Shortest single axioms with neutral element for groups of exponent 2 and 3

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Abstract. In this note, we study identities in product and a constant e only that are valid in all groups of exponent 2 (3) with neutral element e and that imply that a groupoid satisfying one of them is a group of exponent 2 (3) with neutral element e. Such an identity will be called a single axiom with neutral element for groups of exponent 2 (3). We utilize the automated reasoning software Prover9 and Mace4 to attempt to find all shortest single axioms with neutral element for groups of exponent 2 (3). Beginning with a list of 1323 (1716) candidate identities that contains all shortest possible single axioms with neutral element for groups of exponent 2 (3), we find 173 (148) single axioms with neutral element for groups of exponent (2) 3 and eliminate all but 5 (119) of the remaining identities as not being single axioms with neutral element for groups of exponent 3. We also prove that a finite model of any of these 5 (119) identities must be a group of exponent 2 (3) with neutral element e.

1. Introduction

We assume the reader is familiar with the definitions of groupoids, semigroups, and groups. The variables v, w, x, y, and z will always be universally quantified. The letter e will always denote a constant and will denote the neutral element if in the context of a group. The letter n will always denote a natural number. We write x^{n+1} for xx^n, where $x^1 = x$. The capital letters S and T will always denote terms in product or in product and e unless otherwise stated and $T\backslash e$ will denote the corresponding term with all occurrences of e deleted. We denote by $V(T)$ the number of variable occurrences in T. Identities are always in product only or in product and e only.

Definition 1.1. A *group of exponent* n is a group such that $x^n = e$.

Strictly speaking, a group of exponent n is a group such that n is the *smallest* integer for which $x^n = e$. Therefore, our "groups of exponent n" are actually groups of exponent dividing n. Nevertheless, we will refer to any group satisfying the condition in Definition 1.1 as a group of exponent n.

Therefore, groups of exponent n can be axiomatized in terms of product only by

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(1) $xy \cdot z = x \cdot yz$,

(2) $x^n = y^n$, and

(3) $xy^n = x$

or in terms of product and e only by

(1′) $xy \cdot z = x \cdot yz$,

(2′) $x^n = e$, and

(3′) $xe = x$.

Definition 1.2. By an *identity* (*identity with neutral element*), we shall mean an identity in product only (identity in product and e only) unless otherwise stated. We say that an identity (identity with neutral element) $S = T$ is a *single axiom for groups of exponent* n (*single axiom with neutral element for groups of exponent* n) if and only if $S = T$ is true in all groups of exponent n (groups of exponent n with neutral element e) and every model of $S = T$ satisfies (1), (2), and (3) ((1′), (2′), and (3′)). In either case, it is clear that we must have S or T being just a single variable occurrence, otherwise the identity would be valid in any zero semigroup. We sometimes refer to identities and identities with neutral element simply as identities (Note that we do not assume that e is neutral, only that it is a constant. An identity must imply that e is neutral for it to be a single axiom.).

Neumann [12] proved that the variety of groups that satisfy $S = e$, where S is any term in product and inverse only, can be axiomatized by the single identity $T = x$, where T is the term in product and inverse only

$$w(((x^{-1} \cdot w^{-1}y)^{-1}z \cdot (xz)^{-1})(SS'^{-1})^{-1})^{-1}.$$

In the above, w, x, y, and z are variables not occurring in S and S' is a renaming of S using different variables. Taking $S = v^n$ and replacing all occurrences of -1 by $n - 1$ in the above identity, we obtain a single axiom for groups of exponent n of the form $T = x$, where $V(T) = n^4 - 2n^2 + n + 1$. This leaves open the problem of finding shorter and simpler single axioms (with neutral element) for groups of exponent n. No variety of groups can be axiomatized by a single identity in product, inverse, and neutral element [12], [15].

For example, in [11], Meredith and Prior proved that

$$(yx \cdot z) \cdot yz = x$$

is a single axiom for groups of exponent 2 (*Boolean groups*), in [10], Mendelsohn and Padmanabhan proved that

$$x \cdot (xy \cdot z)y = z$$

and

$$(xy \cdot xz)y = z$$

are also single axioms for Boolean groups, and, in [9], McCune and Wos proved that

$$x((y \cdot ez) \cdot xz) = y$$

is a single axiom with neutral element for Boolean groups.

As another example, in [9], it is proved that

$$y \cdot (y \cdot y(x \cdot zz))z = x,$$

$$y(y(yx \cdot z) \cdot zz) = x,$$

and

$$y((yy \cdot xz)z \cdot z) = x$$

are single axioms for groups of exponent 3 and that

$$x(x(xy \cdot z) \cdot (e \cdot zz)) = y$$

is a single axiom with neutral element for groups of exponent 3.

As a final example, in [5], Kunen proved that

$$y(y(yy \cdot xz) \cdot (z \cdot zz)) = x,$$

$$(yy \cdot y) \cdot ((y \cdot xz) \cdot zz)z = x,$$

and

$$y(((yy \cdot y) \cdot xz)z) \cdot zz = x$$

are single axioms for groups of exponent 4 and, in [9], it is proved that

$$x(x(x(e(xy \cdot z) \cdot z) \cdot z) \cdot z) = y$$

along with nine others are single axioms with neutral element for groups of exponent 4.

In the present note, we endeavor to find all shortest (with respect to the number of variable and constant occurrences) single axioms with neutral element for groups of exponent 2 (3) using the automated theorem-prover Prover9 and the model-finder Mace4. Beginning with a list of 1323 (1716) candidate identities that contains all shortest possible single axioms with neutral element for groups of exponent 3, we find 173 (148) single axioms with neutral element for groups of exponent 2 (3) and eliminate all but 5 (119) of the remaining identities as not being single axioms with neutral element for groups of exponent 2 (3). We also prove that a finite model of any of these 5 (119) identities must be a group of exponent 2 (3) with neutral element e, hence obtaining the same type of classification as was achieved in [10] ([4]) for shortest single axioms (without neutral element) for groups of exponent 2 (3).

2. Preliminary Results

In this section, we present some preliminary results that will be needed in the subsequent sections. We begin with an obvious observation.

