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Left quasi-regular and intra-regular ordered semigroups using fuzzy ideals

Niovi Kehayopulu

Abstract. As a continuation of our paper in [6], we characterize here the ordered semigroups which are both intra-regular and left (right) quasi-regular also the ordered semigroups which are both regular and intra-regular in terms of fuzzy right, fuzzy left and fuzzy bi-ideals using first the first and then the second definitions of fuzzy ideals. As in [6], comparing the proofs of the results using the two definitions, we see that with the second definitions the proofs of the results are drastically simplified.

1. Introduction and prerequisites

In [6], we characterized the ordered semigroups in which $f \wedge h \wedge g \preceq g \circ h \circ f$, $f \wedge h \wedge g \preceq h \circ f \circ g$ and $f \wedge h \wedge g \preceq f \circ h \circ g$ as the ordered semigroups which are intra-regular, both regular and intra-regular, and regular, respectively. It would be interesting to characterize the rest, that is the ordered semigroups in which $f \wedge h \wedge g \preceq f \circ g \circ h$, $f \wedge h \wedge g \preceq h \circ g \circ f$ and $f \wedge h \wedge g \preceq g \circ f \circ h$. In this respect, we characterize the ordered semigroups which are both intra-regular and left (or right) quasi-regular, also the ordered semigroups which are both regular and intra-regular in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. We prove that the property $f \wedge h \wedge g \preceq g \circ f \circ h$ characterizes the ordered semigroups which are both intra-regular and left quasi-regular, and the property $f \wedge h \wedge g \preceq h \circ g \circ f$ the ordered semigroups which are both intra-regular and right quasi-regular. We also prove that the property $f \wedge h \wedge g \preceq f \circ g \circ h$ characterizes the ordered semigroups which are both regular and intra-regular adding an additional characterization to the characterization of the same type of semigroups already considered in [6]. The left (resp. right) quasi-regular ordered semigroups are the ordered semigroups in which the left (resp. right) ideals are idempotent. According to the present paper, if an ordered semigroup $(S, ., \leqslant)$ is intra-regular and the left (resp. right) ideals of S are idempotent, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy subset h of $(S, \cdot)$ we have $f \wedge h \wedge g \preceq g \circ f \circ h$ (resp. $f \wedge h \wedge g \preceq h \circ g \circ f$) which shows that the corresponding results in [5] hold not

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only for the fuzzy right, left, bi-ideals of $(S,\cdot,\leqslant)$ but for the fuzzy right, left, bi-ideals of $(S,\cdot)$, not only for the bi-ideals h but for any fuzzy subset h of S. Moreover, if an ordered semigroup $(S,\cdot,\leqslant)$ is both regular and intra-regular, then for every fuzzy right ideal f, every fuzzy subset g and every fuzzy bi-ideal h of $(S,\cdot)$ we have $f\wedge h\wedge g\preceq f\circ g\circ h$. We notice that investigations in the existing bibliography are based on the first definitions. Characterizations of semigroups (without order) which are intra-regular and left quasi-regular have been given by Kuroki in [7].

These are the first definitions:

Definition 1.1. Let $(S,\cdot,\leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) ideal of $(S,\cdot,\leqslant)$ if

(1) $f(xy)\geqslant f(y)$ (resp. $f(xy)\geqslant f(x)$) for all $x,y\in S$ and

(2) if $x\leqslant y$, then $f(x)\geqslant f(y)$.

In particular, if $(S,\cdot,\leqslant)$ is an ordered semigroup, then a fuzzy subset f of S is called a *fuzzy bi-ideal* of $(S,\cdot,\leqslant)$ if

(1) $f(xyz)\geqslant\min\{f(x),f(z)\}$ for all $x,y\in S$ and

(2) if $x\leqslant y$, then $f(x)\geqslant f(y)$.

These are the second definitions:

Definition 1.2. Let $(S,\cdot,\leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) ideal of $(S,\cdot,\leqslant)$ if

(1) $1\circ f\preceq f$ (resp. $f\circ 1\preceq f$) and

(2) if $x\leqslant y$, then $f(x)\geqslant f(y)$.

