop isotopic to the group $(Q;*)$. Hence, by Albert's theorem, it is a group. Hence every $X\in\Sigma$ each $(Q;\underset{X}{+})$ is a group. So, (1) (where $X=A,\ Y=B$) can be rewritten in the form:

$$A(B(x,L_{B^{-1},u}y),z)=A(R_{B,B^{-1}(u,u)}x,L_{A^{-1},u}A(y,z)),$$

$$R_{A,a}(R_{B,a}x\underset{B}{+}L_{B,b}L_{B^{-1},u}y)\underset{A}{+}L_{A,b}z=R_{A,a}R_{B,B^{-1}(u,u)}x\underset{A}{+}L_{A,b}L_{A^{-1},u}(R_{A,a}y\underset{A}{+}L_{A,b}z).$$

Taking $z=L^{-1}_{A,b}0_A$ in the last equality, we have

$$R_{A,a}(R_{B,a}x\underset{B}{+}L_{B,b}L_{B^{-1},u}y)=R_{A,a}R_{B,B^{-1}(u,u)}x\underset{A}{+}L_{A,b}L_{A^{-1},u}R_{A,a}y,$$

$$R_{A,a}(x\underset{B}{+}y)=\alpha_{A,B}x\underset{A}{+}\beta_{A,B}y,\tag{5}$$

where $\alpha_{A,B}=R_{A,a}R_{B,B^{-1}(u,u)}R^{-1}_{B,a}$, $\beta_{A,B}=L_{A,b}L_{A^{-1},u}R_{A,a}L^{-1}_{B^{-1},u}L^{-1}_{B,b}$ are permutations of the set Q. Since the operations A and B are arbitrary we can take $A=B$ in (5). Hence

$$R_{A,a}(x\underset{A}{+}y)=\alpha_{A,A}x\underset{A}{+}\beta_{A,A}y.\tag{6}$$

From (5) and (6) we have

$$\alpha^{-1}_{A,B}x\underset{B}{+}\beta^{-1}_{A,B}y=\alpha^{-1}_{A,A}x\underset{A}{+}\beta^{-1}_{A,A}y,$$

$$x\underset{A}{+}y=\gamma_{A,B}x\underset{B}{+}\delta_{A,B}y,\tag{7}$$

where $\gamma_{A,B} = \alpha^{-1}_{A,B}\alpha_{A,A}$ and $\delta_{A,B} = \beta^{-1}_{A,B}\beta_{A,A}$ are permutations of the set Q. Hence, according to (7), we get

$$R_{A,a}(x \underset{B}{+} y) = \gamma_{A,B}\alpha_{A,B}x \underset{B}{+} \delta_{A,B}\beta_{A,B}y,$$

i.e., $R_{A,a}$ is a quasiautomorphism of the group $(Q; \underset{B}{+})$. Since A is arbitrary we have that $R_{A,a}$ is a quasiautomorfism of the group $(Q; \underset{B}{+})$ for all operations $A \in \Sigma$.

According to (4) we have

$$A(x, y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

This, according to (7), gives:

$$A(x, y) = \theta^1_{A,B}x \underset{B}{+} \theta^2_{A,B}y, \tag{8}$$

where $\theta^1_{A,B} = \gamma_{A,B}R_{A,a}$ and $\theta^2_{A,B} = \delta_{A,B}L_{A,b}$ are permutations of the set Q. Thus, we can represent every operation from Σ by the operation $\underset{B}{+}$.

Let $+ = \underset{B}{+}$. We prove that $\theta^1_{A,B}$ is a quasiautomorphism of the group $(Q; +)$. To do it we take $z = (\theta^2_{A,B})^{-1}0_B,\ X = A,\ Y = B$ in (1) and rewrite this equality in the form:

$$\theta^1_{A,B}(R_{B,a}x+L_{B,b}L_{B^{-1},u}y)+\theta^2_{A,B}z = \theta^1_{A,B}R_{B,B^{-1}(u,u)}x+\theta^2_{A,B}L_{A^{-1},u}(\theta^1_{A,B}y+\theta^2_{A,B}z),$$

$$\theta^1_{A,B}(R_{B,a}x + L_{B,b}L_{B^{-1},u}y) = \theta^1_{A,B}R_{B,B^{-1}(u,u)}x + \theta^2_{A,B}L_{A^{-1},u}\theta^1_{A,B}y.$$

The last equality shows that $\theta^1_{A,B}$ is a quasiautomorphism of the group $(Q; +)$. According to [2, Lemma 2.5] we have

$$\theta^1_{A,B}x = \varphi_A x + s_A,$$

where φ_A is an automorphism of the group $(Q, +)$ and s_A is some element of the set Q. Hence, it follows from (8) that

$$A(x, y) = \varphi_A x + \beta_A y, \tag{9}$$

where $\beta_A y = s_A + \theta^2_{A,B}y$. Since A is an arbitrary operation we obtain that all operations from Σ can be represented in the form (9), i.e., the algebra $(Q; \Sigma)$ is left linear. □

Similarly, we can prove the following theorem.

