Quasigroups and Related Systems **20** (2012), 197 − 202

Prime and weakly prime ideals in semirings

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Abstract. We study the concept of subtractive prime and weakly prime ideals in a semiring and prove some results analogous to ring theory.

1. Introduction

The notion of a semiring was first introduced by H. S. Vandiver in 1935. After that several authors have generalized and characterized the results in many ways. By a *semiring*, we mean a semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0)$ in which 0 is the additive identity and $s \cdot 0 = 0 \cdot s = 0$ for all $s \in S$, both are connected by ring-like distributivity. In this paper, all semirings are considered to be semirings with zero.

A nonempty subset I of a semiring S is called an (*left, right*) *ideal* if $a, b \in I$ and $s \in S$ implies $a + b \in I$ and ($sa \in I$, $as \in I$ respectively) $as \in S$ and $sa \in I$. A *subtractive ideal* I of S is an ideal such that if $a, a + b \in I$ then $b \in I$.

For the remaining definition of a semiring we refer [6].

2. Weakly prime ideals

D. D. Anderson and E. Smith [3] have introduced and studied the concept of a weakly prime ideal of an associative ring with unity. After that several authors have focused on the study of this concept to extend the results to commutative ring and commutative semiring theory.

Definition 2.1. A proper ideal P of a semiring S is said to be *prime* if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A, B of S.

Definition 2.2. A proper ideal P of a semiring S is said to be *weakly prime* if $\{0\} \neq AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of S.

It is clear that every prime ideal is weakly prime. If S be a semiring with zero, then $I = \{0\}$ is a weakly prime ideal of S. It is easy to see that in Z_6 an ideal $I = \{0\}$ is weakly prime but not prime.

2010 Mathematics Subject Classification: 16Y60
Keywords: Subtractive ideal, prime ideal, weakly prime ideal.

Definition 2.3. An element s in a semiring S is said to be *nilpotent* if there exists a positive integer n (depending on s), such that $s^n = 0$ for $s \in S$. *Nil* S denote the set of all nilpotent element of S. An ideal I in a semiring S is said to be *nilpotent* if there exists a positive integer n (depending on I), such that $I^n = 0$.

Theorem 2.4. *Let I be a subtractive ideal in a semiring S with $1 \neq 0$. The following statements are equivalent:*

(i) *I is a weakly prime ideal.*

(ii) *If A, B are right (left) ideals of S such that $\{0\} \neq AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$.*

(iii) *If $a, b \in S$ such that $\{0\} \neq aSb \subseteq I$, then $a \in I$ or $b \in I$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that I is a weakly prime ideal of S and A, B are two right (left) ideals of S such that $\{0\} \neq AB \subseteq I$. Let $\langle A \rangle, \langle B \rangle$ be the ideals generated by A, B respectively. Then $\{0\} \neq \langle A \rangle \langle B \rangle \subseteq I$ implies $\langle A \rangle \subseteq I$ or $\langle B \rangle \subseteq I$ and $A \subseteq \langle A \rangle \subseteq I$ or $B \subseteq \langle B \rangle \subseteq I$. Therefore $A \subseteq I$ or $B \subseteq I$.

$(ii) \Rightarrow (iii)$ Let $\{0\} \neq aSb \subseteq I$. Since S has an identity, therefore $\{0\} \neq (aS)(bS) \subseteq I$ implies $a \in aS \subseteq I$ or $b \in bS \subseteq I$.

$(iii) \Rightarrow (i)$ Suppose that $AB \subseteq I$ for ideals A and B of S, where $A \nsubseteq I$ and $B \nsubseteq I$. Let $a \in A \setminus I$, $b \in B \setminus I$. Also let $a' \in A \cap I$, $b' \in B \cap I$ be chosen arbitrary. Since $a + a', b + b' \notin I$, we must have $\{0\} = (a + a')S(b + b')$. Now if we are letting $a' = 0$ or $b' = 0$ or $a' = 0$ and $b' = 0$ and considering all combinations we get $0 = ab = a'b = a'b' = a'b'$ and hence $AB = \{0\}$. □

Proposition 2.5. *Every ideal of a semiring S is weakly prime if and only if for any ideals A, B in S, we have $AB = A$, $AB = B$, or $AB = 0$.*