In particular, if $(S,\cdot,\leqslant)$ is an ordered semigroup, then a fuzzy subset f of S is called a *fuzzy bi-ideal* of $(S,\cdot,\leqslant)$ if

(1) $f\circ 1\circ f\preceq f$ and

(2) if $x\leqslant y$, then $f(x)\geqslant f(y)$.

A fuzzy subset f of $(S,\cdot,\leqslant)$ is said to be a fuzzy left (resp. right) ideal or fuzzy bi-ideal of $(S,\cdot)$ if the following assertions, respectively hold in $(S,\cdot,\leqslant)$: $f(xy)\geqslant f(y)$ (resp. $f(xy)\geqslant f(x)$), $f(xyz)\geqslant\min\{f(x),f(z)\}$ for all $x,y,z\in S$. The fuzzy set $1:S\to[0,1]\mid a\to 1$ is the greatest element in the set of fuzzy subsets of S. We have $1\circ 1\preceq 1$. In particular in intra-regular, also in regular ordered semigroups we have $1\circ 1=1$. If $(S,\cdot,\leqslant)$ is an ordered groupoid, f,g fuzzy subsets of $(S,\cdot)$ and $f\preceq g$ then, for any fuzzy subset h of $(S,\cdot)$, we have $f\circ h\preceq g\circ h$ and $h\circ f\preceq h\circ g$. If the multiplication on S is associative, then the multiplication " $\circ$ " on fuzzy subsets of S is also associative. An ordered semigroup $(S,\cdot,\leqslant)$ is called *regular* if for every $a\in S$ there exists $x\in S$ such that $a\leqslant axa$, equivalently if $A\subseteq(ASA]$ for every $A\subseteq S$. It is called *intra-regular* if for every $a\in S$ there exist $x,y\in S$ such that $a\leqslant xa^2y$, equivalently if $A\subseteq(SA^2S]$ for every $A\subseteq S$. An ordered semigroup $(S,\cdot,\leqslant)$ is regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S,\cdot,\leqslant)$, we have $f\wedge g=f\circ g$ equivalently $f\wedge g\preceq f\circ g$. It is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S,\cdot,\leqslant)$, we have $f\wedge g\preceq g\circ f$.

Moreover, an ordered semigroup S is regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f$. It is intra-regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f^2 \circ 1$. For further information we refer to [6]. The next two lemmas can be proved using only sets, which shows their pointless character.

Lemma 1.1. (cf. also [1]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is intra-regular, then for every right ideal X and every left ideal Y of $(S, \cdot)$ we have $X \cap Y \subseteq (YX]$. "Conversely", if for every right ideal X and every left ideal Y of $(S, \cdot, \leqslant)$ we have $X \cap Y \subseteq (YX]$, then S is intra-regular.*

Proof. $\Longrightarrow$. Let X be a right ideal and Y a left ideal of $(S, \cdot)$. Since S is intra-regular, we have

$$X \cap Y \subseteq (S(X \cap Y)^2 S] = (S(X \cap Y)(X \cap Y)S] \subseteq ((SY)(XS)] \subseteq (YX].$$

$\Longleftarrow$. Let $A \subseteq S$. Since $R(A)$, $L(A)$ are right and left ideals of $(S, \cdot, \leqslant)$, respectively, by hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq (L(A)R(A)] = ((A \cup SA](A \cup AS]] \\ &= ((A \cup SA)(A \cup AS)] = (A^2 \cup SA^2 \cup A^2 S \cup SA^2 S], \end{aligned}$$

$$\begin{aligned} A^2 &\subseteq (A^2 \cup SA^2 \cup A^2 S \cup SA^2 S](A] \\ &\subseteq (A^3 \cup SA^3 \cup A^2 SA \cup SA^2 SA] \\ &\subseteq (SA^2 \cup A^2 S \cup SA^2 S], \end{aligned}$$

$$\begin{aligned} A &\subseteq ((SA^2 \cup A^2 S \cup SA^2 S] \cup SA^2 \cup A^2 S \cup SA^2 S] \\ &= ((SA^2 \cup A^2 S \cup SA^2 S]] = (SA^2 \cup A^2 S \cup SA^2 S], \end{aligned}$$