Definition 2.1. Define the *mirror* of T, denoted $M(T)$, as follows: $M(T'T'') = M(T'')M(T')$ for subterms T' and T'' of T, $M(x) = x$ for variables x occurring in T, and $M(e) = e$ for constants e occurring in T.

Theorem 2.2. *The identity* (*with neutral element*) $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n *if and only if* $M(T) = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n.

The next result demonstrates that the structure of the single axioms (with neutral element) for groups of exponent n presented in Section is no accident.

Theorem 2.3. [4], [5] *Suppose* $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n, $n \geq 2$. *Then* $V(T) \geq 2n+1$. *If* $n = 2$ *and* $V(T) = 5$, *then a renaming of* $T\backslash e$ *is an association of an arrangement of* y^2xz^2 *If* $n \geq 3$ *and* $V(T) = 2n+1$, *then a renaming of* $T\backslash e$ *is an association of* y^nxz^n. (*in the latter case, it is clear that we must not have* x *being the left-most* (*right-most*) *symbol in* T, *otherwise the identity would be valid in any left-zero* (*right-zero*) *semigroup*).

In light of Theorem 2.3, the single axioms (with neutral element) for groups of exponent n presented in Section are as short as possible. In the case of exponent 3, in [4], it is proved that the three examples from Section are the only shortest single axioms (up to renaming, mirroring, and symmetry), with the possible exceptions of

$$y \cdot y((y \cdot xz) \cdot zz) = x$$

and

$$yy \cdot (y(xz \cdot z) \cdot z) = x.$$

The status of these two identities is unknown. It is known that a non-group model of either identity must be infinite [4]. In the case of exponent 4, it is proved in [5] that the three examples from Section are the only shortest single axioms (up to mirroring, renaming, and symmetry). It is known that there are shortest possible single axioms (with neutral element) for groups of exponent n, n odd [4]. It is unknown if there are shortest possible single axioms (with neutral element) for groups of exponent n, $n \geq 6$ even. An exhaustive search for shortest possible single axioms for groups of exponent 6 was attempted in [3]. The search failed to find any single axioms but did reduce the number of candidates to 204.

We need two more results.

Theorem 2.4. [4], [5] *Suppose a renaming of* $T\backslash e$ *is an association of an arrangement of* y^2xz^2 *with* T *containing at most one occurrence of* e. *Then any associative and commutative model of* $T = x$ *is a group of exponent* 2 (*in particular, if all* *models of* $T = x$ *are associative and commutative, then* $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* 2). *Suppose a renaming of* $T\backslash e$ *is an association of* $y^n x z^n$, $n \geq 3$, *with* T *containing at most one occurrence of* e. *Then any associative model of* $T = x$ *is a group of exponent* n (*in particular, if all models of* $T = x$ *are associative, then* $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n).

Theorem 2.5. [4] *If* G *is a non-trivial group, then there exists a non-associative groupoid* H *such that* H *satisfies every identity that contains at most two distinct variables and that is valid in* G.

3. Prover9 and Mace4

In this section, we briefly describe the software Prover9 and Mace4.

Prover9 [8] is a resolution-style [1], [13] automated theorem-prover for first-order logic with equality that was developed by McCune at Argonne National Laboratory. Prover9 is the successor to the well-known OTTER [7] theorem-prover and, like OTTER, utilizes the *set of support strategy* [1], [16].

The language of Prover9 is the language of *clauses*, a clause being a disjunction of (possible one or zero) literals in which all variables whose names begin with u, v, w, x, y, or z are implicitly universally quantified and all other symbols represent constants, functions, or predicates (relations). An axiom may also be given to Prover9 as an explicitly quantified first-order formula which is immediately transformed by Prover9 into a set of clauses by a *Skolemization* [1], [2] procedure. The conjunction of these clauses is not necessarily logically equivalent to the formula, but they will be *equisatisfiable* [1], [2]. Therefore, the set of clauses can be used by Prover9 in place of the formula in proofs by contradiction.

Prover9 can be asked to prove a potential theorem by giving it clauses or formulas expressing the hypotheses and a clause or formula expressing the negation of the conclusion. Prover9 finds a proof when it derives the *empty clause*, a contradiction.

Prover9 has an *autonomous mode* [8] in which all inference rules, settings, and parameters are automatically set based upon a syntactic analysis of the input clauses (the mechanisms of inference for purely equational problems being *demodulation* and *paramodulation* [1], [14]).

One very important parameter used by Prover9 is the *maximum weight* [8] of a clause. By default, the weight of a literal is the number of occurrences of constants, variables, functions, and predicates in the literal and the weight of a clause is the sum of the weights of its literals. Prover9 discards derived clauses whose weight exceeds the maximum weight specified. By specifying a maximum weight, we sacrifice *refutation-completeness* [1], [13], although in practice it is frequently necessary. We will use the autonomous mode throughout this paper, sometimes overriding Prover9's assignment to the maximum weight parameter.

A useful companion to Prover9 is Mace4 [6], also developed by McCune. Mace4 is a finite first-order model-finder. With possibly some minor modifications, the same input can be given to Mace4 as to Prover9, Prover9 searching for a proof by contradiction and Mace4 searching for counter-examples of specified sizes (a groupoid found by Mace4 would be returned as an $n \times n$ Cayley table with the elements of the structure assumed to be $0, 1, \ldots, n-1$ and the element in the ith row and jth column being ij).

Remark 3.1. The reader should note that Mace4 interprets non-negative integers as *distinct* constants and other constants as not necessarily distinct unless otherwise stated. This is in contrast to Prover9 which interprets all constants as not necessarily distinct unless otherwise stated. The use of non-negative integers for constants in Mace4 can have the advantage of speeding up the search for a model.

The scripting language Perl was also used to further automate the process.

4. The Search

In this section, we describe our search for shortest single axioms with neutral element for groups of exponent 3.

First, all identities $T = x$ such that T contains exactly one occurrence of e and $T\backslash e$ is an association of an arrangement of y^3xz^3 were generated up to renaming and mirroring. This resulted in 1716 identities.

