Soft intersection Lie algebras

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Abstract. In 1999, Molodtsov introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty and vagueness, and many researchers have created some models to solve problems in decision making and medical diagnosis. In this paper, we introduce the concept of soft Lie subalgebras (resp. soft Lie ideals) and state some of their fundamental properties. We also introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) and investigate some of their properties.

1. Introduction

The theory of Lie algebras is an area of mathematics in which we can see a harmonious between the methods of classical analysis and modern algebra. This theory, a direct outgrowth of a central problem in the calculus, has today become a synthesis of many separate disciplines, each of which has left its own mark. Theory of Lie groups were developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. Lie algebras were also discovered by Sophus Lie when he first attempted to classify certain smooth subgroups of general linear groups. The groups he considered are called Lie groups. The importance of Lie algebras for applied mathematics and for applied physics has also become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid-state physics, cosmology and control theory. Lie algebra is also used by electrical engineers, mainly in the mobile robot control. For the basic information of Lie algebras, the readers are referred to [7, 10, 12].

Most of the problems in engineering, medical science, economics, environments, and so forth, have various uncertainties. The problems in system identification involve characteristics which are essentially non probabilistic in nature. In response to this situation Zadeh [19] introduced fuzzy set theory as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [20]. Molodtsov [16] initiated the concept of soft set theory as a new mathematical

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tool for dealing with uncertainties and vagueness. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields including game theory, operations research, Riemann-integration, Perron integration. At present, work on soft set theory is progressing rapidly. After Molodtsov's work, some operations and application of soft sets were studied by many researchers including Ali et al. [6], Aktas et al. [5], Chen et al. [9] and Maji et al. [15]. Maji et al. [15] gave first practical application of soft sets in decision making problems. The algebraic structure of soft set theories has been studied increasingly in recent years. Aktas and Cagman [5] defined the notion of soft groups. Feng et al. [13] initiated the study of soft semirings and soft rings were defined by Acar et al. [1], Cağman et al. [8] introduced the concept of soft int-groups, Yamark et al. [17] introduced soft hyperstructure. In this paper, we introduce the concept of soft Lie subalgebras (resp. soft Lie ideals) and state some of their fundamental properties. We also introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) and investigate some of their properties.

2. Review of literature

In this paper by L will be a Lie algebra. We note that the multiplication in a Lie algebra is not associative, but it is *anti commutative*, i.e., [x, y] = -[y, x] for all $x, y \in L$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

In 1999, Molodtsov [16] initiated soft set theory as a new approach for modelling uncertainties. Later on, Maji et al.[14] expanded this theory to fuzzy soft set theory. Based on the idea of parameterization, a soft set gives a series of approximate descriptions of a complicate object from various different aspects. Each approximate description has two parts, namely predicate and approximate value set. A soft set can be determined by a set-valued mapping assigning to each parameter exactly one crisp subset of the universe. More specifically, we can define the notion of soft set in the following way. Let U be an initial universe and E be a set of parameters. Let P(U) denote the power set of U and let A be a non-empty subset of E.

Definition 2.1. A pair $F_A = (F, A)$ is called a *soft set* over U, where $A \subseteq E$ and $F: A \to P(U)$ is a set-valued mapping, called the *approximate function* of the soft set F_A . It is easy to represent a soft set F_A by a set of ordered pairs as follows:

$$F_A = (F, A) = \{ (x, F(x)) \mid x \in A \}.$$

It is clear that a soft set is a parameterized family of subsets of the set U.

Definition 2.2. Let F_A and G_B be two soft sets over a common universe U. F_A is a said to be *soft subset* of G_B , denoted by $F_A \subset G_B$, if $F(x) \subseteq G(x)$ for all $x \in E$.

We refer the readers to the papers [2-4, 6, 9, 12, 13, 15, 16, 18] for further information regarding soft set theory and the theory of fuzzy Lie algebras.

3. Soft Intersection Lie algebras

Definition 3.1. Let $F_A = (F, A)$ be a soft set over L. Then F_A is called a *soft* Lie subalgebra (resp. soft Lie ideal) over L if F(x) is a Lie subalgebra (resp. Lie ideal) of a Lie algebra L for all $x \in A$.