$$A^2 \subseteq (SA^2 \cup A^2 S \cup SA^2 S](A] \subseteq (SA^3 \cup A^2 SA \cup SA^2 SA],$$
$$SA^2 \subseteq (S](SA^3 \cup A^2 SA \cup SA^2 SA] \subseteq (SA^3 \cup SA^2 SA] \subseteq (SA^2 S], ++$$
$$A \subseteq ((SA^2 S] \cup A^2 S \cup SA^2 S] = (A^2 S \cup (SA^2 S]],$$
$$A^2 \subseteq (A](A^2 S \cup (SA^2 S]] \subseteq (A^3 S \cup A(SA^2 S]].$$

Since $A(SA^2 S] \subseteq (A](SA^2 S] \subseteq (ASA^2 S] \subseteq (SA^2 S]$, we have

$$A^2 \subseteq (A^3 S \cup (SA^2 S]] \subseteq (SA^2 S \cup (SA^2 S]] = ((SA^2 S]] = (SA^2 S].$$

Then we have $A^2 S \subseteq (SA^2 S](S] \subseteq (SA^2 S]$, and $A \subseteq ((SA^2 S]] = (SA^2 S]$. $\square$

In a similar way, the following lemma holds.

Lemma 1.2. (cf. also [2]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is regular, then for every right ideal X and every left ideal Y of $(S, \cdot)$ we have $X \cap Y = (XY]$. "Conversely", if for every right ideal X and every left ideal Y of $(S, \cdot, \leqslant)$ we have $X \cap Y \subseteq (XY]$, then S is regular.*

2. Main results

The first theorem characterizes the ordered semigroups which are both intra-regular and left quasi-regular in terms of fuzzy ideals. These are the ordered semigroups for which $f \wedge h \wedge g \preceq g \circ f \circ h$. Let us prove this theorem using first the first and then the second definitions.

Definition 2.1. An ordered semigroup S is called *left quasi-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xaya$.

Equivalent Definitions:

1) $a \in (SaSa]$ for every $a \in S$.
2) $A \subseteq (SASA]$ for every $A \subseteq S$.

Recall that this type of ordered semigroups are the ordered semigroups in which the left ideals are idempotent.

Theorem 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If $(S, \cdot, \leqslant)$ is intra-regular and left quasi-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy subset h of $(S, \cdot)$, we have*

$$f \wedge h \wedge g \preceq g \circ f \circ h.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, \cdot, \leqslant)$ we have $f \wedge h \wedge g \preceq g \circ f \circ h$, then S is intra-regular and left quasi-regular.

Proof of Theorem 2.1 using the first definitions

We need the following lemmas.

Lemma 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered groupoid. If A is a left (resp. right) ideal of $(S, \cdot, \leqslant)$, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of $(S, \cdot, \leqslant)$. "Conversely", if A is a nonempty set and f_A a fuzzy left (resp. right) ideal of $(S, \cdot, \leqslant)$, then A is a left (resp. right) ideal of $(S, \cdot, \leqslant)$. In particular, let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then, if B is a bi-ideal of $(S, \cdot, \leqslant)$, then the characteristic function f_B is a fuzzy bi-ideal of $(S, \cdot, \leqslant)$. "Conversely", if B is a nonempty set and f_B a fuzzy bi-ideal of $(S, \cdot, \leqslant)$, then B is a bi-ideal of $(S, \cdot, \leqslant)$.*

Lemma 2.2. *If S is an ordered groupoid (or groupoid) and $\{A_i \mid i \in I\}$ a family of subsets of S, then we have*

$$\bigwedge_{i \in I} f_{A_i} = f_{\bigcap\limits_{i \in I} A_i}.$$

Lemma 2.3. *Let S be an ordered semigroup, n a natural number, $n \geqslant 2$ and $\{A_1, A_2, ..., A_n\}$ a set of nonempty subsets of S. Then we have*

$$f_{A_1} \circ f_{A_2} \circ ..... \circ f_{A_n} = f_{(A_1 A_2 ..... A_n]}.$$

Lemma 2.4. *If S is an ordered groupoid (or groupoid) and A, B subsets of S, then we have*

$$A \subseteq B \Longleftrightarrow f_A \preceq f_B.$$

Lemma 2.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is intra-regular and left quasi-regular, then for every right ideal X and every left ideal Y of $(S, \cdot)$, and every subset B of S we have*

$$X \cap B \cap Y \subseteq (YXB].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S, \cdot, \leqslant)$ we have $X \cap B \cap Y \subseteq (YXB]$, then S is intra-regular and left quasi-regular.

