# OD-Characterization of almost simple groups related to $U_3(17)$

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**Abstract.** We characterize groups with the same order and degree pattern as an almost simple groups related to $U_3(17)$.

## 1. Introduction

Let $G$ be a finite group. For any group $G$, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. Let $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. The *prime graph* $\Gamma(G)$ of a group $G$ is the graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (we write $p \sim q$) if and only if $G$ contains an element of order $pq$ ($pq \in \pi_e(G)$). For $p \in \pi(G)$, we put $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which is called the *degree* of $p$. If $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ we define $D(G) := (deg(p_1), deg(p_2), \ldots, deg(p_k))$, where $p_1 < p_2 < \ldots < p_k$, to be called the *degree pattern* of $G$. A group $G$ is called *k-fold OD-characterizable* if there exist exactly $k$ non-isomorphic finite groups having the same order and degree pattern as $G$. In particular, a 1-fold $OD$-characterizable group is simply called *OD-characterizable*. A group $G$ is said to be an almost simple group related to $S$ if and only if $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$ for some non-abelian simple group $S$. In a series of articles, it has been proved, up to now, that many finite almost simple groups are $OD$-characterizable or $k$-fold $OD$-characterizable for $k \geqslant 2$, for instance see [2, 3, 5, 7, 8, 9]. In this paper $U := U_3(17)$ and $\mathrm{Aut}(U) \cong U : \mathbb{S}_3$ and we show that $U$ and $U : 2$ are $OD$-characterizable, also $U : 3$ and $U : \mathbb{S}_3$ are 3-fold and 5-fold $OD$-characterizable respectively ($H.K$ means an extension of a group $H$ by a group $K$ and $H : K$ denotes split extension). We denote the socle of $G$ by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\mathrm{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively, all further unexplained notation are standard and can be found in [4].

Throughout this article, all groups under consideration are finite.

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## 2. Lemmas

It is well-known that $\mathrm{Aut}(U_3(17)) \cong U_3(17) : \mathbb{S}_3$, hence the following lemma follows from definition.

**Lemma 2.1.** *If $G$ is an almost simple group related to $U := U_3(17)$, then $G$ is isomorphic to one of the following groups: $U$, $U:2$, $U:3$ or $U:\mathbb{S}_3$.* □

$G$ is said to be *completely reducible group* if and only if either $G=1$ or $G$ is the direct product of a finite number of simple groups. A completely reducible group will be called a *$CR$-group.* A $CR$-group has trivial center if and only if it is a direct product of non-abelian simple groups and in this case, it has been named a *centerless $CR$-group.* The following lemma determines the structure of the automorphism group of a centerless $CR$-group.

**Lemma 2.2.** ([4], Theorem 3.3.20) *Let $R$ be a finite centerless $CR$-group and write $R = R_1 \times R_2 \times \ldots \times R_k$, where $R_i$ is a direct product of $n_i$ isomorphic copies of a simple group $H_i$, and $H_i$ and $H_j$ are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times \ldots \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$*, where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms* $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times \ldots \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$. □

Let $p \geqslant 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most $p$. Clearly, if $q \leqslant p$ then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class $\mathfrak{S}_{17}$ in Table 1 below, taken from [6].

Table 1: Simple groups in $\mathfrak{S}_p$, $p \leqslant 17$.

| $S$ | $\lvert S\rvert$ | $\lvert\mathrm{Out}(S)\rvert$ | $S$ | $\lvert S\rvert$ | $\lvert\mathrm{Out}(S)\rvert$ |
|---|---|---|---|---|---|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | 6 |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2 | $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | 2 |
| $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 1 |
| $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 | $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 |
| $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 2 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $A_{11}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 2 |
| $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | $M^cL$ | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | 2 |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 | $HS$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | 2 |
| $A_{12}$ | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ | 2 | $A_{15}$ | $2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ | 6 | $L_6(3)$ | $2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$ | 4 |