We then sent the negation of each identity (stored in the Perl variable `$negated_identity`) to Prover9 and ran

```
assign(max_seconds, 1).  % one second time limit per identity
formulas(sos).           % set of support clauses
e * x = x.
x * e = x.
x * y = y * x.
(x * x) * x = e.
x * (x * x) = e.
(x * x) * (x * y) = y.
x * (x * (x * y)) = y.
((x * y) * y) * y = x.
(x * y) * (y * y) = x.   % one and two distinct variable identities
                         % valid in Z_3
$negated_identity.       % negation of candidate identity
end_of_list.             % end of set of support clauses
```

to search for a proof that the identity is derivable from the set of one and two distinct variable identities that are valid in $\mathbb{Z}_3$. If this is the case, then by Theorem 2.5, there is a non-associative model for the identity and it can be eliminated. This reduced the number of candidate identities to 546.

Remark 4.1. We determine whether or not Prover9 has found a proof by observing its exit status. Prover9 outputs an exit code of 0 if and only if it finds a proof.

Next, we sent each identity (stored in the variable `$identity`) to Mace4 (`e` will always be renamed `0` in Mace4 input) and ran

```
assign(max_seconds, 60).              % one minute time limit
                                      % per identity
formulas(theory).                     % theory clauses
x * y != x * z | y = z.               % left cancellative
y * x != z * x | y = z.               % right cancellative
(x * y) * (z * u) = (x * z) * (y * u). % medial
0 * 0 = 0.                            % e idempotent
$identity.                            % candidate identity
(a * b) * c != a * (b * c).           % non-associative
end_of_list.                          % end of theory clauses
```

to search for a non-associative, *cancellative* (*left cancellative* ($xy = xz$ implies $y = z$) and *right cancellative* ($yx = zx$ implies $y = z$)), *medial* ($xy \cdot zu = xz \cdot yu$) groupoid with e *idempotent* ($ee = e$) that satisfies the identity. Any identity for which an example was found was eliminated. This reduced the number of candidate identities to 267.

Remark 4.2. We determine whether or not Mace4 has found a model by observing its exit status. Mace4 outputs an exit code of 0 if and only if it finds a model.

We then sent each remaining identity to Prover9 and ran

```
assign(max_seconds, 300).          % five minute time limit
                                   % per weight per identity
assign(max_weight, $max_weight).   % maximum clause weight
formulas(sos).                     % set of support clauses
$identity.                         % candidate identity
a * b = a * c.
b != c.                            % not left cancellative
end_of_list.                       % end of set of support clauses
```

to search for a proof that the identity implies left cancellativity. We made a run for every value of `$max_weight` from 20 to 150 in steps of 5. A proof was found for 186 identities. The mirror of each identity for which a proof was not found was then sent back to Prover9 to search for a proof that it implies left cancellativity. An additional 57 proofs were found.

Next, we sent each of these 243 identities back to Prover9 and ran

```
assign(max_seconds, 300).          % five minute time limit
                                   % per weight per identity
assign(max_weight, $max_weight).   % maximum clause weight
formulas(sos).                      % set of support clauses
```

```
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
b * a = c * a.
b != c.                             % not right cancellative
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies right cancellativity. We made a run for every value of `$max_weight` from 20 to 150 in steps of 5. A proof was found for 148 identities.

We then sent each of these 148 identities back to Prover9 and ran

```
assign(max_seconds, 600).           % 10 minute time limit
                                    % per weight per identity
assign(max_weight, $max_weight).    % maximum clause weight
formulas(sos).                       % set of support clauses
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
y * x != z * x | y = z.             % right cancellative
e * e != e.                         % e not idempotent
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies that e is idempotent. We made a run for every value of `$max_weight` from 18 to 150 in steps of 2. A proof was found for all 148 identities.

Finally, we sent each of these 148 identities back to Prover9 and ran

```
assign(max_seconds, 600).           % 10 minute time limit
                                    % per weight per identity
assign(max_weight, $max_weight).    % maximum clause weight
formulas(sos).                       % set of support clauses
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
y * x != z * x | y = z.             % right cancellative
e * e = e.                          % e idempotent
(a * b) * c != a * (b * c).         % non-associative
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies associativity. We made a run for every value of `$max_weight` from 18 to 150 in steps of 2. A proof was found for all 148 identities. By Theorem 2.4, these 148 identities are all single axioms with neutral element for groups of exponent 3.

5. Finite Models

In this section, we show that a finite model of any of the 119 remaining candidate identities must be a group of exponent 3 with neutral element e.

Consider the following identity (one of the 119 remaining candidate identities).

$$(y \cdot ey)(y(x \cdot zz) \cdot z) = x$$

Let G be a finite groupoid satisfying this identity. Define $L_x, R_x : G \longrightarrow G$ by $L_x(y) = xy$ and $R_x(y) = yx$. Therefore,

$$L_{y \cdot ey} \circ R_z \circ L_y \circ R_{zz} = Id$$

where Id is the identity mapping on G. Thus, R_{zz} is injective and $L_{y \cdot ey}$ is surjective. Since G is finite, R_{zz} is surjective and $L_{y \cdot ey}$ is injective.

Running the third block of code in Section for every value of `$max_weight` from 20 to 150 in steps of 5 with this candidate identity and with the additional lines