Proof. $\Longrightarrow$. Let X be a right ideal, Y a left ideal of $(S, \cdot)$ and B a subset of S. Then we have

$$\begin{aligned} X \cap B \cap Y &\subseteq (S(X \cap B \cap Y)S(X \cap B \cap Y)] \quad \text{(since } S \text{ is left quasi-regular)} \\ &\subseteq (S(S(X \cap B \cap Y)^2 S]S(X \cap B \cap Y)] \quad \text{(since } S \text{ is intra-regular)} \\ &= (S(S(X \cap B \cap Y)^2 S)S(X \cap B \cap Y)] \\ &\subseteq (S(X \cap B \cap Y)(X \cap B \cap Y)S(X \cap B \cap Y)] \\ &\subseteq ((SY)(XS)B] \subseteq (YXB]. \end{aligned}$$

$\Longleftarrow$. Let X be a right ideal and Y a left ideal of $(S, \cdot, \leqslant)$. Since S is a bi-ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $X \cap Y = X \cap S \cap Y \subseteq (YXS] \subseteq (YX]$. By Lemma 1.1, S is intra-regular. Let now A be a left ideal of $(S, \cdot, \leqslant)$. Since S is a right ideal, A a bi-ideal and A a left ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have

$$A = S \cap A \cap A \subseteq (A(SA)] \subseteq (A^2] \subseteq (SA] \subseteq (A] = A.$$

Then $(A^2] = A$, so S is left quasi-regular. □

Lemma 2.6. [4; Prop. 5] *Let S be an ordered groupoid, f, g fuzzy subsets of S, and $a \in S$. The following are equivalent:*

(1) $(f \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.* □

Proof of Theorem 2.1

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy subset of $(S, \cdot)$, and $a \in S$. Since $(S, \cdot, \leqslant)$ is intra-regular, there exist $x, y \in S$ such that $a \leqslant xa^2y$. Since S is left quasi-regular, there exist $s, t \in S$ such that $a \leqslant sata$. Then we have $a \leqslant sata \leqslant s(xa^2y)ta = sxa^2yta$. Since $(sxa^2yt, a) \in A_a$, we have $A_a \neq \emptyset$, and

$$((g \circ f) \circ h)(a) := \bigvee_{(u,v) \in A_a} \min\{(g \circ f)(u), h(v)\} \geqslant \min\{(g \circ f)(sxa^2yt), h(a)\}.$$

Since $(sxa, ayt) \in A_{sxa^2yt}$, we have $A_{sxa^2yt} \neq \emptyset$, and

$$(g \circ f)(sxa^2yt) := \bigvee_{(w,t) \in A_{sxa^2yt}} \min\{g(w), f(t)\} \geqslant \min\{g(sxa), f(ayt)\}.$$

Since g is a fuzzy left ideal of S, $g(sxa) \geqslant g(a)$. Since f is a fuzzy right ideal of S, $f(ayt) \geqslant f(a)$. Therefore we get

$$\begin{aligned}(g \circ f \circ h)(a) &= ((g \circ f) \circ h)(a) \geqslant \min\{\min\{g(sxa), f(ayt)\}, h(a)\} \\ &\geqslant \min\{\min\{g(a), f(a)\}, h(a)\} = \min\{g(a), f(a), h(a)\} \\ &= (f \wedge h \wedge g)(a).\end{aligned}$$

This holds for every $a \in S$, so $f \wedge h \wedge g \preceq g \circ f \circ h$.