Example 3.2. The real vector space \mathbb{R}^3 with the bracket [.,.] defined as the cross product, i.e., $[x, y] = x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$ forms a real Lie algebra over the field R. Now we define a soft set $\langle F, \mathbb{R}^3 \rangle$ as $F : \mathbb{R}^3 \longrightarrow P(\mathbb{R}^3)$ by $F((0,0,0)) = \{(0,0,0)\}, F(x,0,0) = \{(0,0,0), (x,0,0) : x \neq 0\}$ and $F(x,y,z) = \mathbb{R}^3$. By routine computations, it is easy to see that (F, \mathbb{R}^3) is soft Lie subalgebra but not soft Lie ideal of \mathbb{R}^3 .

Example 3.3. Let $\{e_1, e_2, \ldots, e_5\}$ be a basis of a vector space over V over a field F with Lie brackets as follows:

$$\begin{split} & [e_1,e_2]=e_3, \quad [e_1,e_3]=e_5, \quad [e_1,e_4]=e_5, \quad [e_1,e_5]=0, \\ & [e_2,e_3]=e_5, \quad [e_2,e_4]=0, \quad [e_2,e_5]=0, \quad [e_3,e_4]=0, \\ & [e_3,e_5]=0, \quad [e_4,e_5]=0, \quad [e_i,e_j]=-[e_j,e_i] \end{split}$$

and $[e_i, e_j] = 0$ for all i = j. Then V is a Lie algebra over F. Let (F, V) be soft over V and define by

$$F(x) = \begin{cases} \langle e_7 \rangle \text{ if } x = e_1 \\ \langle e_8 \rangle \text{ if } x = e_2, e_3 \\ \langle e_7, e_8 \rangle \text{ if } x = e_4, e_5 \\ V \text{ otherwise} \end{cases}$$

Routine computations show that (F, V) is a soft Lie ideal over V.

Definition 3.4. [9] Let F_A and G_B be two soft sets over a common universe U. Then the *intersection* $F_A \cap G_B$, is defined as $F_A \cap G_B(x) = F(x) \cap G(x)$ for all $x \in E$. The \wedge -product $F_A \wedge G_B$, is defined by $F_A \wedge G_B(x, y) = F(x) \cap G(y)$ for all $(x, y) \in E \times E$.

Proposition 3.5. If F_A and F_B are soft Lie subalgebras (resp. soft Lie ideals) over L. Then $F_A \wedge F_B$ and $F_A \cap F_B$ are soft Lie subalgebras (resp. soft Lie ideals) over L.

Definition 3.6. A soft Lie subalgebra (resp. soft Lie ideal) F_A over L is called *trivial* over L if $F(x) = \{\underline{0}\}$ for all $x \in A$, and whole over L if F(x) = L for all $x \in A$.

Definition 3.7. Let L_1 , L_2 be two Lie algebras and $\varphi : L_1 \to L_2$ a mapping of Lie algebras. If F_A and G_B are soft sets over L_1 and L_2 respectively, then $\varphi(F_A)$ is a soft set over L_2 where $\varphi(F) : E \to P(L_2)$ is defined by $\varphi(F)(x) = \varphi(F(x))$ for all $x \in E$ and $\varphi^{-1}(G_B)$ is a soft set over L_1 where $\varphi^{-1}(G) : E \to P(L_1)$ is defined by $\varphi^{-1}(G)(y) = \varphi^{-1}(G(y))$ for all $y \in E$.

Proposition 3.8. Let $\varphi: L_1 \to L_2$ be an onto homomorphism of Lie algebras.

- (i) If F_A is a soft Lie algebra over L_1 , then $\varphi(F_A)$ is a soft Lie algebra over L_2 .
- (ii) If F_B is a soft Lie algebra over L_2 , then $\varphi^{-1}(F_B)$ is a soft Lie algebra over L_1 if it is non-null.

Theorem 3.9. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. Let F_A and G_B be two soft Lie algebras over L_1 and L_2 , respectively.

