On 0-minimal $(0,2)$-bi-ideals in ordered semigroups

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Abstract In this paper, we study $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in ordered semigroups. The notions of $(0,2)$-bi-ideals in ordered semigroups and 0-$(0,2)$-bisimple ordered semigroups are introduced and described. The results obtained extend the results on semigroups without order.

1. Introduction

In [5], the notion of (m,n)-ideals in semigroups was introduced by S. Lajos as a generalization of ideals in semigroups. In [4], D. N. Krgović described $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals. The author also introduced the notions of $(0,2)$-bi-ideals and 0-$(0,2)$-bisimple semigroups; and showed that a semigroup S with a zero element 0 is 0-$(0,2)$-bisimple if and only if S is left 0-simple.

In the present paper, using the concept of (m,n)-ideals in ordered semigroups defined by J. Sanborisoot and T. Changphas in [7], we extend the results in [4], mentioned above, to ordered semigroups. We begin with investigation $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in ordered semigroups. The notions of $(0,2)$-bi-ideals in ordered semigroups and 0-$(0,2)$-bisimple ordered semigroups will be introduced.

The rest of this section let us recall some definitions and results used throughout the paper.

Definition 1.1. [1] A semigroup $(S,\cdot)$ together with a partial order $\leqslant$ (on S) that is compatible with the semigroup operation, meaning that for $x,y,z \in S$,

$$x \leqslant y \Rightarrow zx \leqslant zy \ \ \& \ \ xz \leqslant yz,$$

is called an *ordered semigroup*.

Let $(S,\cdot,\leqslant)$ be an ordered semigroup. If A,B are nonempty subsets of S, we let

$$AB = \{xy \in S \mid x \in A, y \in B\},$$
$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

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Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let A, B be nonempty subsets of S. The following was proved in [2]:

(1) $(A](B] \subseteq (AB]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A]] = (A]$.

Definition 1.2. [2] Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* of S if

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) for $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *(two-sided) ideal* of S.

It is clear that every left, right and (two-sided) ideals of an ordered semigroup S is a subsemigroup of S.

Definition 1.3. [7] Let $(S, \cdot, \leqslant q)$ be an ordered semigroup and let m, n be non-negative integers. A subsemigroup A of S is called an (m, n)-*ideal* of S if the following hold:

(i) $A^m S A^n \subseteq A$;

(ii) for $x \in A$ and $y \in S$, $y \leqslant qx$ implies $y \in A$.

Here, let $A^0 S = S$ and $SA^0 = S$.

From Definition 1.3, if $m = 1, n = 1$ then A is called a *bi-ideal* of S.

Note that if A is a nonempty subset of an ordered semigroup S, then the set $(A^2 \cup ASA^2]$ is a bi-ideal of S. Indeed: we have $((A^2 \cup ASA^2]] = (A^2 \cup ASA^2]$ and

$$\begin{aligned}
&(A^2 \cup ASA^2]S(A^2 \cup ASA^2] \\
&= (A^2 \cup ASA^2](S](A^2 \cup ASA^2] \\
&\subseteq (A^2SA^2 \cup A^2SASA^2 \cup ASA^2SA^2 \cup ASA^2SASA^2] \\
&\subseteq (ASA^2] \\
&\subseteq (A^2 \cup ASA^2].
\end{aligned}$$

Therefore, $(A^2 \cup ASA^2]$ is a bi-ideal of S.

We define $(0, 2)$-bi-ideals in an ordered semigroup analogue to [4] as follows:

Definition 1.4. A subsemigroup A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a $(0, 2)$-*bi-ideal* of S if A is both a bi-ideal and a $(0, 2)$-ideal of S.

2. Main Results

We give a characterization of $(0, 2)$-ideals of an ordered semigroup in term of left ideals as follows:

Lemma 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let $A \subseteq S$. Then A is a $(0, 2)$-ideal of S if and only if A is a left ideal of some left ideal of S.*

Proof. If A is a $(0, 2)$-ideal of S, then

$$(A \cup SA]A \subseteq (A^2 \cup SA^2] \subseteq (A] = A$$

and $((A]] = (A]$. Hence A is a left ideal of the left ideal $(A \cup SA]$ of S.