For the converse statement we give three proofs. For the first one we use the Lemmas 2.1–2.5. For the second and third proof the Lemmas 2.1, 2.3 and 2.5 and Lemmas 2.1, 2.5 and 2.6, respectively, together with some basic properties of fuzzy sets.

First proof. Let X be a right ideal, Y a left ideal, B a bi-ideal of $(S, \cdot, \leqslant)$. By Lemma 2.1, f_X is a fuzzy right, f_Y a fuzzy left and f_B a fuzzy bi-ideal of $(S, \cdot, \leqslant)$. By hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_Y \circ f_X \circ f_B$. By Lemma 2.2, $f_X \wedge f_B \wedge f_Y = f_{X \cap B \cap Y}$. By Lemma 2.3, $f_Y \circ f_X \circ f_B = f_{(YXB]}$, then $f_{X \cap B \cap Y} \preceq f_{(YXB]}$. By Lemma 2.4, $X \cap B \cap Y \subseteq (YXB]$. By Lemma 2.5, $(S, \cdot, \leqslant)$ is intra-regular and left quasi-regular.

Second proof. Let X be a right ideal, Y a left ideal, B a bi-ideal of $(S, \cdot, \leqslant)$ and $a \in X \cap B \cap Y$. By Lemma 2.5, it is enough to prove that $a \in (YXB]$. As in the first proof, by Lemma 2.1 and hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_Y \circ f_X \circ f_B$. Then

$$(f_Y \circ f_X \circ f_B)(a) \geqslant (f_X \wedge f_B \wedge f_Y)(a) = \min\{f_X(a), f_B(a), f_Y(a)\}.$$

Since $a \in X$, we have $f_X(a) = 1$, since $a \in B$, $f_B(a) = 1$, since $a \in Y$, $f_Y(a) = 1$. Thus we have $(f_Y \circ f_X \circ f_B)(a) \geqslant 1$. Besides, since $f_Y \circ f_X \circ f_B$ is a fuzzy subset of S, we have $(f_Y \circ f_X \circ f_B)(a) \leqslant 1$, then $(f_Y \circ f_X \circ f_B)(a) = 1$. By Lemma 2.3, $f_Y \circ f_X \circ f_B = f_{(YXB]}$, then $f_{(YXB]}(a) = 1$, and $a \in (YXB]$.

Third proof. Let X be a right ideal, Y a left ideal, B a bi-ideal of $(S, \cdot, \leqslant)$ and $a \in X \cap B \cap Y$. As in the second proof, by Lemma 2.1, we have $(f_Y \circ (f_X \circ f_B))(a) = 1 \neq 0$. By Lemma 2.6, there exists $(b, c) \in A_a$ such that $f_Y(b) \neq 0$ and $(f_X \circ f_B)(c) \neq 0$. Since $(f_X \circ f_B)(c) \neq 0$, there exists $(d, e) \in A_c$ such that $f_X(d) \neq 0$ and $f_B(e) \neq 0$. Then $f_Y(b) = f_X(d) = f_B(e) = 1$, $b \in Y$, $d \in X$, $e \in B$, and $a \leqslant bc \leqslant bde \in YXB$, so $a \in (YXB]$. By Lemma 2.5, S is intra-regular and left quasi-regular. □

Proof of Theorem 2.1 using the second definitions

We need the following lemma

Lemma 2.7. [3] *An ordered semigroup $(S, \cdot, \leqslant)$ is left quasi-regular if and only if, for every fuzzy subset f of S, we have*

$$f \preceq 1 \circ f \circ 1 \circ f,$$

equivalently, if the fuzzy left ideals of $(S, \cdot, \leqslant)$ *are idempotent.*

Proof of the theorem.