(continued)

| $S$ | $\|S\|$ | $\|\mathrm{Out}(S)\|$ | $S$ | $\|S\|$ | $\|\mathrm{Out}(S)\|$ |
|---|---|---|---|---|---|
| $L_3(3)$ | $2^4 \cdot 3^3 \cdot 13$ | 2 | $Suz$ | $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $A_{16}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $U_3(4)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $Fi_{22}$ | $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $S_4(5)$ | $2^6 \cdot 3^2 \cdot 5^4 \cdot 13$ | 2 | $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | 4 | $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ | 4 |
| ${}^2F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ | 2 | $S_4(4)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ | 4 |
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 2 | $He$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ | 2 |
| $L_2(27)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | 6 | $O_8^-(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 2 |
| $G_2(3)$ | $2^6 \cdot 3^6 \cdot 7 \cdot 13$ | 2 | $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$ | 4 |
| ${}^3D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | 3 | $S_8(2)$ | $2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ | 1 |
| $Sz(8)$ | $2^6 \cdot 5 \cdot 7 \cdot 13$ | 3 | $O_{10}^-(2)$ | $2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ | 2 |
| $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 | $F_4(2)$ | $2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ | 2 |
| $U_4(5)$ | $2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$ | 4 | $U_4(4)$ | $2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$ | 4 |
| $L_3(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 | $S_6(4)$ | $2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $S_6(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | 2 | $O_8^+(4)$ | $2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ | 12 |
| $O_7(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | 2 | $L_3(16)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$ | 24 |
| $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 | $S_4(13)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$ | 2 |
| $S_4(8)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ | 6 | $L_2(13^2)$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$ | 4 |
| $O_8^+(3)$ | $2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ | 24 | $U_3(17)$ | $2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$ | 6 |
| $L_5(3)$ | $2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$ | 2 | $A_{17}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $A_{13}$ | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 | $A_{18}$ | $2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $A_{14}$ | $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ | 2 | | | |

**Lemma 2.3.** ([1], Theorem 10.3.1) *Let $G$ be a Frobenius group with kernel $K$ and complement $H$. Then*

(a) *$K$ is a nilpotent group,*

(b) $|K| \equiv 1(\mathrm{mod}|H|)$. □

# 3. Almost simple groups related to $U_3(17)$

**Theorem 3.1.** *Let $M$ be an almost simple group related to $U := U_3(17)$. If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:*

(a) *If $M = U$, then $G \cong U$.*

(b) *If $M = U : 2$, then $G \cong U : 2$.*

(c) *If $M = U : 3$, then $G \cong U : 3$, $\mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$.*

(d) *If $M = U : \mathbb{S}_3$, then $G \cong U : \mathbb{S}_3$, $\mathbb{Z}_3 \times (U : 2)$, $\mathbb{Z}_3.(U : 2)$, $(\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.*

*In particular, $U$ and $U : 2$ are OD-characterizable, $U : 3$ is 3-fold OD-characterizable and $U : \mathbb{S}_3$ is 5-fold OD-characterizable.*

*Proof.* We break the proof into a number of separate cases. Note that the set of order elements in each of the following cases is calculated using GAP.

**Case 1.** If $M = U$, then $G \cong U$.

By Table 1, $|G| = |U| = 2^6.3^4.7.13.17^3$ and we have $\pi_e(U) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 24, 32, 34, 48, 51, 91, 96, 102\}$, so by assumption, $D(G) = D(U) = (2, 2, 1, 1, 2)$. Therefore, there exist two possibilities for $\Gamma(G)$ are as follows:

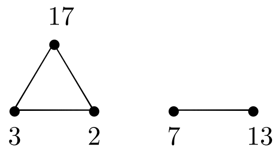


Figure 1-1

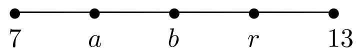


Figure 1-2

where $a, b, r$ are distinct prime numbers that belong to $\{2, 3, 17\}$. We have to show that $G$ is isomorphic to $U := U_3(17)$ and we break the proof into a sequence of steps.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

We consider these two parts separately:

**Part A.** Consider Figure 1-1, and Figure 1-2 where $r \neq 17$.

First, we show that $K$ is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 does not divide the order of $K$ (otherwise, we may suppose that $T$ is a Hall $\{17, 13\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $13.17^i$ for $i = 1, 2$ or 3. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 7, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, $N$ contains an element of order 13, say $\sigma$. Since $G$ has no element of order 13.17, $\langle\sigma\rangle$ should act fixed point freely on $K_{17}$, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle| \,|\, (|K_{17}| - 1)$. It follows that $13 | 17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.