Theorem 2.3. *A binary invertible algebra $(Q; \Sigma)$ is a right linear algebra if and only if for all $X, Y \in \Sigma$ the following formula*

$$X(x, Y(^{-1}Y(y, u), z)) = X(^{-1}X(X(x, y), u), Y(^{-1}Y(u, u), z)), \tag{10}$$

is valid in the algebra $(Q; \Sigma \cup {}^{-1}\Sigma)$. □

Proposition 2.4. *A left and right linear invertible algebra is linear.* □

Corollary 2.5. *The class of all invertible linear algebras is characterized by the second order formulaes* (1) *and* (10). □

A linear invertible algebra over an abelian group is called an *invertible T-algebra* (see [7]). The class of medial invertible algebras is a special subclass of invertible T-algebras. An invertible algebra $(Q;\Sigma)$ is called a *left* (*right*) *T-algebra*, briefly a *LT-algebra* (*RT-algebra*) if $(Q;\Sigma)$ is a left (right) linear algebra over an abelian group. It follows from Proposition 2.4, that if an invertible algebra is a LT-algebra and RT-algebra, then it is a T-algebra.

Using the same arguments as in the proof of Theorem 1 from [6] and applying our Theorems 2.2 and 2.3 we obtain

Theorem 2.6. *A binary invertible algebra $(Q;\Sigma)$ is a LT-algebra if and oly if for all $X, Y \in \Sigma$ the following formulaes*

$$X(Y(x, Y^{-1}(u, y)), z) = X(Y(x, Y^{-1}(u, u)), X^{-1}(u, X(y, z))),$$

$$X({}^{-1}X(x, u), X^{-1}(u, y)) = X({}^{-1}X(y, u), X^{-1}(u, x)), \tag{11}$$

are valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$. □

Theorem 2.7. *A binary invertible algebra $(Q;\Sigma)$ is a RT-algebra if and only if for all $X, Y \in \Sigma$ the following formulaes*

$$X(x, Y({}^{-1}Y(y, u), z)) = X({}^{-1}X(X(x, y), u), Y({}^{-1}Y(u, u), z)),$$

$$X({}^{-1}X(x, u), X^{-1}(u, y)) = X({}^{-1}X(y, u), X^{-1}(u, x))$$

are valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$.

Corollary 2.8. *The class of all invertible T-algebras is characterized by the second order formulaes* (1), (10) *and* (11). □

3. Invertible algebras of mixed type of linearity

Definition 3.1. An invertible algebra $(Q;\Sigma)$ is called an *invertible algebra of mixed type of linearity of the first* (*second*) *kind* over a group $(Q;+)$, if every operation $A \in \Sigma$ has the form

$$A(x, y) = \varphi_A x + c_A + \overline{\psi}_A y \quad (A(x, y) = \overline{\varphi}_A x + c_A + \psi_A y),$$

where $\varphi_A, \psi_A \in Aut(Q;+)$, $\overline{\psi}_A, \overline{\varphi}_A$ are antiautomorphisms of $(Q;+)$, and c_A is a fixed element from Q.

Theorem 3.2. *An invertible algebra $(Q;\Sigma)$ is of mixed type of linearity of the first kind if and only if for all $X, Y \in \Sigma$ the following second order formulaes*

$$X(Y(x, Y^{-1}(u, y)), z) = X(Y(x, Y^{-1}(u, u)), X^{-1}(u, X(y, z))), \tag{12}$$

$$X(x, {}^{-1}Y(Y(y, Y^{-1}(u, v)), u)) = X({}^{-1}X(X(x, {}^{-1}Y(v, u)), u), y) \tag{13}$$

are valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$.

