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Generalized IP-loops

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Abstract. Some generalization of the inverse identities for loops are presented and it is proved that loops of order $n < 7$ satisfy one of these generalized identities. Included examples presented method of computation of these identities. Some universal relations between left, right and middle translations are described.

1. Introduction

Let $Q = \{1, 2, \ldots, n\}$ be a finite set, S_n - the set of all permutations of Q. The multiplication (composition) of permutations $\varphi, \psi \in S_n$ is defined as $\varphi\psi(x) = \varphi(\psi(x))$. All permutations will be written in the form of cycles, cycles will be separated by dots, e.g.

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 1 & 5 & 2 & 3 \end{pmatrix} = (1\,4)(2\,7\,3\,6)(5) = (1\,4.\,2\,7\,3\,6.\,5.).$$

By a *cyclic type* of permutation φ we mean the sequence $\{l_1, l_2, \ldots, l_n\}$, where l_i denotes the number of cycles of the length i. In this case we will write

$$C(\varphi) = \{l_1, l_2, \ldots, l_n\} \quad \text{and} \quad P(\varphi) = \{x_1^{l_1}, x_2^{l_2}, \ldots, x_n^{l_n}\}.$$

For example, forthe above permutation φ we have $C(\varphi) = \{1, 1, 0, 1, 0, 0, 0\}$ and $P(\varphi) = \{x_1^1, x_2^1, x_3^0, x_4^1, x_5^0, x_6^0, x_7^0\}$.

Let $Q(\cdot)$ be a quasigroup with the multiplication denoted by juxtaposition. Then $L_a(x) = a \cdot x$ is called a *left translation*, $R_a(x) = x \cdot a$ is called a *right translation*. By a *middle translation* (shortly: *track*) we mean a permutation φ_a such that $x \cdot \varphi_a(x) = a$ for every $x \in Q$. The permutation φ_a^{-1} is denoted by λ_a, i.e., $\lambda_a(x) \cdot x = a$ for every $x \in Q$. Moreover, for all $i, j \in Q$, $i \neq j$, we define *left spins* $L_{ij} = L_i L_j^{-1}$, *right spins* $R_{ij} = R_i R_j^{-1}$ and *middle spins* $\varphi_{ij} = \varphi_i \varphi_j^{-1}$.

The "matrices" $L = [L_{ij}]$, $R = [R_{ij}]$ and $\Phi = [\varphi_{ij}]$ are called the *left* (*right*, *middle*) *spectrum* of a quasigroup $Q(\cdot)$, respectively. By the *indicator of the spectrum* L (cf. [5]) we mean the polynomial $L^* = \sum_{i=1}^n P(\overline{L}_i)$, where $\overline{L}_i$ is the ith row of L and $P(\overline{L}_i) = \sum_{j=1,\, i\neq j}^n P(L_{ij})$.

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The indicators R^*, Φ^* of R and Φ are defined analogously.

As it is well known (cf. [6]), two permutations $\varphi, \psi \in S_n$ are *conjugated* if there exists a permutation $\rho \in S_n$ such that $\rho\varphi\rho^{-1} = \psi$.

Theorem 1.1. (Theorem 5.1.3. in [6]) *Two permutations are conjugate if and only if they have the same cyclic type.* □

We will use the following notation: $L'_{ij} = L_i^{-1}L_j$, $R'_{ij} = R_i^{-1}R_j$, $\varphi'_{ij} = \varphi_i^{-1}\varphi_j$.

2. IP-identities

As it is well known (cf. for example [1]), IP-loops satisfy the following two identities:

$$x^{-1} \cdot (x \cdot y) = y, \qquad (y \cdot x) \cdot x^{-1} = y. \tag{1}$$

In any IP-loop we also have:

$$(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}. \tag{2}$$

Let $Q(\cdot)$ be a quasigroup, $\alpha, \beta, \gamma, \rho, \sigma, \tau$ – fixed permutations of Q. Consider the following identities:

$$\alpha(x) \cdot \beta(x \cdot y) = \gamma(y) \tag{3}$$

$$\beta(y \cdot x) \cdot \alpha(x) = \gamma(y) \tag{4}$$

$$\alpha(x) \cdot \beta(y \cdot x) = \gamma(y) \tag{5}$$

$$\beta(x \cdot y) \cdot \alpha(x) = \gamma(y) \tag{6}$$

$$\rho(x \cdot y) = \sigma(y) \cdot \tau(x). \tag{7}$$

Identities (3) – (6) generalize (1), (7) is a generalization of (2).