Next, we show that $K$ is a $p'$-group for $p \in \{13, 7\}$. Let $x$ be an element of $K$ of order $p$ and set

$$C := C_G(x), \quad N := N_G(< x >).$$

Let $p = 13$. According to Figure 1-1, $C$ is a $\{7, 13\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence $N$ is a $\{2, 3, 7, 13\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of $K$, which is a contradiction. According to Figure 1-2, $C$ is a $\{r, 13\}$-group, where $r = 2$ or 3. Therefore, by the same argument, we conclude that $N$ is a $\{2, 3, 13\}$-group and by the Frattini argument, 17 must divide the order of $K$, which is a contradiction, so $K$ is a $\{2, 3, 7\}$-group.

Let $p = 7$. According to Figure 1-1, $C$ is a $\{7, 13\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence $N$ is a $\{2, 3, 7, 13\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of $K$, which is a contradiction. According to Figure 1-2, $C$ is a $\{7, a\}$-

group, where $a = 2, 3$ or 17. Then by the same argument, we conclude that $N$ is a $\{2, 3, 7\}$-group for $a = 2, 3$, and $\{2, 3, 7, 17\}$-group for $a = 17$. Now by the Frattini argument, $G = KN$ then 13 must divide the order of $K$, which is a contradiction. Therefore, $K$ is a $\{2, 3\}$-group.

**Part B.** Consider Figure 1-2 where $r = 17$.

First, we show that $K$ is a $17^{'}$-group. Assume the contrary and let $17 \in \pi(K)$. Then 7 does not divide the order of $K$ (otherwise, we may suppose that $T$ is a Hall $\{7, 17\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $7.17^i$ for $i = 1, 2$ or 3. Thus, $7.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 13, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, $N$ contains an element of order 7, say $\sigma$. Since $G$ has no element of order 7.17, $\langle\sigma\rangle$ should act fixed point freely on $K_{17}$, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle||(|K_{17}| - 1)$. It follows that $7|17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.
Next, we show that $K$ is a $p^{'}$-group for $p \in \{13, 7\}$. Let $x$ be an element of $K$ of order $p$ and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 7$. By the prime graph of $G$, $C$ is a $\{7, a\}$-group, where $a = 2$ or 3. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence $N$ is a $\{2, 3, 7\}$-group and by the Frattini argument, $G = KN$, so 17 must divide the order of $K$, which is a contradiction. Therefore, $K$ is a $\{2, 3, 13\}$-group.

Let $p = 13$. By the prime graph of $G$, $C$ is a $\{13, 17\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence $N$ is a $\{2, 3, 13, 17\}$-group and by the Frattini argument, 7 must divide the order of $K$, which is a contradiction, so $K$ is a $\{2, 3\}$-group. In addition since $G \neq K$, $G$ is non-solvable, and this completes the proof of Step 1.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$, see [3, Proposition 3.1, Step 2]. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by Step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some $j$, 13 divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see Table 1), so 13 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$ . Therefore, $t \geqslant 13$ and so $2^{26}$ must divide the order of $G$, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof is completed.

**Step 3.** $G$ is isomorphic to $U_3(17)$.

By Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 6$ and $1 \leqslant \beta \leqslant 4$. Now, using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. As $|G| = |U|$, we deduce $K = 1$, so $G \cong U$, and the proof is completed.