Proof. Assume that every ideal of S is weakly prime. Let A, B be ideals of S. Suppose $AB \neq S$. Then AB is weakly prime. If $\{0\} \neq AB \subseteq AB$; then we have $A \subseteq AB$ or $B \subseteq AB$ (since AB is weakly prime ideal of S), that is, $A = AB$ or $B = AB$. If $AB = S$, then we have $A = B = S$ whence $S^2 = S$. Conversely, let I be any proper ideal of S and suppose that $\{0\} \neq AB \subseteq I$ for ideals A and B of S. Then we have either $A = AB \subseteq I$ or $B = AB \subseteq I$. □

Now we can easily prove the following results based on the above proposition. *Let S be a semiring in which every ideal of S is weakly prime. Then for any ideal A of S, we have either $A^2 = A$ or $A^2 = 0$.*

Lemma 2.6. *Let P be a subtractive ideal of semiring S. Let P be a weakly prime ideal but not a prime ideal of semiring S. Suppose $ab = 0$ for some $a, b \notin P$, then we have $aP = Pb = \{0\}$.*

Proof. Suppose $ap_1 \neq 0$, for some $p_1 \in P$. Then $0 \neq a(b + p_1) \in P$. Since P is a weakly prime ideal of S, therefore $a + p_1 \in P$ or $b \in P$, that is, $a \in P$ or $b \in P$, a contradiction. Therefore $aP = \{0\}$. Similarly, we can show that $Pb = \{0\}$ □

Theorem 2.7. *Suppose that P is a subtractive ideal in a semiring S. If P is weakly prime but not prime, then $P^2 = \{0\}$.*

Proof. Suppose that $p_1p_2 \neq 0$ for some $p_1, p_2 \in P$ and $ab = 0$ for some $a, b \notin P$, where P is not a prime ideal of S. Then by Lemma 2.6 we have $(a+p_1)(b+p_2) = p_1p_2 \neq 0$. Hence either $(a + p_1) \in P$ or $(b + p_2) \in P$, and thus either $a \in P$ or $b \in P$, a contradiction. Hence $P^2 = \{0\}$. □

Corollary 2.8. *Let P be a weakly prime ideal of S. If P is not a prime ideal of S, then $P \subseteq Nil\, S$.*

A subtractive ideal in a commutative semiring S satisfying $P^2 = \{0\}$ may not be weakly prime.

Example 2.9. Let $S = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_{12}^{+}\right\}$. Then S is a commutative semiring and $P = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is its ideal such that $P^2 = \{0\}$. In this semiring $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in P$ but $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin P$ and $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \notin P$. Therefore P is not a weakly prime ideal of S.

Theorem 2.10. *Let P be a weakly subtractive prime ideal of a commutative semiring S that is not prime. Then if $z \in Nil\, S$, then either $z \in P$ or $zP = \{0\}$.*

Proof. Let $z \in$*Nil* S. To show that if $zP \neq \{0\}$, then $z \in P$, suppose that $zP \neq \{0\}$. Let n be the least positive integer such that $z^n = 0$. Then for $n \geqslant 2$ and for some $p \in P$ we have $0 \neq z(p + z^{n-1}) = zp \in P$. Hence either $z \in P$ or $(p + z^{n-1}) \in P$. If $z \in P$ then nothing to prove. So let $(p + z^{n-1}) \in P$. Then $z^{n-1} \in P$ and thus $z \in P$. Hence for each $z \in$ *Nil* S, we have either $z \in P$ or $zP = \{0\}$. Again we suppose that $z \notin P$ for some $z \in$ *Nil* S. Then we will show that $zP = \{0\}$. Now let $zp \neq 0$ for some $p \in P$. Let n be the least positive integer such that $z^n = 0$. Since $z \notin P, n \geqslant 2$ and $zP \neq 0$. Hence $z(z^{n-1}+p) = zp \neq 0$. Since $0 \neq z(z^{n-1} + p) \in P$, therefore we have either $z \in P$ or $z^{n-1} \neq 0$ and $z^{n-1} \in P$. Hence in both cases, we have $z \in P$, a contradiction. Thus $zP = \{0\}$. □

3. Prime ideals

The following lemma is obvious.

