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Simple ternary semigroups

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Abstract. Simple ternary semigroups are studied using idempotent pairs. The concept of primitive idempotent pairs is introduced and the connection between them and minimal left (right) ideals are studied. An example of ternary semigroups containing primitive idempotent pairs is given. Some simple ternary semigroups containing a primitive idempotent pair are characterized.

1. Introduction

Investigation of ideals is an essential part of the study of any algebraic system. Investigation of ideals and radicals in ternary semigroups was initiated by Sioson [16]. The study has been continued by many authors for ternary semigroups and more generally for n-ary semigroups [8, 9, 10]. Cyclic ternary groups are described by Dörnte [3]. The n-ary power was introduced by Post [12]. The notion of minimal (maximal) left and right ideals in a ternary semigroups has been studied in [9] and a characterization has been obtained. In this paper we study some aspects of ternary semigroups such as Green's relations and simplicity. The definition of $\mathcal{D}$- and $\mathcal{H}$-equivalences given here are more general than those defined in [2]. In this paper a 0-t-simple ternary semigroup is defined and a characterization is obtained. Primitive idempotent pairs in a ternary semigroup are defined. Some results for 0-t-simple ternary semigroup which contains primitive idempotent pairs are proved. A connection between primitive idempotents and minimal (left and right) ideals are established. Completely 0-t-simple ternary semigroups are introduced and characterized.

2. (0)-simple ternary semigroups

A ternary semigroup is called (*right, left*) *simple* if it does not contains any proper (right, left) ideals. A ternary semigroup T is called t-*simple* if it does not contain any proper two-sided ideal. A t-simple ternary semigroup is simple. A simple ternary semigroup is *surjective*, i.e., $T = T^{<1>} = [TTT]$.

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For other definitions we refer [8, 9, 16].

We start with the following simple lemma proved in [8].

Lemma 2.1. *A ternary semigroup T is a right (left) simple if and only if $[aTT] = T$ (respectively, $[TTa] = T$) for all $a \in T$.* □

From this lemma we deduce

Corollary 2.2. *A ternary semigroup T is a right (left) simple if and only if for given $a, b \in T$ there exist $u, v \in T$ such that $[auv] = b$.* □

The following facts are almost obvious

Lemma 2.3. *A ternary semigroup T is a right (left) simple if and only if $[abT] = T$ (resp. $[Tab] = T$) for all $a, b \in T$.* □

Corollary 2.4. *A ternary semigroup T is a right (left) simple if and only if for given $a, b, c \in T$ there exist $x \in T$ such that $[abx] = c$ (resp. $[xab] = c$).* □

Lemma 2.5. *A ternary semigroup T is simple if and only if $T = [TaT] \cup [TTaTT]$ for any $a \in T$.* □

Lemma 2.6. *A ternary semigroup T is t-simple if and only if $[TTaTT] = T$ for any $a \in T$.* □

An element $z \in T$ is called a *zero element* if $[abz] = [zab] = [azb] = z$ for all $a, b \in T$. A zero element is uniquely determined and is denoted by 0. If T has no zero element, then a zero element can be adjoined by putting $[abc] = 0$ if any of a, b, c is a zero. We denote this fact by $T^0 = T \cup \{0\}$. If a ternary semigroup has a zero, then clearly $\{0\}$ is an ideal of T. It is denoted by (0). A ternary semigroup T with 0 is called a *null ternary semigroup* if $[abc] = 0$ for all $a, b, c \in T$. It is clear that a ternary semigroup with 0 has at least two ideals: 0 and T. If it has no other ideals (two-sided ideals) and $T^{<1>} \neq (0)$, then it is called 0-*simple* (resp. 0-t-*simple*.