Conversely, assume that A is a left ideal of a left ideal L of S. Then

$$SA^2 \subseteq SLA \subseteq LA \subseteq A.$$

Let $x \in A$ and $y \in S$ be such that $y \leqslant x$. Since $x \in L$, we have $y \in L$. The assumption applies $y \in A$. □

The following result give some characterizations of $(1, 2)$-ideals of an ordered semigroup.

Theorem 2.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let $A \subseteq S$. The following statements are equivalent:*

(i) *A is a $(1, 2)$-ideal of S;*

(ii) *A is a left ideal of some bi-ideal of S;*

(iii) *A is a bi-ideal of some left ideal of S;*

(iv) *A is a $(0, 2)$-ideal of some right ideal of S;*

(v) *A is a right ideal of some $(0, 2)$-ideal of S.*

Proof. (i) ⇒ (ii). If A is a $(1, 2)$-ideal of S, then

$$(A^2 \cup ASA^2]A = (A^2 \cup ASA^2](A] \subseteq (A^3 \cup ASA^3] \subseteq (A^2 \cup ASA^2] \subseteq (A] = A.$$

Clearly, if $x \in A, y \in (A^2 \cup ASA^2]$ such that $y \leqslant x$ then $y \in A$. Hence A is a left ideal of the bi-ideal $(A^2 \cup ASA^2]$ of S.

(ii) ⇒ (iii). Let A be a left ideal of a bi-ideal B of S. Note that $(A \cup SA]$ is a left ideal of S. By assumption, we have

$$A(A \cup SA]A \subseteq (A](A \cup SA](A] \subseteq (A^3 \cup ASA^2] \subseteq (A \cup BSBA] \subseteq (A \cup BA] \subseteq (A] = A.$$

Let $x \in A, y \in (A \cup SA]$ such that $y \leqslant qx$. Since $x \in A$, $x \in B$. Thus $y \in B$, so $y \in A$. Therefore, A is a bi-ideal of the left ideal $(A \cup SA]$ of S.

(iii) ⇒ (iv). Assume that A is a bi-ideal of a left ideal L of S. Note that $(A \cup AS]$ is a right ideal of S. We have

$$(A\cup AS]A^2 \subseteq (A\cup AS](A^2] \subseteq (A^3\cup ASA^2] \subseteq (A\cup ASLA] \subseteq (A\cup ALA] \subseteq (A] = A.$$

Let $x \in A, y \in (A \cup AS]$ such that $y \leqslant x$, then $x \in L$. Thus $y \in L$, so $y \in A$. Hence A is a $(0,2)$-ideal of the right ideal $(A \cup AS]$ of S.

(iv) $\Rightarrow$ (v). If A is a $(0,2)$-ideal of a right ideal R of S, then $(A \cup SA^2]$ is a $(0,2)$-ideal of S and

$$A(A\cup SA^2] \subseteq (A](A\cup SA^2] \subseteq (A^2\cup ASA^2] \subseteq (A\cup RSA^2] \subseteq (A\cup RA^2] \subseteq (A] = A.$$

Assume that $x \in A, y \in (A \cup SA^2]$ such that $y \leqslant x$. Then $x \in R$, so $y \in R$, thus $y \in A$. Hence (v) holds.

(v) $\Rightarrow$ (i). If A is a right ideal of a $(0,2)$-ideal R of S, then

$$ASA^2 \subseteq ASR^2 \subseteq AR \subseteq A.$$

Assume that $x \in A, y \in S$ such that $y \leqslant x$. Since $x \in R$, so $y \in R$, thus $y \in A$. Hence A is a $(1,2)$-ideal of S. □

The following characterize $(1,2)$-ideals in term of left ideals and right ideals of an ordered semigroup.