```
R(z,u) = u * (z * z).          % R_zz definition
L(y,u) = (y * (e * y)) * u.    % L_y(ey) definition
R(z,f(z,u)) = u.               % R_zz surjective
L(y,u) != L(y,v) | u = v.      % L_y(ey) injective
```

in the set of support, Prover9 finds a proof that this identity implies that G is left cancellative. Running the fourth block of code in Section for every value of `$max_weight` from 20 to 150 in steps of 5 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that G is right cancellative. Running the fifth block of code in Section for every value of `$max_weight` from 18 to 150 in steps of 2 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that e is idempotent in G. Running the sixth block of code in Section for every value of `$max_weight` from 18 to 150 in steps of 2 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that G is associative. By Theorem 2.4, G must be a group of exponent 3 with neutral element e.

The above procedure was automated and carried out for each of the 119 remaining candidate identities and each one was shown to imply that a finite model of it must be a group of exponent 3 with neutral element e. Therefore, if any one of these 119 identities fails to be a single axiom with neutral element for groups of exponent 3 (the authors feel that it is likely that most if not all of them fail to be), then it can only be eliminated from contention through the construction of an *infinite* non-associative model.

6. Conclusion

In this section, we summarize our results.

Theorem 6.1. *The following* 148 *identities with neutral element* (*and their mirrors*) *are single axioms with neutral element for groups of exponent* 3.

$$ey \cdot ((yy \cdot xz)z \cdot z) = x$$
$$(yy \cdot ((ye \cdot x)z \cdot z))z = x$$
$$(ey \cdot ((yy \cdot x)z \cdot z))z = x$$
$$(yy \cdot (ey \cdot xz)z)z = x$$
$$yy \cdot ((ye \cdot xz)z \cdot z) = x$$
$$y(y \cdot (y \cdot (e \cdot xz)z)z) = x$$
$$(ey \cdot (yy \cdot x)z) \cdot zz = x$$
$$(ye \cdot ((yy \cdot x) \cdot zz))z = x$$
$$(y \cdot (y \cdot y(ex \cdot z))z)z = x$$
$$((yy \cdot e)(yx \cdot z) \cdot z)z = x$$
$$y(e(y \cdot (yx \cdot z)z) \cdot z) = x$$
$$(y \cdot y((e \cdot yx)z \cdot z))z = x$$
$$((e \cdot yy) \cdot (y \cdot xz)z)z = x$$
$$(y \cdot e(y(y \cdot xz) \cdot z))z = x$$
$$y(y \cdot (y(e \cdot xz) \cdot z)z) = x$$
$$(y \cdot ((y \cdot ye) \cdot xz)z)z = x$$
$$y(((e \cdot yy) \cdot xz)z \cdot z) = x$$
$$e((y \cdot y(yx \cdot z))z \cdot z) = x$$
$$(y \cdot ((e \cdot yy)x \cdot z)z)z = x$$
$$y((y \cdot y(ex \cdot z))z \cdot z) = x$$
$$(y \cdot ((ye \cdot y) \cdot xz)z)z = x$$
$$(yy \cdot e)((y \cdot xz)z \cdot z) = x$$
$$y((y \cdot y(e \cdot xz)) \cdot zz) = x$$
$$y(y \cdot e(yx \cdot z)) \cdot zz = x$$
$$y(ey \cdot (yx \cdot zz)) \cdot z = x$$
$$(y \cdot y((ye \cdot x) \cdot zz))z = x$$
$$y(e(yy \cdot xz) \cdot zz) = x$$
$$e(yy \cdot (yx \cdot z)) \cdot zz = x$$
$$y \cdot (ey \cdot (yx \cdot zz))z = x$$
$$(yy \cdot e)(y(x \cdot zz) \cdot z) = x$$
$$(ey \cdot y)(yx \cdot zz) \cdot z = x$$
$$y((ey \cdot y)x \cdot z) \cdot zz = x$$
$$(yy \cdot e)((y \cdot xz) \cdot zz) = x$$
$$y \cdot (yy \cdot e(x \cdot zz))z = x$$
$$yy \cdot (e \cdot y(x \cdot zz))z = x$$
$$y((ye \cdot y)(x \cdot zz) \cdot z) = x$$
$$(y \cdot y((ey \cdot x) \cdot zz))z = x$$
$$y(e \cdot (yy \cdot x)z) \cdot zz = x$$
$$y((y \cdot e(yx \cdot z)) \cdot zz) = x$$
$$e((y \cdot (y \cdot yx)z) \cdot zz) = x$$
$$ey \cdot (y \cdot y(x \cdot zz))z = x$$
$$(y \cdot ye)(yx \cdot z) \cdot zz = x$$
$$y(y((e \cdot yx)z \cdot z) \cdot z) = x$$
$$y(y \cdot (y \cdot (ex \cdot z)z)z) = x$$

$$y(e(y(yx \cdot z) \cdot z) \cdot z) = x$$
$$e(y(y \cdot (y \cdot xz)z) \cdot z) = x$$
$$y(e(y(y \cdot xz) \cdot z) \cdot z) = x$$
$$((yy \cdot e) \cdot (y \cdot xz)z)z = x$$
$$y(((yy \cdot e) \cdot xz)z \cdot z) = x$$