- (a) If $F(x) = \ker(\varphi)$ for all $x \in A$, then $\varphi(F_A)$ is the trivial soft Lie algebra over L_2 .
- (b) If φ is onto and F_A is whole, then $\varphi(F_A)$ is the whole soft Lie algebra over L_2 .
- (c) If $G(y) = \varphi(L_1)$ for all $y \in B$, then $\varphi^{-1}(G_B)$ is the whole soft Lie algebra over L_1 .
- (d) If φ is injective and G_B is trivial, then $\varphi^{-1}(G_B)$ is the trivial soft Lie algebra over L_1 .

We now introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals).

Definition 3.10. Let L = E be a Lie algebra and let A be a subset of L. Let F_A be a soft set over U. Then, F_A is called a *soft intersection Lie subalgebra* over U if it satisfies the following conditions:

- (a) $F(x+y) \supseteq F(x) \cap F(y)$,
- (b) $F(mx) \supseteq F(x)$,
- (c) $F([x,y]) \supseteq F(x) \cap F(y)$

for all $x, y \in A$, $m \in K$. A soft set F_A is called a *soft intersection Lie ideal* over U if it satisfies (a), (b) and

(d) $F([x,y]) \supseteq F(x)$

for all
$$x, y \in A$$
.

Example 3.11. Assume that $U = \mathbb{Z}$ is the universal set. The vector space $E = \mathbb{R}^2$ with the bracket [.,.] defined as the usual cross product, i.e., $[x, y] = x \times y = xy - yx$ forms a real Lie algebra. Let $A = \{(0,0), (0,x), x \neq 0\}$ be a subset of E. Let F_A be a soft set over U. Then $F(0,0) = \mathbb{Z}$ and $F(0,x) = \{-2,-1,0,1,2\}$. It is easy to see that F_A is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.

From now on, we will always assume L = E unless otherwise specified.

Proposition 3.12. Let L be a Lie algebra and let A be Lie subalgebra (resp. Lie ideal) of L. If F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U. Then $F(\underline{0}) \supseteq F(x)$ and F(-x) = F(x) for all $x \in A$.

Proposition 3.13. Let L be a Lie algebra and let A and B be Lie subalgebras (resp. Lie ideals) of L. If F_A and G_B are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A \wedge G_B$ is a soft intersection Lie subalgebras (resp. soft intersection Lie ideal) over U.

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$ and $m \in K$. Then

$$\begin{split} (F_A \widetilde{\wedge} G_B)((x_1, y_1) + (x_2, y_2)) &= (F_A \widetilde{\wedge} G_B)((x_1 + x_2, y_1 + y_2)) \\ &= F(x_1 + x_2) \cap G(y_1 + y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) \cap (G(y_1) \cap G(y_2)) \\ &= (F(x_1) \cap G(y_1)) \cap (F(x_2) \cap G(y_2)) \\ &= (F_A \widetilde{\wedge} G_B)(x_1, y_1) \cap (F_A \widetilde{\wedge} G_B)(x_2, y_2), \\ (F_A \widetilde{\wedge} G_B)(m(x_1, y_1)) &= (F_A \widetilde{\wedge} G_B)(mx_1, my_1) \\ &= F(mx_1) \cap G(my_1) \\ &\supseteq F(x_1) \cap G(y_1) = (F_A \widetilde{\wedge} G_B)(x_1, y_1), \\ (F_A \widetilde{\wedge} G_B)([(x_1, y_1), (x_2, y_2)]) &= (F_A \widetilde{\wedge} G_B)([x_1, x_2], [y_1, y_2]) \\ &= F([x_1, x_2]) \cap G([y_1, y_2]) \\ &\supseteq ([F(x_1), F(x_2)] \cap [G(y_1), G(y_2)]) \\ &= [F(x_1), G(y_1)] \cap [F(x_2), G(y_2)] \\ &= (F_A \widetilde{\wedge} G_B)([(x_1, y_1), (x_2, y_2)]) &= (F_A \widetilde{\wedge} G_B)([x_1, x_2], [y_1, y_2]) \\ &= F([x_1, x_2]) \cap G([y_1, y_2]) \\ &= F([x_1, x_2]) \cap G([y_1, y_2]) \\ &= F([x_1, x_2]) \cap G([y_1, y_2]) \\ &\subseteq F(x_1) \cap G(y_1) = (F_A \widetilde{\wedge} G_B)[x_1, y_1]. \end{split}$$

Hence $F_A \wedge G_B$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

Theorem 3.14. Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\widetilde{\bigwedge}_{i \in \Lambda}(F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

Proposition 3.15. Let L be a Lie algebra and let A be a Lie subalgebra (resp. Lie ideal) of L. If F_A and G_A are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A \cap G_A$ is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.