Lemma 2.7. *If a ternary semigroup T with 0 has only one two-sided ideal $A \neq T$, then either T is 0-t-simple or T is the null ternary semigroup of order 2.*

Proof. Clearly $A = (0)$. Since $T^{<1>}$ is an ideal of T, we have $T^{<1>} = T$ or $T^{<1>} = (0)$. In the first case $T = (0)$, which means that T is 0-t-simple. In the second case for any non-zero element $t \in T$ the set $\{0, t\}$ is a non-zero two-sided ideal of T and so $\{0, t\} = T$. Thus T is a null ternary semigroup of order 2. □

Lemma 2.8. *A ternary semigroup T is 0-simple if and only if for every non-zero $a \in T$ we have $T = [TaT] \cup [TTaTT]$.*

Proof. Suppose that T is 0-simple. Then $T^{<1>}$ is a non-zero ideal of T and so $T^{<1>} = T$. Hence $T = T^{<1>} = T^{<2>}$. For any non-zero element $a \in T$ the subset $[TaT] \cup [TTaTT]$ is an ideal of T. Hence we have either $[TaT] \cup [TTaTT] = (0)$ or $[TaT] \cup [TTaTT] = T$. Suppose $[TaT] = (0)$. Then the set $M = \{m \in T : [TmT] = (0)\}$ contains the nonzero element a. M is a non-zero ideal and so $M = T$. This means that $T^{<1>} = (0)$, a contradiction. Therefore $[TaT] \cup [TTaTT] = T$ for every non-zero element $a \in T$. The converse is obvious. □

Lemma 2.9. *A ternary semigroup T is 0-t-simple if and only if $T = [TTaTT]$ for all $a \neq 0 \in T$.*

Proof. Suppose that T is 0-t-simple. Then $T^{<1>} \neq 0$ and $T^{<1>}$ is an ideal of T. Hence $T = T^{<1>} = T^{<2>}$. For any non-zero element $a \in T$, the subset $[TTaTT]$ of T is a two-sided ideal. Thus we have either $[TTaTT] = T$ or $[TTaTT] = (0)$. If $[TTaTT] = (0)$, then as in Lemma 2.8 we obtain a contradiction. Thus $T = [TTaTT]$. □

In a similar way we can prove

Lemma 2.10. *If a ternary semigroup T is 0-t-simple, then $T = [TaT]$ for all $a \neq 0 \in T$.* □

3. Green's equivalence on ternary semigroups

The Green's equivalence relation $\mathcal{L}$ and $\mathcal{R}$ on a ternary semigroup T are defined as follows (see [2]):

$$a\mathcal{L}b \Longleftrightarrow a \cup [TTa] = b \cup [TTb],$$
$$a\mathcal{R}b \Longleftrightarrow a \cup [aTT] = b \cup [bTT],$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

In other words $a\mathcal{L}b$ if and only if a and b generate the same left ideal, i.e., $a = b$ or $a = [xyb]$, $b = [uva]$ for some $x, y, u, v \in T$. Similarly, $a\mathcal{R}b$ if and only if a and b generate the same right ideal, i.e., $a = b$ or $a = [bpq]$, $b = [ars]$ for some $p, q, r, s \in T$.

Note that our definition of $\mathcal{H}$ is different from that found in [2].

Lemma 3.1. *$\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.* □

Proposition 3.2. *In ternary semigroups $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.*

Proof. Let $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then there exists $c \in T$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$ so, there exist $x, y, u, v, p, q, r, s \in T$ such that $a = [xyc]$, $c = [uva]$ and $c = [bpq], b = [crs]$. Put $d = [[xyc]rs]$. Then $[ars] = [[xyc]rs] = d$, and, $[dpq] = [[xyc]rs]pq] = [[xy[crs]]pq] = [[xyb]pq] = [xy[bpq]] = [xyc] = a$. Therefore $a\mathcal{R}d$. Also $[xyb] = [xy[crs]] = d$ and $[uvd] = [[uva]rs] = [crs] = b$, and so $d\mathcal{L}b$. Hence $(a, b) \in \mathcal{R} \circ \mathcal{L}$. Thus $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. Similarly we can prove $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. Therefore $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is an equivalence relation on T. □

Proposition 3.3. *$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is the smallest equivalence on T containing $\mathcal{R}$ and $\mathcal{L}$.* □

The equivalence $\mathcal{D}$ defined by us is contained in the equivalence $\mathcal{D}$ defined by Dixit and Dewan [2].

Recall [13, 14] that an element $t \in T$ is called *regular* if $[tut] = t$ for some $t \in T$. If $[tut] = t$ and $[utu] = u$, then u and t are *inverses* of one another.