Lemma 2.3. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup and let A be a subsemigroup of S such that $A = (A]$. Then A is a $(1,2)$-ideal of S if and only if there exist a $(0,2)$-ideal L of S and a right ideal R of S such that $RL^2 \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-ideal of S. We have $(A \cup SA^2]$ and $(A \cup AS]$ are $(0,2)$-ideal and right ideal of S, respectively. Setting $L = (A \cup SA^2]$ and $R = (A \cup AS]$, we obtain

$$RL^2 \subseteq (A^3 \cup A^2SA^2 \cup ASA^2 \cup ASASA^2] \subseteq (A^3 \cup ASA^2] \subseteq (A] = A.$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right ideal of S and L be a $(0,2)$-ideal of S such that $RL^2 \subseteq A \subseteq R \cap L$. Then

$$ASA^2 \subseteq (R\cap L)S(R\cap L)(R\cap L) \subseteq RSL^2 \subseteq RL^2 \subseteq A.$$

Hence A is a $(1,2)$-ideal of S. □

Definition 2.4. A $(0,2)$-bi-ideal A of an ordered semigroup $(S,\cdot,\leqslant)$ with a zero element 0 will be said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only $(0,2)$-bi-ideal of S properly contained in A.

Assume that $(S,\cdot,\leqslant)$ is an ordered semigroup with a zero element 0. It is easy to see that every left ideal of S is a $(0,2)$-ideal of S. Hence if L is a 0-minimal $(0,2)$-ideal of S and A is a left ideal of S contained in L then $A = \{0\}$ or $A = L$. What can we say about $(0,2)$-ideals contained in some 0-minimal left ideal of S ? The answer to the same question for a semigroup without order was given in [4].

Lemma 2.5. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with a zero element* 0*. Suppose that* L *is a* 0*-minimal left ideal of* S *and* A *is a subsemigroup of* L *such that* $A = (A]$*. Then* A *is a* $(0,2)$*-ideal of* S *contained in* L *if and only if* $(A^2] = \{0\}$ *or* $A = L$*.*

Proof. Assume that A is a $(0,2)$-ideal of S contained in L. Then $(SA^2] \subseteq L$. Since $(SA^2]$ is a left ideal of S, we have $(SA^2] = \{0\}$ or $(SA^2] = L$. If $(SA^2] = L$, then $L = (SA^2] \subseteq (A]$. Hence $A = L$. Let $(SA^2] = \{0\}$. Since $S(A^2] \subseteq (SA^2] = \{0\} \subseteq (A^2]$, it follows that $(A^2]$ is a left ideal of S contained in L. By the minimality of L, $(A^2] = \{0\}$ or $(A^2] = L$. If $A^2 = L$, then $A = L$. The opposite direction is clear. □

Lemma 2.6. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with a zero element* 0 *and let* L *be a* 0*-minimal* $(0,2)$*-ideal of* S*. Then* $(L^2] = \{0\}$ *or* L *is a* 0*-minimal left ideal of* S*.*

Proof. We have

$$S(L^2]^2 = S(L^2](L^2] \subseteq (SL^2](L^2] \subseteq (L](L^2] \subseteq (L^2].$$

Then $(L^2]$ is a $(0,2)$-ideal of S contained in L, hence $(L^2] = \{0\}$ or $(L^2] = L$. Suppose that $(L^2] = L$. Since $SL = S(L^2] \subseteq (SL^2] \subseteq (L] = L$, we obtain L is a left ideal of S. Let B be a left ideal of S contained in L. It follows that $SB^2 \subseteq B^2 \subseteq B \subseteq L$. This shows that B is a $(0,2)$- ideal of S contained in L, so $B = \{0\}$ or $B = L$. □

The following corollary follows from Lemma 2.5 and Lemma 2.6:

Corollary 2.7. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup without zero. Then* L *is a minimal* $(0,2)$*-ideal of* S *if and only if* L *is a minimal left ideal of* S*.*