$\Longrightarrow$. Let f be a fuzzy right, g a fuzzy left and h a fuzzy bi-ideal of $(S, \cdot)$. By Lemma 2.7, we have

$$\begin{aligned} f \wedge h \wedge g &\preceq 1 \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \quad \text{(since } S \text{ is left quasi-regular)} \\ &\preceq 1 \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ 1 \circ (f \wedge h \wedge g) \\ &\qquad\qquad \text{(since } S \text{ is intra-regular)} \\ &\preceq (1 \circ g) \circ (f \circ 1) \circ h \preceq g \circ f \circ h. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S, \cdot, \leqslant)$. Since 1 is a fuzzy bi-ideal of S, by hypothesis, we have

$$f \wedge g = f \wedge 1 \wedge g \preceq g \circ (f \circ 1) \preceq g \circ f,$$

so S is intra-regular. Let now g be a fuzzy left ideal of $(S, \cdot, \leqslant)$. Since 1 is a fuzzy right ideal and g at the same time a fuzzy bi-ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $g = 1 \wedge g \wedge g \preceq g \circ (1 \circ g) \preceq g^2 \preceq 1 \circ g \preceq g$, so $g^2 = g$. By Lemma 2.7, S is left quasi-regular. $\square$

The next theorem characterizes the ordered semigroups which are both intra-regular and right quasi-regular in terms of fuzzy left, right and fuzzy bi-ideals. These are the ordered semigroups for which $f \wedge h \wedge g \preceq h \circ g \circ f$.

Definition 2.2. An ordered semigroup S is called *right quasi-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant axay$.

Equivalent Definitions:

1) $a \in (aSaS]$ for every $a \in S$.
2) $A \subseteq (ASAS]$ for every $A \subseteq S$.

Theorem 2.2. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. If* $(S, \cdot, \leqslant)$ *is intra-regular and right quasi-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy subset h of* $(S, \cdot)$*, we have*

$$f \wedge h \wedge g \preceq h \circ g \circ f.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, \cdot, \leqslant)$ *we have* $f \wedge h \wedge g \preceq h \circ g \circ f$*, then* $(S, \cdot, \leqslant)$ *is intra-regular and right quasi-regular.*

Proof of Theorem 2.2 using the first definitions

In addition to Lemmas 2.1–2.4 (or 2.1 and 2.3 or 2.1 and 2.6), we need the following lemma.

Lemma 2.8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is intra-regular and right quasi-regular, then for every right ideal X and every left ideal Y of $(S, \cdot)$, and every subset B of S we have*

$$X \cap B \cap Y \subseteq (BYX].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S, \cdot, \leqslant)$ we have $X \cap B \cap Y \subseteq (BYX]$, then S is intra-regular and right quasi-regular.

Proof of Theorem 2.2.

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy subset of $(S, \cdot)$, and $a \in S$. Since $(S, \cdot, \leqslant)$ is intra-regular, there exist $x, y \in S$ such that $a \leqslant xa^2y$. Since S is right regular, there exist $s, t \in S$ such that $a \leqslant asat$. Then we have $a \leqslant asat \leqslant as(xa^2y)t = asxa^2yt$. Then $(asxa, ayt) \in A_a$, $A_a \neq \emptyset$, and

$$((h \circ g) \circ f)(a) := \bigvee_{(u,v) \in A_a} \min\{(h \circ g)(u), f(v)\} \geqslant \min\{(h \circ g)(asxa), f(ayt)\}.$$

Since $(a, sxa) \in A_{asxa}$, we have $A_{asxa} \neq \emptyset$ and

$$(h \circ g)(asxa) := \bigvee_{(w,t) \in A_{asxa}} \min\{(h(w), g(t)\} \geqslant \min\{h(a), g(sxa)\}.$$

Therefore we get

$$((h \circ g) \circ f)(a) \geqslant \min\{\min\{h(a), g(sxa)\}, f(ayt)\} = \min\{h(a), g(sxa), f(ayt)\}.$$

Since g is a fuzzy left ideal of S, we have $g(sxa) \geqslant g(a)$. Since f is a fuzzy right ideal of S, we have $f(ayt) \geqslant f(a)$. Then we get

$$(h \circ g \circ f)(a) = ((f \circ g) \circ f)(a) \geqslant \min\{h(a), g(a), f(a)\} = (f \wedge h \wedge g)(a).$$

Thus we obtain $f \wedge h \wedge g \preceq h \circ g \circ f$.