Proposition 3.4. *If a $\mathcal{D}$-class D of T contains a regular element, then every element of D is regular.*

Proof. Let D be a $\mathcal{D}$-class in T and $a \in T$ be a regular element in D. Let b be an arbitrary element of D. Since $b \in D$, for some $c \in T$ we have $a\mathcal{L}c\mathcal{R}b$. From $a\mathcal{L}c$ we obtain either $a = c$ or $a = [efc]$, $c = [uva]$ for some $e, f, u, v \in T$. Similarly, $c\mathcal{R}b$ gives either $c = b$ or $c = [bpq]$, $b = [crs]$ for some $p, q, r, s \in T$. Let x be an inverse of a. Then $[axa] = a$, $[xax] = x$. Take $y = [[pqx]ef]$ we get $[byb] = [b[[pqx]ef]b] = [[bpq]x[efb]] = [cx[efb]] = [cx[ef[crs]]] = [cx[efc]rs] = [[uva]xa]rs] = [uv[axa]rs] = [uv[ars]] = [[uva]rs] = [crs] = b$. Similarly if $a = c$, or $c = b$, then by taking $y = [pqx]$ $(y = [xef])$ we can show that b is regular element. □

Let $\mathrm{L_a}$ $(\mathrm{R_a},\ \mathrm{D_a},\ \mathrm{H_a})$ be the $\mathcal{L}$ $(\mathcal{R},\ \mathcal{D},\ \mathcal{H})$-class containing $a \in T$.

The following Lemmas are found in [2].

Lemma 3.5. *Let a, b be $\mathcal{R}$-equivalent elements in a ternary semigroup T and let $p, q, r, s \in T$ be such that $a = [brs]$, $b = [apq]$. Then the right translations $\rho_{pq}|L_a$, $\rho_{rs}|L_b$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and from L_b onto L_a respectively.* □

Lemma 3.6. *Let a, b be $\mathcal{L}-$equivalent elements in a ternary semigroup T and let $x, y, u, v \in T$ be such that $a = [xyb]$, $b = [uva]$. Then the left translations $\lambda_{xy}|R_a$, $\lambda_{rs}|R_b$ are mutually inverse $\mathcal{L}$- class preserving bijections from R_a onto R_b and from R_b onto R_a respectively.* □

Using the above maps the following lemma can be proved.

Lemma 3.7. *Let a, b be $\mathcal{D}$-equivalent elements in a ternary semigroup T. Then $|H_a| = |H_b|$.*

Proof. If c is such that $a\mathcal{R}c$, $c\mathcal{L}b$, then there exists $p, q, r, s, x, y, u, v \in T$ such that $a = [crs]$, $c = [apq]$ and $b = [xyc]$, $c = [uvb]$. Then by Lemmas 3.5 and 3.6 we see that $\rho_{pq}|H_a$ is a bijection onto H_c and $\lambda_{xy}|H_c$ is a bijection onto H_b. Thus $\rho_{pq}\lambda_{xy}$ is a bijection from H_a onto H_b. Therefore, $|H_a| = |H_b|$. □

Corollary 3.8. *If $x, y, z \in T$ are such that $[xyz] \in H_x$, then ρ_{yz} is a bijection of H_x onto itself. If $[xyz] \in H_z$, then λ_{xy} is a bijection of H_z onto itself.* □

Theorem 3.9. *If H is an $\mathcal{H}$-class of a ternary semigroup T, then we have either $H^{<1>} \cap H = \emptyset$ or $H^{<1>} = H$ and H is a ternary subgroup of T.*

Proof. Suppose that $H^{<1>} \cap H \neq \emptyset$. Then there exists $a, b, c \in H$ such that $[abc] \in H$. By Corollary 3.8, the right translation ρ_{bc} and the left translation λ_{ab} are bijections of H onto itself. Hence $[hbc] \in H$ and $[abh] \in H$ for every $h \in H$. Also ρ_{bh} and λ_{hb} are bijections of H onto itself. Therefore $[hbH] = H$ and $[Hbh] = H$. By Lemma 2.1, H is a right and left simple ternary semigroup. Therefore by Theorem 1.1 in [13], H is a ternary group. $\square$