Proof. Similarly as Proposition 3.13.

Theorem 3.16. Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\widetilde{\bigcap}_{i \in \Lambda}(F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

In the same manner we can prove

Proposition 3.17. Let A and B be Lie subalgebras (resp. Lie ideals) of a Lie algebra L. If F_A and G_B are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A \times G_B$ defined by $F_A \times G_B(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$, is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.

Theorem 3.18. Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\prod_{i \in \Lambda} (F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

Proposition 3.19. Let L be a Lie algebra and let A, B and C be Lie subalgebras (resp. Lie ideals) of L. If F_A , G_B and F_C are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U such that $F_A \cong G_B$ and $F_C \cong G_B$, then $F_A \cap F_C \cong G_B$ over U.

Proof. Straightforward.

Definition 3.20. Let F_A and G_B be two soft sets over the common universe U and let φ be a function from A to B. The soft image $\varphi(F_A)$ of F_A under φ is a soft set over U defined by

$$\varphi(F)(y) = \begin{cases} \bigcup \{F(x) \mid x \in A \text{ and } \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $y \in B$. The soft pre-image (or soft inverse image) $\varphi^{-1}(G_B)$ of G_B under φ is a soft set over U such that $\varphi^{-1}(G)(x) = G(\varphi(x))$ for all $x \in A$.

Proposition 3.21. Let L be a Lie algebra and let A be Lie ideal of L. If F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $A_F = \{x \in A \mid F(x) = F(\underline{0})\}$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

Theorem 3.22. Let A and B be Lie ideals of a Lie algebra L and φ be a Lie homomorphism from A to B. If G_B is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $\varphi^{-1}(G_B)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

Proof. Straightforward.

Theorem 3.23. Let A and B be Lie ideals of a Lie algebra L. If $\varphi : A \to B$ is a surjective Lie homomorphism and F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $\varphi(F_A)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U. *Proof.* Since φ is surjective, for all $a, b \in B$ there exist $x, y \in A$ such that $a = \varphi(x)$ and $b = \varphi(y)$. Then

$$\begin{split} \varphi(F_A)(x+y) &= \cup \{F(z) \mid z \in A, \varphi(z) = a + b\} \\ &= \cup \{F(x+y) \mid x, y \in A, \varphi(x) = a, \varphi(y) = b\} \\ &\supseteq \cup \{F(x) \cap F(y) \mid x, y \in A, \varphi(x) = a, \varphi(y) = b\} \\ &= (\cup \{F(x) \mid x \in A, \varphi(x) = a\}) \cap (\cup \{F(y) \mid y \in A, \varphi(y) = b\}) \\ &= \varphi(F_A)(a) \cap \varphi(F_A)(b), \\ \\ \varphi(F_A)(mx) &= \cup \{F(z) \mid z \in A, \varphi(z) = ma\} \\ &= \cup \{F(mx) \mid x \in A, \varphi(x) = a\} \\ &\supseteq \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &\supseteq \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \varphi(F_A)(a), \\ \\ \varphi(F)([x,y]) &= \cup \{F(z) \mid z \in A, \varphi(z) = [a,b]\} \\ &= \cup \{F(x) \cap F(y) \mid x, y \in A, \varphi(x) = a, \varphi(y) = b\} \\ &\supseteq \cup \{F(x) \mid x \in A, \varphi(x) = a\}) \cap (\cup \{F(y) \mid y \in A, \varphi(y) = b\}) \\ &= \varphi(F_A)(a) \cap \varphi(F_A)(b), \\ \\ \varphi(F_A)([x,y]) &= \cup \{F(z) \mid z \in A, \varphi(z) = [a,b]\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \cup \{F(x) \mid x \in A, \varphi(x) = a\} \\ &= \varphi(F_A)(a). \end{split}$$

Hence $\varphi(F_A)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.

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