Proof. Let $(Q;\Sigma)$ be an invertible algebra of mixed type of linearity of the first kind, then for every $X \in \Sigma$ we have

$$\begin{aligned} X(x,y) &= \varphi_X x + c_X + \overline{\psi}_X y, \\ {}^{-1}X(x,y) &= \varphi_X^{-1}(x - \overline{\psi}_X y - c_X), \\ X^{-1}(x,y) &= \overline{\psi}_X^{-1}(-c_X - \varphi_X x + y), \end{aligned}$$

where $\varphi_X \in Aut(Q;+)$, $\overline{\psi}_X$ is an antiautomorphism of $(Q;+)$ and $c_X \in Q$.

Using the above identities we can prove that the left and right sides of (12) and (13) are the same.

Conversely, let (12) and (13) be valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$ for all $X, Y \in \Sigma$. We prove that an algebra $(Q;\Sigma)$ is an algebra of mixed type of linearity of the first kind.

As in the proof of Theorem 2.2 we can see that from (12) we obtain

$$A(x,y) = \theta^1_{A,B} x + \theta^2_{A,B} y, \tag{14}$$

for any operation $A \in \Sigma$, where $\theta^1_{A,B}$ is a quasiautomorfism of the group $(Q;+)$.

Thus,

$$\theta^1_{A,B} x = \varphi_A x + t_A, \tag{15}$$

where $\varphi_A \in Aut(Q;+)$ and t_A is some element of the set Q [2, Lemma 2.5].

To prove that $\theta^2_{A,B}$ is an antiquasiautomorphism of the group $(Q;+)$ observe that (13) for $X = A$, $Y = B$ and fixed $u \in Q$ gives

$$A(x, R_{^{-1}B,u} B(y, L_{B^{-1},u} v)) = A(R_{^{-1}A,u} A(x, R_{^{-1}B,u} v), y),$$

$$\theta^1_{A,B} x + \theta^2_{A,B} R_{^{-1}B,u}(R_{B,a} y + L_{B,b} L_{B^{-1},u} v) = \theta^1_{A,B} R_{^{-1}A,u}(\theta^1_{A,B} x + \theta^2_{A,B} R_{^{-1}B,u} v) + \theta^2_{A,B} y.$$

Taking $x = (\theta^1_{A,B})^{-1}0$ in the last equality, we obtain

$$\theta^2_{A,B} R_{^{-1}B,u}(y+v) = \theta^1_{A,B} R_{^{-1}A,u} \theta^2_{A,B} R_{^{-1}B,u} L^{-1}_{B^{-1},u} L^{-1}_{B,b} v + \theta^2_{A,B} R^{-1}_{B,a} y.$$

Thus, the triplet

$$(\theta^1_{A,B} R_{^{-1}A,u} \theta^2_{A,B} R_{^{-1}B,u} L^{-1}_{B^{-1},u} L^{-1}_{B,b},\ \theta^2_{A,B} R^{-1}_{B,a},\ \theta^2_{A,B} R_{^{-1}B,u})$$

is an antiautotopy of the group $(Q;+)$. Since any component of an antiautotopy of a group is an antiquasiautomorphism (see [1]), then $\theta^2_{A,B}R^{-1}_{B,a}$ is an antiquasi-automorphism of the group $(Q;+)$. Similarly as in the proof of Theorem 2.2 we can see that $R^{-1}_{B,a}$ is a quasiautomorphism of the group $(Q;+)$. Therefore $\theta^2_{A,B}$ is an antiquasiautomorphism of the group $(Q;+)$.

Thus,

$$\theta^2_{A,B}x = s_A + \overline{\psi}_A x, \tag{16}$$

where $\overline{\psi}_A$ is an antiautomorphism of $(Q;+)$, and s_A is an element of the set Q.

Hence, from (14), (15) and (16) we get

$$A(x,y) = \varphi_A x + c_A + \overline{\psi}_A x, \tag{17}$$

where $c_A = t_A + s_A$. □

Theorem 3.3. *An invertible algebra $(Q;\Sigma)$ is an invertible algebra of mixed type of linearity of the second kind if and only if for all $X,Y \in \Sigma$ the following formulaes*

$$X(x, Y({}^{-1}Y(y,u),z)) = X({}^{-1}X(X(x,y),u), Y({}^{-1}Y(u,u),z)), \tag{18}$$

$$X(Y^{-1}(u, Y({}^{-1}Y(x,u),y)), v) = X(y, X^{-1}(u, X(Y^{-1}(u,x),v))), \tag{19}$$

are valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$.

Proof. The proof is similar to the proof of Theorem 3.2. □

Note, that the equalities (1), (10), (11), (12), (13), (18) and (19) are the identities of the second order (in the sense of Yu.M. Movsisyan [11]).

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