4. Minimal ideals

If A, B are two-sided ideals of a ternary semigroup T, then A and B both contain the product $[ATB]$. Therefore there can be almost one minimal two-sided ideal of T. Similarly we see that if A and B are ideals of a ternary semigroup, then $[ATB] \cup [TATBT]$ is an ideal of T contained in A and B. Therefore a minimal ideal (if it exists) is unique. If $T = [a] = \{a^{<n>}, n \geqslant 0\}$ is a cyclic ternary semigroup, then $[a] = (a) \supset (a)^3 \supset \dots$ is an infinite descending chain of ideals of T and T does not have a minimal two-sided ideal. If T is a finite cyclic ternary semigroup, then $T = \{a^{<n>} : a^{<m>} = a^{<m+r>} : m = \text{index},\ r = \text{period}\}$ and $T = (a) \supset (a)^3 \supset \cdots \supset K_a$, where $K_a = \{a^{<m>}, \dots, a^{<m+r-1>}\}$ is the unique minimal ideal of T.

If a non-zero ideal M of a ternary semigroup T with 0 ,is said to be 0-*minimal* if $M \neq (0)$ and (0) is the only ideal of T contained in M. Similarly 0-minimal left (right, two-sided) ideals are defined.

Lemma 4.1. *Let L be a minimal left ideal of a ternary semigroup T and let $x, y \in T$. Then $[Lxy]$ is a minimal left ideal of T.*

Proof. $[Lxy]$ is a left ideal of T. Let M be a left ideal of T contained in $[Lxy]$. Consider the set $N = \{n \in L : [nxy] \in M\}$. Then $[Nxy] = M$. For $t_1, t_2 \in T$, and $n \in N$ $[[t_1t_2n]xy] = [t_1t_2[nxy]] \in [TTM] \subseteq M$. Therefore $[t_1t_2n] \in N$ and so N is a left ideal of T contained in L. From the minimality of L we obtain $N = L$. Therefore $M = [Lxy]$ and so $[Lxy]$ is minimal. $\square$

Theorem 4.2. *Let M be a minimal two-sided ideal of a ternary semigroup T. Then M is a t-simple ternary subsemigroup of T.*

Proof. $M^{<1>}$ is a two-sided ideal of T contained in M. Therefore $M^{<1>} = M$. For any $a \in M$, $(a)_t = a \cup [TTa] \cup [aTT] \cup [TTaTT]$ is a two-sided ideal of T contained in M. Therefore $(a)_t = M$. Consequently, $M = M^{<1>} = M^{<2>} = [MM(a)_tMM] = [MM(a \cup [TTa] \cup [aTT] \cup [TTaTT])MM] = [MMaMM] \subseteq M^{<2>} = M$. Thus, $M = [MMaMM]$ for all $a \in M$ and so M is a t-simple ternary semigroup by Lemma 2.9. $\square$

Let K denote the intersection of all two-sided ideals and K^* the intersection of all ideals of a ternary semigroup T. Clearly $K \subset K^*$. Suppose $K \neq \emptyset$.

Lemma 4.3. *K is a t-simple ternary semigroup.*

Proof. For $a \in K$, $(a)_t = a \cup [TTa] \cup [aTT] \cup [TTaTT]$ is a two-sided ideal of T contained in K and so $K = (a)_t$ for all $a \in K$. Thus K is the unique minimal two-sided ideal of T and so K is t-simple by Theorem 4.2. □

Lemma 4.4. *K^* is a simple ternary semigroup.*

Proof. For $a \in K^*$, $(a) = a \cup [TTa] \cup [aTT] \cup [TaT] \cup [TTaTT] \subset K$. Therefore $K^* = (a)$ for all $a \in K^*$. Hence K^* is a simple ternary semigroup. □

Lemma 4.5. $K^* = [TKT]$.

Proof. Since $K \subseteq K^*$, we have $[TKT] \subseteq [TK^*T] \subseteq K^*$. Thus $[TT(K \cup [TKT])] = [TTK] \cup [TTTKT] = [TTK] \cup [TKT] \subseteq K \cup [TKT]$. Therefore $K \cup [TKT]$ is a left ideal of T. Similarly, $K \cup [TKT]$ is an ideal and so $K^* \subseteq K \cup [TKT]$. Since $K \subseteq K^*$ we have $K^* \subseteq [TKT]$. Therefore $K^* = [TKT]$. □

Theorem 4.6. $K = K^*$.