**Case 2.** If $M = U : 2$, then $G \cong U : 2$.

$|G| = 2|U| = 2^7.3^4.7.13.17^3$ and $\pi_e(U:2) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 24, 32, 34, 36, 48, 51, 68, 91, 96, 102\}$, so $D(G) = D(U:2) = (2, 2, 1, 1, 2)$, and therefore we conclude that the possibilities for $\Gamma(G)$ are as in Figure 1-1 and Figure 1-2, where $a, b, r$ are distinct prime numbers that belong to $\{2, 3, 17\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

By an argument similar to that used in Case 1, we can obtain this assertion.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where $S$ is a finite non-abelian simple group.

The proof is similar to Step 2, in Case 1.
By Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 7$ and $1 \leqslant \beta \leqslant 4$. Now, using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$. Therefore by Step 2, $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$, which implies that $|K| = 1$ or $2$.

If $|K| = 1$, then $G \cong U : 2$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 4$, which is a contradiction.

**Case 3.** If $M = U : 3$, then $G \cong U : 3$, $\mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$.

$|G| = 3|U| = 2^6.3^5.7.13.17^3$ and $\pi_e(U:3) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 21, 24, 32, 34, 36, 39, 48, 51, 72, 91, 96, 102, 144, 153, 273, 288, 306\}$. Thus, we get $D(G) = D(U:3) = (2, 4, 2, 2, 2)$. Therefore we have two possibilities for $\Gamma(G)$:

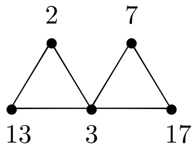


Figure 2-1

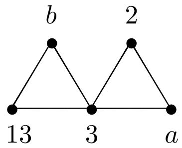


Figure 2-2

where $a, b$ are distinct prime numbers which belong to $\{7, 17\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.
We consider these two parts separately:

**Part A.** Consider Figure 2-1, and Figure 2-2 where $a = 17$ and $b = 7$.

First, we claim $K$ is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 does not divide the order of $K$ (otherwise, we may suppose that $T$ is a Hall $\{17, 13\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order

$13.17^i$ for $i = 1, 2$ or 3. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 7, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, $N$ contains an element of order 13, say $\sigma$. Since $G$ has no element of order 13.17, $\langle\sigma\rangle$ should act fixed point freely on $K_{17}$, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle||(|K_{17}| - 1)$. It follows that $13|17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.

Next, we show that $K$ is a $p'$-group for $p \in \{13, 7\}$. Let $x$ be an element of $K$ of order $p$ and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 13$. So $C$ is a $\{2, 3, 13\}$ and $\{3, 7, 13\}$-group, in Figure 2-1 and Figure 2-2 respectively. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence $N$ is a $\{2, 3, 13\}$-group in Figure 2-1, and $\{2, 3, 7, 13\}$-group in Figure 2-2. On the other hand, by the Frattini argument, $G = KN$. Then 17 must divide the order of $K$, which is a contradiction.

Let $p = 7$. According to Figure 2-1, $C$ is a $\{3, 7, 17\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence $N$ is a $\{2, 3, 7, 17\}$-group and by the Frattini argument, $G = KN$ then 13 must divide the order of $K$, which is a contradiction. According to Figure 2-2, $C$ is a $\{3, 7, 13\}$-group. Then by a same argument, we conclude that $N$ is a $\{2, 3, 7, 13\}$-group. Now by the Frattini argument, $G = KN$ then 17 must divide the order of $K$, which is a contradiction. Therefore, $K$ is a $\{2, 3\}$-group.

**Part B.** Consider Figure 2-2, where $a = 7$ and $b = 17$.

First, we claim $K$ is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 7 does not divide the order of $K$ (otherwise, we may suppose that $T$ is a Hall $\{7, 17\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $7.17^i$ for $i = 1, 2$ or 3. Thus, $7.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 13, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, $N$ contains an element of order 7, say $\sigma$. Since $G$ has no element of order 7.17, $\langle\sigma\rangle$ should act fixed point freely on $K_{17}$, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle||(|K_{17}| - 1)$. It follows that $7|17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction. Therefore, $K$ is a $17'$-group.

Next, we show that $K$ is a $p'$-group for $p \in \{7, 13\}$. Let $x$ be an element of $K$ of order $p$ and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 7$. So $C$ is a $\{2, 3, 7\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence $N$ is a $\{2, 3, 7\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of $K$, which is a contradiction.