$$(e \cdot (y \cdot y(yx \cdot z))z)z = x$$
$$(ey \cdot ((yy \cdot x) \cdot zz))z = x$$
$$(yy \cdot (ye \cdot x)z) \cdot zz = x$$
$$(yy \cdot ((ye \cdot x) \cdot zz))z = x$$
$$e(y(y \cdot (yx \cdot z)z) \cdot z) = x$$
$$(y \cdot e((y \cdot yx)z \cdot z))z = x$$
$$e(y \cdot (y(y \cdot xz) \cdot z)z) = x$$
$$((ey \cdot y) \cdot (y \cdot xz)z)z = x$$
$$((y \cdot ey) \cdot (y \cdot xz)z)z = x$$
$$(y \cdot y(y(e \cdot xz) \cdot z))z = x$$
$$(y \cdot ye)((y \cdot xz)z \cdot z) = x$$
$$y(((ey \cdot y) \cdot xz)z \cdot z) = x$$
$$y((y \cdot e(yx \cdot z))z \cdot z) = x$$
$$(y \cdot ((ey \cdot y)x \cdot z)z)z = x$$
$$y(y \cdot (e(y \cdot xz) \cdot z)z) = x$$
$$(y \cdot ((yy \cdot e) \cdot xz)z)z = x$$
$$((ye \cdot y) \cdot (yx \cdot z)z)z = x$$
$$ey \cdot (y(yx \cdot z) \cdot zz) = x$$
$$y(y(ey \cdot (x \cdot zz)) \cdot z) = x$$
$$y((ye \cdot (yx \cdot z)) \cdot zz) = x$$
$$(ye \cdot (y \cdot yx)z) \cdot zz = x$$
$$(ey \cdot y)((y \cdot xz) \cdot zz) = x$$
$$yy \cdot (y(ex \cdot zz) \cdot z) = x$$
$$ye \cdot (y(yx \cdot z) \cdot zz) = x$$
$$y(y \cdot (ey \cdot x)z) \cdot zz = x$$
$$(yy \cdot e)(yx \cdot z) \cdot zz = x$$
$$y((yy \cdot e)x \cdot zz) \cdot z = x$$
$$(ye \cdot y)((y \cdot xz) \cdot zz) = x$$
$$y((y \cdot (ye \cdot x)z) \cdot zz) = x$$
$$(ey \cdot y)(y(x \cdot zz) \cdot z) = x$$
$$(ye \cdot y(yx \cdot zz))z = x$$
$$e((y \cdot (yy \cdot x)z) \cdot zz) = x$$
$$e((y \cdot y(yx \cdot z)) \cdot zz) = x$$
$$y \cdot e(y(yx \cdot z) \cdot zz) = x$$
$$y(y \cdot e(yx \cdot zz)) \cdot z = x$$
$$y \cdot (y \cdot e(yx \cdot zz))z = x$$
$$y((e \cdot y(y \cdot xz))z \cdot z) = x$$
$$(y \cdot e(y \cdot (y \cdot xz)z))z = x$$
$$y((y \cdot e(y \cdot xz))z \cdot z) = x$$

$$y(y(e(yx \cdot z) \cdot z) \cdot z) = x$$
$$e(y(y(y \cdot xz) \cdot z) \cdot z) = x$$
$$(ye \cdot (yy \cdot xz)z)z = x$$
$$y(y(y \cdot (e \cdot xz)z) \cdot z) = x$$
$$y(y \cdot (e \cdot (y \cdot xz)z)z) = x$$
$$(y \cdot y(e(yx \cdot z) \cdot z))z = x$$
$$(ye \cdot (yy \cdot x)z) \cdot zz = x$$
$$(y \cdot y(y(ex \cdot z) \cdot z))z = x$$
$$(y \cdot ((yy \cdot e)x \cdot z)z)z = x$$
$$(e \cdot y((y \cdot yx)z \cdot z))z = x$$
$$y(y(e \cdot (yx \cdot z)z) \cdot z) = x$$
$$(e \cdot y(y(y \cdot xz) \cdot z))z = x$$
$$y(e \cdot (y(y \cdot xz) \cdot z)z) = x$$
$$(y \cdot y(e(y \cdot xz) \cdot z))z = x$$
$$(y \cdot y((y \cdot ex)z \cdot z))z = x$$
$$y(y(y \cdot (ex \cdot z)z) \cdot z) = x$$
$$y(((y \cdot ey) \cdot xz)z \cdot z) = x$$
$$((y \cdot ye) \cdot (yx \cdot z)z)z = x$$
$$(y \cdot ((y \cdot ey)x \cdot z)z)z = x$$
$$y((e \cdot y(yx \cdot z))z \cdot z) = x$$
$$(ye \cdot y)((y \cdot xz)z \cdot z) = x$$
$$((yy \cdot e) \cdot (yx \cdot z)z)z = x$$
$$y \cdot e(y(yx \cdot zz) \cdot z) = x$$
$$e(y((y \cdot yx) \cdot zz) \cdot z) = x$$
$$ye \cdot (y \cdot y(x \cdot zz))z = x$$
$$ey \cdot (y(y \cdot xz) \cdot zz) = x$$
$$y(((yy \cdot e) \cdot xz) \cdot zz) = x$$
$$y(y(ye \cdot (x \cdot zz)) \cdot z) = x$$
$$y(ye \cdot (yx \cdot zz)) \cdot z = x$$
$$y((ey \cdot y)(x \cdot zz) \cdot z) = x$$
$$(e \cdot yy)(yx \cdot zz) \cdot z = x$$
$$y(((ey \cdot y) \cdot xz) \cdot zz) = x$$
$$y \cdot (ye \cdot y(x \cdot zz))z = x$$
$$y((yy \cdot (ex \cdot z)) \cdot zz) = x$$
$$y((yy \cdot e)(x \cdot zz) \cdot z) = x$$
$$(yy \cdot e(yx \cdot zz))z = x$$
$$e \cdot (yy \cdot (yx \cdot zz))z = x$$
$$e \cdot (y \cdot y(yx \cdot zz))z = x$$
$$y(y(y \cdot e(x \cdot zz)) \cdot z) = x$$
$$y((ey \cdot (yx \cdot z)) \cdot zz) = x$$
$$y((y \cdot ye)x \cdot zz) \cdot z = x$$
$$(e \cdot y(y \cdot (y \cdot xz)z))z = x$$
$$e(y \cdot (y \cdot (yx \cdot z)z)z) = x$$
$$(y \cdot y(e \cdot (y \cdot xz)z))z = x$$

$y(y((y \cdot ex)z \cdot z) \cdot z) = x$ $e(y \cdot (y(yx \cdot z) \cdot z)z) = x$ $y((y \cdot y(e \cdot xz))z \cdot z) = x$
$(y \cdot (e \cdot y(y \cdot xz))z)z = x$ $(e \cdot y(y \cdot (yx \cdot z)z))z = x$ $y(e \cdot (y \cdot (yx \cdot z)z)z) = x$