Theorem 2.1. *If* (3) *and* (5) (*or* (4) *and* (6)) *hold for some* α, β, γ, *then* (7) *holds for some* ρ, σ, τ.

Proof. Let (3) and (5) be satisfied, i.e., let

$$\alpha_1(x) \cdot \beta_1(x \cdot y) = \gamma_1(y) \tag{8}$$

$$\alpha_2(x) \cdot \beta_2(y \cdot x) = \gamma_2(y) \tag{9}$$

for some $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$. Multiplying the second identity by β_1 and $\alpha_1(\alpha_2(x))$ we obtain

$$\alpha_1(\alpha_2(x)) \cdot \beta_1(\alpha_2(x) \cdot \beta_2(y \cdot x)) = \alpha_1(\alpha_2(x)) \cdot \beta_1(\gamma_2(y)),$$

which for $\alpha_2(x) = u$ and $\beta_2(y \cdot x) = v$ gives

$$\alpha_1(u) \cdot \beta_1(u \cdot v) = \alpha_1(u) \cdot \beta_1(\gamma_2(y)).$$

From this, applying (8), we get

$$\alpha_1(u) \cdot \beta_1(\gamma_2(y)) = \gamma_1(v).$$

So,

$$\gamma_1(\beta_2(y \cdot x)) = \alpha_1(\alpha_2(x)) \cdot \beta_1(\gamma_2(y)).$$

This shows that (7) is satisfied for $\rho = \gamma_1\beta_2$, $\sigma = \alpha_1\alpha_2$, $\tau = \beta_1\gamma_2$.

Analogously we can show that (4) and (6)) imply (7). $\square$

Theorem 2.2. *In any quasigroup:*

- (3) *holds if and only if* $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$,
- (4) *holds if and only if* $L_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$,
- (5) *holds if and only if* $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$,
- (6) *holds if and only if* $R_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$,
- (7) *holds if and only if* $L_i = \rho^{-1}R_{\tau(i)}\sigma$

for all $i \in Q$.

Proof. We prove only the first equivalence. The proof of other equivalences is very similar.

Let (3) holds. Then for $y = i$ we have

$$\alpha(x) \cdot \beta R_i(x) = \gamma(i) = \alpha(x) \cdot \varphi_{\gamma(i)}\alpha(x),$$

which means that $\beta R_i = \varphi_{\gamma(i)}\alpha$, whence we obtain $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$.

The converse statement is obvious. $\square$

Theorem 2.3. *In a quasigroup* $Q(\cdot)$ *we have:*

(a) $R^* = \Phi^*$ *if* (3) *or* (6) *holds,*

(b) $L^* = \Phi^*$ *if* (4) *or* (5) *holds,*

(c) $L^* = R^*$ *if* (7) *holds.*

Proof. Let (3) holds. Then $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$, whence $R_{ij} = \beta^{-1}\varphi_{\gamma(i)\gamma(j)}\beta$. This, by Theorem 2.2, gives $R'_{ij} = \beta^{-1}\varphi_{\gamma(i)\gamma(j)}\beta$. So, $R^* = \Phi^*$.

In other cases the proof is similar. $\square$

Corollary 2.4. *If in a quasigroup* $Q(\cdot)$ *for every* $i \in Q$

(a) $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ *or* $R_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$, *then* $R^* = \Phi^*$,

(b) $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ *or* $L_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$, *then* $L^* = \Phi^*$,

(c) $L_i = \rho^{-1}R_{\tau(i)}\sigma$, *then* $L^* = R^*$. $\square$

Theorem 2.5. *Relations* $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$, $L_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$, $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$, $R_i = \beta^{-1}\varphi^{-1}_{\gamma(i)}\alpha$ *and* $L_i = \rho^{-1}R_{\tau(i)}\sigma$ *are universal, i.e., they are saved by isotopy.*

Proof. Assume that quasigroups $Q(\circ)$ and $Q(\cdot)$ are isotopic, i.e.,

$$\delta(x \circ y) = \mu(x) \cdot \eta(y)$$

for some permutations δ, μ, η of Q.