$\Longleftarrow$. Let X be a right ideal, Y a left ideal and B a bi-ideal of $(S, \cdot, \leqslant)$. Since f_X is a fuzzy right, f_Y a fuzzy left and f_B a fuzzy bi-ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_B \circ f_Y \circ f_X$. Since $f_X \wedge f_B \wedge f_Y = f_{X \cap B \cap Y}$ and $f_B \circ f_Y \circ f_X = f_{(BYX]}$, we have $f_{X \cap B \cap Y} \preceq f_{(BYX]}$. Then $X \cap B \cap Y \subseteq (BYX]$ and, by Lemma 2.8, S is intra-regular and right quasi-regular. $\square$

Proof of Theorem 2.2 using the second definition.

We need the following lemma.

Lemma 2.9. [3] *An ordered semigroup $(S, \cdot, \leqslant)$ is right quasi-regular if and only if, for every fuzzy subset f of S, we have*

$$f \preceq f \circ 1 \circ f \circ 1,$$

equivalently, if the fuzzy right ideals of $(S, \cdot, \leqslant)$ *are idempotent.*

Proof of the theorem.

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy subset of $(S, \cdot)$. By Lemma 2.9, we have

$$\begin{aligned} f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ 1 \text{ (since } S \text{ is right quasi-regular)} \\ &\preceq (f \wedge h \wedge g) \circ 1 \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ 1 \\ &\qquad\qquad \text{(since } S \text{ is intra-regular)} \\ &= (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \\ &\preceq h \circ (1 \circ g) \circ (f \circ 1) \preceq h \circ g \circ f. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S, \cdot, \leqslant)$. Since 1 is a fuzzy bi-ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $f \wedge g = f \wedge 1 \wedge g \preceq (1 \circ g) \circ f \preceq g \circ f$, so S is intra-regular. Let now f be a fuzzy right ideal of $(S, \cdot, \leqslant)$. Since f is at the same time a fuzzy bi-ideal and 1 a fuzzy left ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have

$$f = f \wedge f \wedge 1 \preceq (f \circ 1) \circ f \preceq f \circ f \preceq f \circ 1 \preceq f,$$

so $f^2 = f$. By Lemma 2.9, S is right quasi-regular. □

The last theorem characterizes the ordered semigroups which are both regular and intra-regular in terms of fuzzy left, right and fuzzy bi-ideals. These are the ordered semigroups for which $f \wedge h \wedge g \preceq f \circ g \circ h$.

Theorem 2.3. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. If* S *is both regular and intra-regular, then for every fuzzy right ideal* f*, every fuzzy subset* g *and every fuzzy bi-ideal* h *of* $(S, \cdot)$ *we have*

$$f \wedge h \wedge g \preceq f \circ g \circ h.$$

"Conversely", if for every fuzzy right ideal f*, every fuzzy left ideal* g *and every fuzzy bi-ideal* h *of* $(S, \cdot, \leqslant)$ *we have* $f \wedge h \wedge g \preceq f \circ g \circ h$*, then* S *is both regular and intra-regular.*

Proof of Theorem 2.3 using the first definitions

In addition to Lemmas 2.1–2.4 (or 2.1 and 2.3 or 2.1 and 2.6), we need the following lemma.

Lemma 2.10. (cf. also [8]) *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. If* $(S, \cdot, \leqslant)$ *is both regular and intra-regular, then for every right ideal* X*, every subset* Y *and every bi-ideal* B *of* $(S, \cdot)$*, we have*

$$X \cap B \cap Y \subseteq (XYB].$$

"Conversely", if for every right ideal X*, every left ideal* Y *and every bi-ideal* B *of* $(S, \cdot, \leqslant)$*, we have* $X \cap B \cap Y \subseteq (XYB]$*, then* S *is both regular and intra-regular.*

Proof. Condition "$A \subseteq (ASA^2SA]$ for all $A \subseteq S$" characterizes the ordered semigroups which are both regular and intra-regular. Let now X be a right ideal, Y a subset and B a bi-ideal of $(S,\cdot)$. Then we have

$$\begin{aligned}X \cap B \cap Y &\subseteq ((X \cap B \cap Y)S(X \cap B \cap Y)(X \cap B \cap Y)S(X \cap B \cap Y)]\\ &\subseteq ((XS)Y(BSB)] \subseteq (XYB].\end{aligned}$$