Lemma 3.1. *Let f be a homomorphism of semiring S_1 onto a semiring S_2. Then each of the following is true:*

(*i*) *If I is an ideal (subtractive ideal) in S_1, then $f(I)$ is an ideal (subtractive ideal) in S_2.*

(*ii*) *If J is an ideal (subtractive ideal) in S_2, then $f^{-1}(J)$ is an ideal (subtractive ideal) in S_1.*

Proposition 3.2. *If $f : S_1 \to S_2$ is a homomorphism of semirings and P is a prime ideal in S_2, then $f^{-1}(P)$ is a prime ideal in S_1.*

Proof. By Lemma $f^{-1}(P)$ is an ideal of S_1. Let $xy \in f^{-1}(P)$. Then $f(xy) \in P$ implies $f(x)f(y) \in P$. Since P is a prime ideal of S_2 therefore it follows that either $f(x) \in P$ or $f(y) \in P$ and thus either $x \in f^{-1}(P)$ or $y \in f^{-1}(P)$. Hence $f^{-1}(P)$ is a prime ideal of S_1. □

Theorem 3.3. *Let I be an arbitrary ideal of a semiring S and $P_1, P_2, \dots, P_n$ be subtractive prime ideals of S. If $I \nsubseteq P_i$ for all i, then there exists an element $a \in I$such that $a \notin \bigcup P_i$. Hence, $I \nsubseteq \bigcup P_i$.*

Proof. We will prove it by induction. Clearly the result is true for $n = 1$. Suppose that the theorem holds for $n-1$ subtractive prime ideals. Then, for each i, where $1 \leqslant i \leqslant n$, there exists $x_i \in I$ with $x_i \notin \bigcup_{j \neq i} P_j$. If $x_i \notin P_i$, then $x_i \notin \cup P_j$ and then we are done. Now suppose that $x_i \in P_i$ for all i. Let $a_i = x_1 \cdots x_{i-1}x_{i+1} \cdots x_n$. We claim that $a_i \notin P_i$. Suppose $a_i \in P_i$ and since P_i is prime therefore $x_j \in P_i$ for some $j \neq i$, which is not possible by original choice of x_j. If $j \neq i$, then the element $a_j \in P_i$ because x_i being a factor of a_j. Consider $a = \sum_{j=1}^{n} a_j$. Since each $a_j \in I$ where $1 \leqslant j \leqslant n$, therefore $a \in I$. As $a = a_i + \sum_{j \neq i} a_j$ with $\sum_{j \neq i} a_j \in P_i$ implies that $a \in P_i$; otherwise we would obtain $a_i \in P_i$ (as P_i is a subtractive ideal), which is a contradiction. Thus we get an existence of an element $a = \sum a_j \in I$ and $a \notin P_i$, which proves the theorem. □

Corollary 3.4. *Let I be an arbitrary ideal of a semiring S and $P_1, P_2, \dots, P_n$ be subtractive prime ideals of S. If $I \subseteq \bigcup P_i$, then $I \subseteq P_i$ for some I.*

Theorem 3.5. *Let I be a subset of a commutative semiring S which is closed under addition and multiplication.*

(*i*) *Let $P_1, \dots, P_n$ be subtractive ideals in S, at least $n-2$ of which are primes. If $I \subseteq P_1 \cup \ldots \cup P_n$, then I is contained in some P_i.*

(*ii*) *Let J be an ideal of S with $J \subset I$. If there are subtractive prime ideals $P_1, \dots, P_n$ such that $I \setminus J \subseteq P_1 \cup \ldots \cup P_n$, then $I \subseteq P_i$ for some i.*

Proof. (*i*) The proof is by induction $n \geqslant 2$. If we consider $n = 2$, that is, $I \subseteq P_1 \cup P_2$ implies $I \subseteq P_1$ or $I \subseteq P_2$. In this case P_1 and P_2 need not be prime because if $I \nsubseteq P_2$, then there is $x_1 \in I$ with $x_1 \notin P_2$; since $I \subseteq P_1 \cup P_2$, we must have $x_1 \in P_1$. Similarly, if $I \nsubseteq P_1$, there is $x_2 \in I$ with $x_2 \notin P_1$ and $x_2 \in P_2$. However, if $a = x_1 + x_2$, then $a \notin P_1$ (because if $a \in P_1$ then $x_2 \in P_1$), a contradiction. Similarly, $a \notin P_2$ which contradicts to fact that $I \subseteq P_1 \cup P_2$.