Lemma 2.8. *Let* $(S, \cdot, \leqslant q)$ *be an ordered semigroup without zero and let* A *be a nonempty subset of* S*. Then* A *is a minimal* $(2,1)$*-ideal of* S *if and only if* A *is a minimal bi-ideal of* S*.*

Proof. Assume that A is a minimal $(2,1)$-ideal of S. Then $(A^2SA]$ is a $(2,1)$-ideal of S contained in A, and hence $(A^2SA] = A$. Since

$$ASA = (A^2SA]SA \subseteq (A^2SASA] \subseteq (A^2SA] = A,$$

it follows that A is a bi-ideal of S. Suppose that there exits a bi-ideal B of S contained in A. Then $B^2SB \subseteq B^2 \subseteq B \subseteq A$, so B is a $(2,1)$-ideal of S contained in A. Using the minimality of A we get $B = A$.

Conversely, assume that A is a minimal bi-ideal of S. Then A is a $(2,1)$-ideal of S. Let D be a $(2,1)$-ideal of S contained in A. Since $(D^2SD]S(D^2SD] \subseteq (D^2(SDSD^2S)D] \subseteq (D^2SD]$, we have $(D^2SD]$ is a bi-ideal of S. This implies that $(D^2SD] = A$. Since $A = (D^2SD] \subseteq (D] = D$, $A = D$. Therefore A is a minimal $(2,1)$-ideal of S. □

Lemma 2.9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let $A \subseteq S$. Then A is a $(0,2)$-bi-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Assume that A is a $(0,2)$-bi-ideal of S. Then

$$S(A^2 \cup SA^2] \subseteq (SA^2 \cup S^2A^2] \subseteq (SA^2] \subseteq (A^2 \cup SA^2],$$

hence $(A^2 \cup SA^2]$ is a left ideal of S. Since

$$A(A^2 \cup SA^2] \subseteq (A^3 \cup ASA^2] \subseteq (A] = A, \ (A^2 \cup SA^2]A \subseteq (A^3 \cup SA^3] \subseteq (A] = A$$

we obtain A is an ideal of $(A^2 \cup SA^2]$.

Conversely, if A is an ideal of a left ideal L of S then $ASA \subseteq ASL \subseteq AL \subseteq A$. Hence, by Lemma 2.1, A is a $(0,2)$-bi-ideal of S. □

Theorem 2.10. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with a zero element 0. If A is a 0-minimal $(0,2)$-bi-ideal of S, then exactly one of the following cases occurs:*

(i) $A = \{0\}, \ \big(aS^1a\big] = \{0\}$;

(ii) $A = (\{0,a\}], \ a^2 = 0, \ (aSa] = A$;

(iii) $\forall a \in A \backslash \{0\}, \ \big(Sa^2\big] = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a \in A\backslash\{0\}$. Then $(Sa^2] \subseteq A$. Moreover, $(Sa^2]$ is a $(0,2)$-bi-ideal of S. Hence $(Sa^2] = \{0\}$ or $(Sa^2] = A$.

Suppose that $(Sa^2] = \{0\}$. Since $a^2 \in A$, we have either

$$a^2 = a \text{ or } a^2 = 0 \text{ or } a^2 \in A\backslash\{0,a\}.$$

If $a^2 = a$, then $a = 0$. This is a contradiction. Suppose that $a^2 \in A\backslash\{0,a\}$. We have

$$S^1(\{0\} \cup a^2]^2 \subseteq (\{0\} \cup Sa^2] = (\{0\}] \cup (Sa^2] = \{0\} \subseteq (\{0\} \cup a^2],$$

$$(\{0\} \cup a^2]S(\{0\} \cup a^2] \subseteq (a^2Sa^2] \subseteq (Sa^2] = \{0\} \subseteq \{0, a^2\}.$$