Translations L_i, R_i, φ_i of $Q(\cdot)$ and translations L_i°, R_i°, φ_i° of $Q(\circ)$ are connected by formulas:

$$L_i = \delta L^\circ_{\mu^{-1}(i)}\eta^{-1}, \qquad R_i = \delta R^\circ_{\eta^{-1}(i)}\mu^{-1}, \qquad \varphi_i = \eta\varphi^\circ_{\delta^{-1}(i)}\mu^{-1}$$

(for details see [5]). Hence, if the formula $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ is satisfied in $Q(\cdot)$, then in $Q(\circ)$ it has the form

$$\delta L^\circ_{\mu^{-1}(i)}\eta^{-1} = \beta^{-1}(\eta\varphi^\circ_{\delta^{-1}\gamma(i)}\mu^{-1})\alpha\,.$$

Thus

$$L^\circ_{\mu^{-1}(i)} = \delta^{-1}\beta^{-1}\eta\varphi^\circ_{\delta^{-1}\gamma(i)}\mu^{-1}\alpha\eta\,,$$

which for $j = \mu^{-1}(i)$ gives

$$L^\circ_j = (\delta^{-1}\beta^{-1}\eta)\varphi^\circ_{\delta^{-1}\gamma\mu(i)}(\mu^{-1}\alpha\eta)\,.$$

So, the formula $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ is universal.

In other cases the proof is analogous. □

3. Examples

We will use universal relations mentioned in Theorem 2.5 to determine conditions under which identities (3) – (7) are satisfied by quasigroups belonging to the isotopy classes of quasigroups listed in the book [2]. We omit classes of quasigroups isotopic to groups since groups satisfy each of these identities for some permutations.

1. The first class is represented by the loop No. 2.1.1:

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	1	2	4
4	4	3	5	1	2
5	5	4	2	3	1

$L_1 = \varphi_1 = (1.2.3.4.5.)$ $\quad R_1 = (1.2.3.4.5.)$
$L_2 = \varphi_2 = (12.345.)$ $\quad R_2 = (12.354.)$
$L_3 = \varphi_3 = (13.254.)$ $\quad R_3 = (13.245.)$
$L_4 = \varphi_4 = (14.235.)$ $\quad R_4 = (14.253.)$
$L_5 = \varphi_5 = (15.243.)$ $\quad R_5 = (15.234.)$

In this loop $L_i = \varphi_i$ for all i, so from the first universal relation $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ we see that this is possible for $\alpha = \beta = \gamma = \varepsilon$, which, by Theorem 2.2, means that this loop satisfies the identity:

$$x \cdot (y \cdot x) = y.$$

This universal relation is possible also for other α and β. Indeed, for $\gamma(1) = 1$, from Theorem 2.2 and (3), we obtain $\alpha(x) \cdot \beta(x) = 1$, which for this loop implies $\alpha = \beta$. Hence for $\gamma(2) = 3$ we have $L_2 = \alpha^{-1}\varphi_3\alpha$. This is possible only for $\alpha = (1.23.45.)$. Then $\gamma = \alpha$. So, the identity

$$\alpha(x) \cdot \alpha(y \cdot x) = \alpha(y), \tag{10}$$

where $\alpha = (1.23.45.)$ also is possible in this loop.

Now we check connections between R_i and φ_i. For this we use indicators R^* and Φ^*. In the case $R^* \neq \Phi^*$ no any connections, in the case $R^* = \Phi^*$ connections are possible.

For this loop we have

$\overline{\Phi}_1 = \{\varphi_{12}, \varphi_{13}, \varphi_{14}, \varphi_{15}\} = \{(12.354.), (13.245.), (14.253.), (15.234.)\}$
$\overline{\Phi}_2 = \{\varphi_{21}, \varphi_{23}, \varphi_{24}, \varphi_{25}\} = \{(12.345.), (14325.), (15423.), (13524.)\}$
$\overline{\Phi}_3 = \{\varphi_{31}, \varphi_{32}, \varphi_{34}, \varphi_{35}\} = \{(13.254.), (15234.), (12435.), (14532.)\}$
$\overline{\Phi}_4 = \{\varphi_{41}, \varphi_{42}, \varphi_{43}, \varphi_{45}\} = \{(14.235.), (13245.), (15342.), (12543.)\}$
$\overline{\Phi}_5 = \{\varphi_{51}, \varphi_{52}, \varphi_{53}, \varphi_{54}\} = \{(15.243.), (14235.), (12354.), (13452.)\}$

and

$P(\overline{\Phi}_1) = \sum_{j=2}^{5} P(\varphi_{1j}) = x_2x_3 + x_2x_3 + x_2x_3 + x_2x_3 = 4x_2x_3$
$P(\overline{\Phi}_2) = \sum_{j=1,\, j\neq 2}^{5} P(\varphi_{2j}) = x_2x_3 + x_5 + x_5 + x_5 = x_2x_3 + 3x_5$
$P(\overline{\Phi}_3) = P(\overline{\Phi}_4) = P(\overline{\Phi}_5) = P(\overline{\Phi}_2) = x_2x_3 + 3x_5$
$\Phi^* = \sum_{i=1}^{5} P(\overline{\Phi}_i) = 5x_2x_3 + 12x_5$