$y(y \cdot (y(ex \cdot z) \cdot z)z) = x$ $e(y((y \cdot yx)z \cdot z) \cdot z) = x$ $y((y \cdot (e \cdot yx)z)z \cdot z) = x$
$(y \cdot (y \cdot e(y \cdot xz))z)z = x$ $(y \cdot e(y \cdot (yx \cdot z)z))z = x$ $y((y \cdot (y \cdot ex)z)z \cdot z) = x$
$y(e \cdot (y(yx \cdot z) \cdot z)z) = x$ $(y \cdot y(e \cdot (yx \cdot z)z))z = x$ $e((y \cdot (y \cdot yx)z)z \cdot z) = x$
$(y \cdot (y \cdot y(e \cdot xz))z)z = x$

A finite model of any of the following 119 *identities with neutral element* (*or their mirrors*) *is a group of exponent* 3 *with neutral element* e.

$y \cdot y(y(x \cdot zz) \cdot ez) = x$ $yy \cdot (y(x \cdot ze) \cdot zz) = x$ $y((yy \cdot (x \cdot ze)) \cdot zz) = x$
$y \cdot y(y(x \cdot ze) \cdot zz) = x$ $yy \cdot (y \cdot (x \cdot ez)z)z = x$ $y \cdot (yy \cdot (xe \cdot z)z)z = x$
$y(y \cdot (y(xe \cdot z) \cdot z)z) = x$ $y(y \cdot (y \cdot (xe \cdot z)z)z) = x$ $y(y \cdot (y \cdot x(ez \cdot z))z) = x$
$y(y(y(xz \cdot z) \cdot z) \cdot e) = x$ $y(y \cdot (y(xz \cdot z) \cdot z)e) = x$ $y(y(y(xz \cdot z) \cdot e) \cdot z) = x$
$y(y \cdot (y(xz \cdot z) \cdot e)z) = x$ $y(y \cdot (y(xz \cdot e) \cdot z)z) = x$ $ey \cdot ((yy \cdot xz) \cdot zz) = x$
$y \cdot e(y(y \cdot xz) \cdot zz) = x$ $ye \cdot ((yy \cdot xz) \cdot zz) = x$ $ye \cdot (yy \cdot (xz \cdot z))z = x$
$y(ye \cdot ((y \cdot xz) \cdot zz)) = x$ $y((ye \cdot y)(xz \cdot z) \cdot z) = x$ $yy \cdot ((ey \cdot xz) \cdot zz) = x$
$(yy \cdot e)(y(xz \cdot z) \cdot z) = x$ $y \cdot y(y(e \cdot xz) \cdot zz) = x$ $y((y \cdot ye)(xz \cdot z) \cdot z) = x$
$yy \cdot ((ye \cdot xz) \cdot zz) = x$ $ey \cdot (yy \cdot (x \cdot zz))z = x$ $ye \cdot (yy \cdot (x \cdot zz))z = x$
$yy \cdot (ey \cdot (x \cdot zz))z = x$ $y(y \cdot (y \cdot e(x \cdot zz))z) = x$ $yy \cdot (ye \cdot (x \cdot zz))z = x$
$yy \cdot (ey \cdot (xz \cdot z))z = x$ $y \cdot (yy \cdot (x \cdot zz)e)z = x$ $e(yy \cdot (y(x \cdot zz) \cdot z)) = x$
$yy \cdot (y \cdot (x \cdot ze)z)z = x$ $y(y \cdot (y \cdot x(z \cdot ez))z) = x$ $y(y \cdot (y \cdot x(e \cdot zz))z) = x$
$y(y \cdot (y \cdot xz)(zz \cdot e)) = x$ $y(y \cdot (y \cdot xz)(z \cdot ze)) = x$ $y(y \cdot (y \cdot xz)(ze \cdot z)) = x$
$e(y \cdot y(y(xz \cdot z) \cdot z)) = x$ $(e \cdot yy)(y(xz \cdot z) \cdot z) = x$ $(ey \cdot y)(y(xz \cdot z) \cdot z) = x$
$y(e \cdot y(y(xz \cdot z) \cdot z)) = x$ $y(e \cdot (y \cdot y(xz \cdot z))z) = x$ $ye \cdot y((y \cdot xz) \cdot zz) = x$
$(y \cdot ey)(y(xz \cdot z) \cdot z) = x$ $y(y \cdot e((y \cdot xz) \cdot zz)) = x$ $y(y \cdot (e \cdot y(xz \cdot z))z) = x$
$y \cdot y((ye \cdot xz) \cdot zz) = x$ $y(y \cdot (y \cdot e(xz \cdot z))z) = x$ $y((yy \cdot e)(xz \cdot z) \cdot z) = x$
$e(y \cdot (y \cdot y(xz \cdot z))z) = x$ $y \cdot (yy \cdot x(zz \cdot e))z = x$ $y(ey \cdot ((y \cdot xz) \cdot zz)) = x$
$yy \cdot e((y \cdot xz) \cdot zz) = x$ $yy \cdot (y \cdot x(z \cdot ez))z = x$ $y \cdot (yy \cdot x(z \cdot ez))z = x$
$y((e \cdot yy)(x \cdot zz) \cdot z) = x$ $yy \cdot (y \cdot xz)(zz \cdot e) = x$ $yy \cdot (y \cdot xz)(z \cdot ze) = x$
$y \cdot (yy \cdot xz)(z \cdot ez) = x$ $y(((e \cdot yy) \cdot xz) \cdot zz) = x$ $y \cdot y((ey \cdot xz) \cdot zz) = x$
$yy \cdot (y(e \cdot xz) \cdot zz) = x$ $e \cdot y((yy \cdot xz) \cdot zz) = x$ $y(y \cdot e(y(x \cdot zz) \cdot z)) = x$
$y(y \cdot (ey \cdot (x \cdot zz))z) = x$ $(ye \cdot y)(y(x \cdot zz) \cdot z) = x$ $yy \cdot (y(xe \cdot zz) \cdot z) = x$
$y((yy \cdot (xz \cdot z)) \cdot ze) = x$ $yy \cdot (y(xz \cdot ez) \cdot z) = x$ $yy \cdot (y(xz \cdot e) \cdot zz) = x$
$(e \cdot yy)((y \cdot xz) \cdot zz) = x$ $yy \cdot (y \cdot x(zz \cdot e))z = x$ $y \cdot (yy \cdot x(z \cdot ze))z = x$
$yy \cdot (y \cdot x(e \cdot zz))z = x$ $(e \cdot yy)(y(x \cdot zz) \cdot z) = x$ $yy \cdot (y(xz \cdot z) \cdot ze) = x$