Then $(\{0\} \cup a^2]$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $(\{0\} \cup a^2] \neq \{0\}$ and $(\{0\} \cup a^2] \neq A$. This is a contradiction because A is 0-minimal $(0,2)$-bi-ideal of S. Therefore, $a^2 = 0$, hence, by Lemma 2.9, $A = (\{0,a\}]$. Now, using $(aSa]$ is a $(0,2)$-bi-ideal of S contained in A we obtain $(aSa] = \{0\}$ or $(aSa] = A$. Therefore, $(Sa^2] = \{0\}$ implies either $A = \{0,a\}$ and $(aS^1a] = \{0\}$ or $A = \{0,a\}$, $a^2 = \{0\}$ and $(aSa] = A$. If $(Sa^2] \neq \{0\}$, then $(Sa^2] = A$. □

Corollary 2.11. *Let A be a 0-minimal $(0,2)$-bi-ideal of an ordered semigroup $(S, \cdot, \leqslant)$ with a zero element 0. If $(A^2] \neq \{0\}$, then $A = (Sa^2]$ for every $a \in A\backslash\{0\}$.*

Definition 2.12. An ordered semigroup $(S, \cdot, \leqslant)$ with a zero element 0 is said to be 0-$(0,2)$-*bisimple* if $(S^2] \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S.

Corollary 2.13. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with zero* 0. *Then* S *is* 0-$(0, 2)$*-bisimple if and only if* $(Sa^2] = S$ *for every* $a \in S\backslash\{0\}$.

Proof. Assume that $(Sa^2] = S$ for all $a \in S\backslash\{0\}$. Let A be a $(0, 2)$-bi-ideal of S such that $A \neq \{0\}$. Let $a \in A\backslash\{0\}$. Since $S = (Sa^2] \subseteq (SA^2] \subseteq (A] = A$, so $S = A$. Since $S = (Sa^2] \subseteq (SS] = (S^2]$ we have $(S^2] = S \neq \{0\}$. Therefore S is 0-$(0, 2)$-bi-simple.

The converse statement follows from Corollary 2.11. □

Theorem 2.14. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with zero* 0. *Then* S *is* 0-$(0, 2)$*-bisimple if and only if* S *is left* 0*-simple.*

Proof. Assume that S is 0-$(0, 2)$-bisimple. If A is a left ideal of S, then A is a $(0, 2)$-bi-ideal of S, and so $A = \{0\}$ or $A = S$.

Conversely, assume that S is left 0-simple. Let $a \in S\backslash\{0\}$. Then $(Sa] = S$, hence

$$S = (Sa] = ((Sa]a] \subseteq ((Sa^2]] = (Sa^2].$$

By Corollary 2.13, S is 0-$(0, 2)$-bisimple. □

Theorem 2.15. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with a zero element* 0. *If* A *is a* 0*-minimal* $(0, 2)$*-bi-ideal of* S*, then either* $(A^2] = \{0\}$ *or* A *is left* 0*-simple.*

Proof. Assume that $(A^2] \neq \{0\}$. Using Corollary 2.11, $(Sa^2] = A$ for every $a \in A\backslash\{0\}$. Since $a^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$, we have $a^4 = (a^2)^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$. Let $a \in A\backslash\{0\}$. Since

$$(Aa^2]S^1(Aa^2] \subseteq (AAa^2] \subseteq (Aa^2],$$

$$S(Aa^2]^2 \subseteq (SAa^2Aa^2] \subseteq (SA^2a^2] \subseteq (Aa^2],$$

we obtain $(Aa^2]$ is a $(0, 2)$-bi-ideal of S contained in A. Hence $(Aa^2] = \{0\}$ or $(Aa^2] = A$. Since $a^4 \in Aa^2 \subseteq (Aa^2]$ and $a^4 \in A\backslash\{0\}$, we get $(Aa^2] = A$. We conclude by Corollary 2.13 that A is 0-$(0, 2)$-bisimple. Theorem 2.14 applies A is left 0-simple. □

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