Analogously,

$\overline{R}_1 = \{R_{12}, R_{13}, R_{14}, R_{15}\} = \{(12.345.), (13.254.), (14.235.), (15.243.)\}$
$\overline{R}_2 = \{R_{21}, R_{23}, R_{24}, R_{25}\} = \{(12.354.), (15324.), (13425.), (14523.)\}$
$\overline{R}_3 = \{R_{31}, R_{32}, R_{34}, R_{35}\} = \{(13.245.), (14234.), (15432.), (12534.)\}$
$\overline{R}_4 = \{R_{41}, R_{42}, R_{43}, R_{45}\} = \{(14.253.), (15243.), (12345.), (13542.)\}$
$\overline{R}_5 = \{R_{51}, R_{52}, R_{53}, R_{54}\} = \{(15.234.), (13245.), (14352.), (12453.)\}$

and

$P(\overline{R}_1) = 4x_2x_3,\ P(\overline{R}_2) = P(\overline{R}_3) = P(\overline{R}_4) = P(\overline{R}_5) = x_2x_3 + 3x_5$
$R^* = 5x_2x_3 + 12x_5$

So, $R^* = \Phi^*$. Thus, the relation $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ or $R_i = \beta^{-1}\varphi_{\gamma(i)}^{-1}\alpha$ is possible (Corollary 2.4). If the first relation holds, then $R_{ij} = \beta^{-1}\varphi_{\gamma(i)\gamma(j)}\beta$. For $i = 1$ must be $\gamma(1) = 1$ since two conjugated permutations have the same cyclic type (Theorem 1.1). So, $R_{1j} = \beta^{-1}\varphi_{1\gamma(j)}\beta$, which for $\gamma(2) = 2$ gives $R_{12} = \beta^{-1}\varphi_{12}\beta$. The last equation has three solutions: $\beta_1 = (1.2.3.45.)$, $\beta_2 = (1.2.35.4.)$ and $\beta_3 = (1.2.34.5.)$. Hence, in view of Theorem 2.3, the identity (3) may be true in this class of quasigroups for $\alpha = \beta = \gamma = \beta_i$. Comparing this fact with (10), where $\alpha = (1.23.45.)$, and the end of the proof of Theorem 2.1 ($\rho = \gamma_1\beta_2$, $\sigma = \alpha_1\alpha_2$, $\tau = \beta_1\gamma_2$), we see that $\alpha_1 = \beta_1 = \gamma_1 = (1.23.45.)$, $\alpha_2 = \beta_2 = \gamma_2 = (1.2.3.45.)$ and

$\rho = \sigma = \tau = (1.23.4.5.)$. So, in this loop we have

$$\rho(x \cdot y) = \rho(y) \cdot \rho(x)$$

for $\rho = (1.23.4.5.)$.

2. Using the same method we can see that $L^* = R^* = \Phi^*$ for loops no. 3.1.1, 4.1.1, 5.1.1, 6.1.1 and 7.1.1. For example, for the loop no. 3.1.1 we have $L^* = R^* = \Phi^* = 6(2x_3^2 + 3x_6)$ and $L_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$, where $\alpha = (1.2.3.465.)$, $\beta = \gamma = (1.23.4.5.6.)$. So, in this loop $\alpha(x) \cdot \beta(y \cdot x) = \beta(y)$ for the above α, β. In this loop also $R_i = \beta^{-1}\varphi_{\gamma(i)}\alpha$ for $\alpha = (1.2.3.46.5.)$, $\beta = \gamma = (1.23.4.56.)$, which means that this loop satisfies $\alpha(x) \cdot \beta(x \cdot y) = \beta(y)$ for the above α, β. Hence, it satisfies also $\rho(x \cdot y) = \rho(y) \cdot \rho(x)$ for $\rho = (1.2.3.4.56.)$.

3. For loops no. 8.1.1, 8.2.1, 8.3.1, 9.1.1, 9.2.1, 9.3.1, 10.1.1, 10.2.1, 10.3.1, 11.1.1, 11.2.1, 11.3.1, 12.1.1, 12.2.1 and 12.3.1, one of the following relations take place: $L^* = R^* \neq \Phi^*$, $L^* \neq R^* = \Phi^*$, $R^* \neq L^* = \Phi^*$.

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