Now assume that $I \subseteq P_1 \cup \ldots \cup P_{n+1}$, where at least $n-1 = (n+1)-2$ of the P_i are prime ideals. Let $M_i = P_1 \cup \ldots \cup P_{i-1} \cup P_{i+1} \ldots \cup P_{n+1}$. Since M_i is union of n ideals at least $(n-1)-1=n-2$ of which are prime. By the hypothesis we can suppose that $I \nsubseteq M_i$ for all i. Thus, for all i, there exist $x_i \in I$ with $x_i \notin M_i$; since $I \subseteq M_i \cup P_i$ therefore we must have $x_i \in P_i$. Now $n \geqslant 3$, so that at least one of the P_i are prime ideals; without loss of generality assume that P_1 is prime. Consider the element $a = x_1 + x_2x_3 \cdots x_{n+1}$. Since all $x_i \in I$ and I is closed under addition and multiplication and $a \in I$. Now $a \notin P_1$ because if $a \in P_1$ then $x_2 \cdots x_{n+1} \in P_1$ (as P_1 is subtractive). Since P_1 is a prime ideal in S therefore some $x_i \in P_1$. This is a contradiction, for $x_i \notin P_1 \subseteq M_i$. If $i > 1$ and $a \in P_i$, then $x_2x_3 \cdots x_{n+1} \in P_i$, because P_i is an subtractive ideal and so $x_1 \in P_i$. This cannot be, for $x_1 \notin P_i \subseteq M_1$. Therefore, $a \notin P_i$ for any i, contradicting to fact that $I \subseteq P_1 \cup \ldots \cup P_{n+1}$.

(ii) By hypothesis, we have $I \subseteq J \cup P_1 \cup \ldots \cup P_n$. Therefore ($i$) gives $I \subseteq J$ or $I \subseteq P_i$. Since J is a proper subset of I therefore $I \nsubseteq J$. Hence we must have $I \subseteq P_i$. □

Let I be an ideal of a commutative semiring S. Then the *radical* of I, denoted by $\sqrt{I}$, is defined as the set

$$\sqrt{I} = \{x \in S : x^n \in I \quad \text{for some positive integer}\, n\}.$$

This is an ideal of S containing I, and it is the intersection of all prime ideals of S containing I [2]. It is easy to see that if an ideal I is subtractive then $\sqrt{I}$ is subtractive.

Definition 3.6. An ideal I of the commutative semiring S is said to be *semiprime* if and only if $I = \sqrt{I}$.

Subtractive semiprime ideals of semirings are characterized by the following theorem.

Theorem 3.7. *An subtractive ideal I of a commutative semiring S is semiprime if and only if the quotient semiring S/I has no nonzero nilpotent elements.*

Proof. Suppose that a subtractive ideal I of a semiring S is semiprime. Let $a+\sqrt{I}$ be a nilpotent element of $S/\sqrt{I}$. Then there exists some positive integer $n \in Z^+$ such that $(a+\sqrt{I})^n = a^n + \sqrt{I} = \sqrt{I}$. As $\sqrt{I}$ is subtractive therefore $a^n \in \sqrt{I}$. Hence, $(a^n)^m = a^{nm} \in I$ for some positive integer m. This shows that $a \in I$. Therefore we have $a + \sqrt{I} = \sqrt{I}$, the zero element of $S/\sqrt{I}$.

Conversely, suppose that S/I has no nonzero nilpotent elements and let $a \in \sqrt{I}$. Then for some positive integer n, we have $a^n \in I$. This implies that $(a+I)^n = I$, that is, $a+I$ is nilpotent in S/I. As $a+I=I$ (by hypothesis), therefore $a \in I$. Thus, we have $\sqrt{I} \subseteq I$. The inclusion $I \subseteq \sqrt{I}$ is obvious. Hence $I = \sqrt{I}$, so I is semiprime. □

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Received March 2, 2012

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