Let $p = 13$. Therefore, $C$ is a $\{3, 13, 17\}$-group. Now, using $(N/C)$-Theorem, the factor group $N/C$ is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence $N$ is a $\{2, 3, 13, 17\}$-group and by the Frattini argument, $G = KN$ then 7 must divide the order of $K$, which is a contradiction. So $K$ is a $\{2,3\}$-group. In addition since $G \neq K$, $G$ is non-solvable, and this completes the proof of Step 1.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Similar to Step 1, we consider two parts:

**Part A.** Consider Figure 2-1, and Figure 2-2 when $a = 17$ and $b = 7$.
Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 7 does not divide $|S|$. Assume the contrary and let $7 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 7$ and $3 \sim 7$, which is a contradiction. Now, by Step 1, we observe that $7 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some $j$, 7 divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 7 (see Table 1), so 7 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore, $t \geqslant 7$ and so $2^{14}$ must divide the order of $G$, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof of this part is completed.

**Part B.** Consider Figure 2-2, when $a = 7$ and $b = 17$.
Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by Step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some $j$, 13 divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see Table 1), so 13 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$ . Therefore, $t \geqslant 13$ and so $2^{26}$ must divide the order of $G$, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof is completed.

Now by Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 6$ and $1 \leqslant \beta \leqslant 5$. By using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. Hence, $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong U : 3$.

If $|K| = 3$, then $G/K \cong U$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus, $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $\mathbb{Z}_3$ by $U$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_3 \times U$, otherwise, we have $G \cong \mathbb{Z}_3.U$. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq$

$C_G(K)/K \trianglelefteq G/K \cong U$. Thus, we obtain $G = C_G(K)$ because $U$ is simple, which is a contradiction.

**Case 4.** If $M = U : \mathbb{S}_3$, then $G \cong U : \mathbb{S}_3$, $\mathbb{Z}_3 \times (U : 2)$, $\mathbb{Z}_3.(U : 2)$, $(\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.

$|G| = 6|U| = 2^7.3^5.7.13.17^3$ and $\pi_e(U : \mathbb{S}_3) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 21, 24, 32, 34, 36, 39, 48, 51, 68, 72, 91, 96, 102, 144, 153, 273, 288, 306\}$, so $D(G) = D(U : \mathbb{S}_3) = (2, 4, 2, 2, 2)$, and therefore we conclude that there exist two possibilities for the prime graph of $G$ presented by Figure 2-1 and Figure 2-2, where $a, b$ are distinct prime numbers which belong to $\{7, 17\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

We can prove this by the similar way to that in Case 3.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where $S$ is a finite non-abelian simple group.

The proof is similar to Step 2, in Case 3.

Now by Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 7$ and $1 \leqslant \beta \leqslant 5$. By using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. Hence, $|K| = 1, 2, 3$ or $6$.

If $|K| = 1$, then $G \cong U : \mathbb{S}_3$.

If $|K| = 2$, then $K \leq Z(G)$. It follows that $deg(2) = 4$, which is a contradiction.

If $|K| = 3$, then $G/K \cong U : 2$. In this case, we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus, $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $\mathbb{Z}_3$ by $U : 2$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_3 \times (U : 2)$, otherwise, we have $G \cong \mathbb{Z}_3.(U : 2)$. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong U : 2$ and we obtain $C_G(K)/K \cong U$. Because $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $U$. Thus, $C_G(K) \cong \mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$. Therefore, $G \cong (\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong U$ and $K \cong \mathbb{Z}_6$ or $\mathbb{S}_3$.

If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 4$, a contradiction. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong U$, which is a contradiction because $U$ is simple.

If $K \cong \mathbb{S}_3$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim \mathbb{S}_3$. Thus, $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong U$. It follows that $U \cong G/K \cong C_G(K)$ because $U$ is simple. Therefore, $G \cong \mathbb{S}_3 \times U$, which implies that $deg(2) = 4$, a contradiction. The proof of Theorem 3.1 is completed. □

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