$yy \cdot (y \cdot xz)(z \cdot ez) = x$ $y \cdot (yy \cdot xz)(e \cdot zz) = x$ $y(((ye \cdot y) \cdot xz) \cdot zz) = x$
$y((yy \cdot (x \cdot zz))z \cdot e) = x$ $y(e \cdot (yy \cdot (x \cdot zz))z) = x$ $y((yy \cdot (xe \cdot z)) \cdot zz) = x$
$yy \cdot (y(xe \cdot z) \cdot z)z = x$ $y((yy \cdot (xe \cdot z))z \cdot z) = x$ $yy \cdot (y(xz \cdot z) \cdot e)z = x$
$y(e \cdot (y \cdot y(x \cdot zz))z) = x$ $yy \cdot (y \cdot (xe \cdot z)z)z = x$ $y(y \cdot ((y \cdot xz)z \cdot z)e) = x$
$y(y(y \cdot (xz \cdot z)e) \cdot z) = x$ $y(y \cdot ((y \cdot xz)z \cdot e)z) = x$ $y(y \cdot (y \cdot (xz \cdot z)e)z) = x$
$y(y(y \cdot (xz \cdot e)z) \cdot z) = x$ $y(y \cdot ((y \cdot xz)e \cdot z)z) = x$ $y(y \cdot (y \cdot (xz \cdot e)z)z) = x$
$e(y \cdot y((y \cdot xz)z \cdot z)) = x$ $y(e \cdot y((y \cdot xz)z \cdot z)) = x$ $y(e(y \cdot y(xz \cdot z)) \cdot z) = x$
$y(y \cdot e(y(xz \cdot z) \cdot z)) = x$ $y(y \cdot e((y \cdot xz)z \cdot z)) = x$ $y(y(e \cdot y(xz \cdot z)) \cdot z) = x$
$y(y(y \cdot e(xz \cdot z)) \cdot z) = x$ $y((yy \cdot (xz \cdot z))e \cdot z) = x$ $y(((y \cdot ye) \cdot xz) \cdot zz) = x$

$$
\begin{array}{rrr}
(y\cdot ye)(y(x\cdot zz)\cdot z)=x & yy\cdot(y\cdot x(ez\cdot z))z=x & y((y\cdot ye)(x\cdot zz)\cdot z)=x\\
(y\cdot ye)((y\cdot xz)\cdot zz)=x & y\cdot(yy\cdot xz)(ez\cdot z)=x & (y\cdot ey)((y\cdot xz)\cdot zz)=x\\
y\cdot(yy\cdot x(ze\cdot z))z=x & y((y\cdot ey)(x\cdot zz)\cdot z)=x & y(((y\cdot ey)\cdot xz)\cdot zz)=x\\
yy\cdot(y\cdot x(ze\cdot z))z=x & y\cdot(yy\cdot xz)(ze\cdot z)=x & (y\cdot ey)(y(x\cdot zz)\cdot z)=x\\
yy\cdot(y\cdot xz)(ze\cdot z)=x & y(y(y\cdot(xe\cdot z)z)\cdot z)=x &
\end{array}
$$

Any additional single axioms with neutral element for groups of exponent 3 *are among these* 119 *identities with neutral element* (*up to renaming, mirroring, and symmetry*).

A similar search for shortest single axioms with neutral element for Boolean groups was also carried out with the following results.

Theorem 6.2. *The following* 173 *identities with neutral element* (*and their mirrors*) *are single axioms with neutral element for Boolean groups.*

$$
\begin{array}{rrr}
e(x(xy\cdot z)\cdot y)=x & ex\cdot(xy\cdot zy)=x & e(x\cdot y(x\cdot zy))=x\\
e(xy\cdot(x\cdot zy))=x & ex\cdot y(x\cdot zy)=x & (e\cdot xy)(y\cdot zx)=x\\
e((x\cdot yz)\cdot xy)=x & ex\cdot y(z\cdot xy)=x & e(x\cdot y(z\cdot yx))=x\\
e((x\cdot yz)y\cdot x)=x & ex\cdot y(z\cdot yx)=x & e(x\cdot yz)\cdot yx=x\\
(e\cdot(x\cdot yz)z)x=x & x(ey\cdot(x\cdot zy))=x & xe\cdot y(x\cdot zy)=x\\
xe\cdot(yx\cdot z)y=x & (xe\cdot y)(x\cdot zy)=x & (x\cdot(ey\cdot x)z)y=x\\
(x\cdot(ey\cdot x)z)z=x & (xe\cdot(yx\cdot z))z=x & xe\cdot y(yz\cdot x)=x\\
x(ey\cdot(z\cdot xy))=x & x((e\cdot yz)\cdot xy)=x & x(ey\cdot z)\cdot xy=x\\
x((e\cdot yz)x\cdot z)=x & xe\cdot(y\cdot zx)z=x & xe\cdot(yz\cdot x)z=x\\
((xe\cdot y)\cdot zx)z=x & ((x\cdot ey)z\cdot x)z=x & ((xe\cdot y)z\cdot x)z=x\\
x(e\cdot y(zy\cdot x))=x & x((ey\cdot z)\cdot yx)=x & (xe\cdot y)(zy\cdot x)=x\\
x(ey\cdot z)\cdot yx=x & (x(e\cdot yz)\cdot y)x=x & ((xe\cdot y)z\cdot y)x=x\\
(x(e\cdot yz)\cdot z)x=x & (xe\cdot yz)z\cdot x=x & x(x(ey\cdot z)\cdot y)=x\\
x\cdot(xe\cdot yz)y=x & x(x(y\cdot ez)\cdot y)=x & x(y\cdot e(x\cdot zy))=x\\
x(ye\cdot(x\cdot zy))=x & xy\cdot(ex\cdot z)y=x & x(y\cdot(e\cdot yz)x)=x\\
x(ye\cdot(yz\cdot x))=x & xy\cdot(ey\cdot z)x=x & (x\cdot ye)(y\cdot zx)=x\\
(xy\cdot(e\cdot yz))x=x & ((x\cdot ye)\cdot yz)x=x & x(y\cdot e(z\cdot xy))=x\\
(x\cdot ye)(zx\cdot y)=x & x(y\cdot ez)\cdot xy=x & ((x\cdot ye)z\cdot x)y=x\\
(xy\cdot ez)\cdot xz=x & (x\cdot ye)z\cdot xz=x & (xy\cdot ez)x\cdot z=x\\
x(y\cdot ez)\cdot yx=x & ((x\cdot ye)\cdot zy)x=x & x((y\cdot ez)\cdot zx)=x\\
(xy\cdot ez)\cdot zx=x & (x(y\cdot ez)\cdot z)x=x & e(x\cdot(x\cdot yz)y)=x\\