For the converse statement, suppose X is a right ideal and Y a left ideal of $(S,\cdot,\leqslant)$. Since S is a left ideal and Y a bi-ideal of $(S,\cdot,\leqslant)$, by hypothesis, we have $X \cap Y = X \cap Y \cap S \subseteq (XSY] \subseteq (XY]$ then, by Lemma 1.2, S is regular. Since S is a right ideal and X a bi-ideal of $(S,\cdot,\leqslant)$, by hypothesis, we have $X \cap Y = S \cap X \cap Y \subseteq (SYX] \subseteq (YX]$ then, by Lemma 1.1, S is intra-regular. $\square$

Proof of Theorem 2.3.

$\Longrightarrow$. Let f be a fuzzy right ideal of $(S,\cdot)$, g a fuzzy subset of S, h a fuzzy bi-ideal of $(S,\cdot)$, and $a \in S$. Since S is both regular and intra-regular, there exist $x,y,z \in S$ such that $a \leqslant axa$ and $a \leqslant za^2y$. Then we have $a \leqslant ax(axa) \leqslant ax(za^2y)xa$. As $(axza, ayxa) \in A_a$, we have $A_a \neq \emptyset$, and

$$((f \circ g) \circ h)(a) := \bigvee_{(u,v)\in A_a} \min\{(f \circ g)(u), h(v)\} \geqslant \min\{(f \circ g)(axza), h(ayxa)\}.$$

Since $(axz, a) \in A_{axza}$, we have $A_{axza} \neq \emptyset$, and

$$(f \circ g)(axza) := \bigvee_{(w,t)\in A_{axza}} \min\{f(w), f(t)\} \geqslant \min\{f(axz), g(a)\}.$$

Then we have

$$\begin{aligned}(f \circ g \circ h)(a) &\geqslant \min\{\min\{f(axz), g(a)\}, h(ayxa)\}\\ &= \min\{f(axz), g(a), h(ayxa)\}\\ &\geqslant \min\{f(a), g(a), h(a)\}\\ &= (f \wedge h \wedge g)(a).\end{aligned}$$

Thus we obtain $f \wedge h \wedge g \preceq f \circ g \circ h$.

$\Longleftarrow$. Let X be a right ideal, Y a left ideal and B a bi-ideal of $(S,\cdot,\leqslant)$. Since f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S,\cdot,\leqslant)$, by hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_X \circ f_Y \circ f_B$. Then $f_{X\cap B\cap Y} \preceq f_{(XYB]}$, and $X \cap B \cap Y \subseteq (XYB]$. By Lemma 2.10, S is both regular and intra-regular. $\square$

Proof of Theorem 2.3 using the second definitions

$\Longrightarrow$. Since S is both regular and intra-regular, for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f^2 \circ 1 \circ f$. Indeed: Since S is regular, we have $f \preceq f \circ 1 \circ f$ and, since S is intra-regular, $f \preceq 1 \circ f^2 \circ 1$. Then we have $f \preceq f \circ 1 \circ (f \circ 1 \circ f) \preceq$

$f \circ 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f = f \circ 1 \circ f^2 \circ 1 \circ f$. Let now f be a fuzzy right ideal, g a fuzzy subset and h a fuzzy bi-ideal of $(S, \cdot)$. Then we have

$$\begin{aligned} f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \\ &\preceq (f \circ 1) \circ g \circ (h \circ 1 \circ h) \\ &\preceq f \circ g \circ h. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S, \cdot, \leqslant)$. Since g is a fuzzy bi-ideal and 1 a fuzzy left ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $f \wedge g = f \wedge g \wedge 1 \preceq (f \circ 1) \circ g \preceq f \circ g$, so S is regular. Since 1 is a fuzzy right ideal and f a fuzzy bi-ideal of $(S, \cdot, \leqslant)$, by hypothesis, we have $f \wedge g = 1 \wedge f \wedge g \preceq (1 \circ g) \circ f \preceq g \circ f$, and S is intra-regular. □

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University of Athens
Department of Mathematics
15784 Panepistimiopolis
Athens, Greece

E-mail: nkehayop@math.uoa.gr