(ex\cdot(xy\cdot z))y=x & e(x\cdot(yx\cdot z)y)=x & e(xy\cdot(xz\cdot y))=x\\
ex\cdot(yx\cdot zy)=x & (e\cdot xy)(x\cdot zy)=x & (e\cdot x(y\cdot xz))y=x\\
e(xy\cdot xz)\cdot y=x & (ex\cdot(yx\cdot z))y=x & ((e\cdot xy)\cdot xz)y=x\\
(ex\cdot(y\cdot xz))z=x & (ex\cdot(yx\cdot z))z=x & e(xy\cdot(yz\cdot x))=x\\
ex\cdot y(yz\cdot x)=x & (ex\cdot y)(yz\cdot x)=x & ((e\cdot xy)\cdot yz)z=x\\
e(x\cdot(yz\cdot x)y)=x & ex\cdot(yz\cdot xy)=x & (e\cdot x(y\cdot zx))y=x\\
e((xy\cdot z)x\cdot z)=x & ex\cdot(yz\cdot xz)=x & ex\cdot(yz\cdot x)z=x\\
(e(xy\cdot z)\cdot x)z=x & e(x\cdot y(zy\cdot x))=x & e((x\cdot yz)\cdot yx)=x
\end{array}
$$

$$
\begin{array}{rrr}
ex \cdot (yz \cdot yx) = x & (ex \cdot yz) \cdot yx = x & e((x \cdot yz)y \cdot z) = x \\
e(xy \cdot z) \cdot yz = x & e(xy \cdot zy) \cdot z = x & ((e \cdot xy) \cdot zy)z = x \\
(e(x \cdot yz) \cdot y)z = x & (ex \cdot yz)z \cdot x = x & e((x \cdot yz)z \cdot y) = x \\
e(xy \cdot z) \cdot zy = x & x \cdot e(yx \cdot zy) = x & x((ey \cdot x) \cdot zy) = x \\
x((e \cdot yx)z \cdot y) = x & (x \cdot ey)(xz \cdot y) = x & (x \cdot e(yx \cdot z))y = x \\
x(ey \cdot xz) \cdot y = x & (xe \cdot (y \cdot xz))y = x & ((xe \cdot y) \cdot xz)y = x \\
xe \cdot (y \cdot xz)z = x & (x \cdot e(yx \cdot z))z = x & ((xe \cdot y) \cdot xz)z = x \\
x(e \cdot y(yz \cdot x)) = x & x \cdot e(yz \cdot xy) = x & x(e \cdot (yz \cdot x)y) = x \\
x((ey \cdot z) \cdot xy) = x & x((e \cdot yz)x \cdot y) = x & (x \cdot e(y \cdot zx))y = x \\
((xe \cdot y) \cdot zx)y = x & x(e \cdot (yz \cdot x)z) = x & (xe \cdot yz) \cdot xz = x \\
x \cdot e(yz \cdot yx) = x & x(ey \cdot (z \cdot yx)) = x & (xe \cdot yz) \cdot zx = x \\
x((x \cdot ey) \cdot zy) = x & x((xe \cdot y)z \cdot y) = x & (x \cdot (x \cdot ey)z)y = x \\
x(xy \cdot (e \cdot zy)) = x & x(x(ye \cdot z) \cdot y) = x & x \cdot (xy \cdot ez)y = x \\
x((xy \cdot e)z \cdot y) = x & x(xy \cdot ez) \cdot y = x & xy \cdot e(x \cdot zy) = x \\
xy \cdot e(xz \cdot y) = x & (xy \cdot e)(x \cdot zy) = x & (x \cdot y(e \cdot xz))y = x \\
x(ye \cdot xz) \cdot y = x & (xy \cdot (e \cdot xz))y = x & ((xy \cdot e) \cdot xz)y = x \\
x(y \cdot (ey \cdot z)x) = x & xy \cdot e(y \cdot zx) = x & xy \cdot e(yz \cdot x) = x \\
(xy \cdot e)(y \cdot zx) = x & ((xy \cdot e) \cdot yz)x = x & x((y \cdot ez)x \cdot y) = x \\
(x \cdot y(e \cdot zx))y = x & x((y \cdot ez) \cdot xz) = x & x \cdot (ye \cdot zx)z = x \\
x((y \cdot ez)x \cdot z) = x & (x \cdot y(e \cdot zx))z = x & (x \cdot y(ez \cdot x))z = x \\
x(ye \cdot zx) \cdot z = x & ((xy \cdot e) \cdot zx)z = x & x(y \cdot e(zy \cdot x)) = x \\
x(y \cdot (ez \cdot y)x) = x & (x(y \cdot ez) \cdot y)x = x & x(ye \cdot z) \cdot zx = x \\
e(x \cdot (xy \cdot z)y) = x & e(x(yx \cdot z) \cdot y) = x & (ex \cdot (y \cdot xz))y = x \\
e(x(yx \cdot z) \cdot z) = x & ((e \cdot xy) \cdot yz)x = x & (ex \cdot yz) \cdot xy = x \\
(e(x \cdot yz) \cdot y)x = x & x(ye \cdot z) \cdot xz = x & e((xy \cdot z) \cdot yz) = x \\
x(e(x \cdot yz) \cdot y) = x & x(e(xy \cdot z) \cdot y) = x & xe \cdot (x \cdot yz)y = x \\
x \cdot e(xy \cdot zy) = x & x \cdot (ex \cdot yz)y = x & x((e \cdot xy)z \cdot y) = x \\
xe \cdot (xy \cdot z)y = x & (x \cdot e(xy \cdot z))y = x &
\end{array}
$$

A finite model of any of the following 5 *identities with neutral element (or their mirrors) is a Boolean group with neutral element* e.

$$
\begin{array}{rrr}
(ex \cdot y)z \cdot xz = x & ((ex \cdot y)z \cdot x)z = x & (ex \cdot y)z \cdot yz = x \\
(ex \cdot yz)y \cdot z = x & (ex \cdot yz)z \cdot y = x &
\end{array}
$$

Any additional single axioms with neutral element for Boolean groups are among these 5 *identities with neutral element (up to renaming, mirroring, and symmetry).*

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