Proof. Put $M = [KK^*K]$. Then $M \subset K$; $M \subseteq K^*$. M is an ideal and so $K^* \subset M$. Therefore $K^* = M$. Similarly $K = M$. Hence $K = K^*(= M)$. □

Definition 4.7. If $K = K^*$ is nonempty, then it is called the *kernel* of T.

Lemma 4.8. *If L is a 0-minimal left ideal of a ternary semigroup T with 0 such that $L^{<1>} \neq (0)$, then $L = [TTa]$ for every element $a \neq 0$ of L.*

Proof. For any $a \neq (0)$ in L, $[TTa]$ is clearly a left ideal of T contained in L. If $[TTa] = (0)$ then $a^{<1>} = (0)$ and $\{0, a\}$ is a non-zero left ideal of T contained in L and so $\{0, a\} = L$ and $L^{<1>} = (0)$, a contradiction. Hence $[TTa] \neq (0)$ and so $[TTa] = L$. □

Lemma 4.9. *Let L be a 0-minimal left ideal of a ternary semigroup T with 0 and let $x, y \in T$. Then $[Lxy]$ is either (0) or a 0-minimal left ideal of T.*

Proof. Assume that $[Lxy] \neq (0)$. Then $[Lxy]$ is a left ideal of T. Let M be a left ideal of T contained in $[Lxy]$. Let $N = \{n \in L : [nxy] \in M\}$. Then $[Nxy] = M$. Recalling the proof of Lemma 4.1, it can be shown that N is a left ideal of T so that $N = (0)$ or $N = L$. Therefore either $M = (0)$ or $M = [Lxy]$ proving that $[Lxy]$ is a 0-minimal left ideal. □

Theorem 4.10. *Let M be a 0-minimal two-sided ideal of a ternary semigroup with zero 0. Then either $M^{<1>} = (0)$ or M is a 0-t-simple ternary subsemigroup of T.*

Proof. $M^{<1>}$ is an two-sided ideal of T contained in M. Therefore $M^{<1>} = (0)$ or $M^{<1>} = M$. Suppose $M^{<1>} \neq (0)$. Then $M = M^{<1>} = M^{<2>}$. As in the proof of Theorem 4.2, we can show that for every $a \in M$, $a \neq 0$ $M = [MMaMM]$. Thus M is a 0-t-simple ternary semigroup. □

Theorem 4.11. *Let T be a ternary semigroup with 0. If a 0-minimal two-sided ideal M of T contains at least one 0-minimal left ideal of T, then M is the union of all the 0-minimal left ideals of T contained in M.*

Proof. Let N be the union of all the 0-minimal left ideal of T contained in M. Clearly N is a left ideal of T. We prove that N is a right ideal. Let $n \in N$ and $x, y \in T$. By the definition, $n \in L$ for some 0- minimal left ideal L of T contained in M. By Lemma 4.9, $[Lxy] = (0)$ or $[Lxy]$ is a 0-minimal left ideal of T. Moreover, $[Lxy] \subseteq [Mxy] \subseteq M$ and hence $[Lxy] \subseteq N$. Therefore $[nxy] \in N$, for all $n \in N$. Hence, $N \neq (0)$ since it contains at least one 0-minimal left ideal of T. Thus N is a non-zero two-sided ideal of T contained in M. Therefore $N = M$, by the 0-minimality of M. □

Lemma 4.12. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T with 0 such that $M^{<1>} \neq (0)$. Then also $L^{<1>} \neq (0)$ L for any non-zero left ideal of T contained in M.*

Proof. Since $[LTT]$ is two-sided ideal of T contained in M we have either $[LTT] = M$ or $[LTT] = (0)$. If $[LTT] = (0)$, then L is an ideal of T whence $L = M$, and so $M^{<1>} = [LMM] \subset [LTT] = (0)$, contrary to our hypothesis on M. Hence $[LTT] = M$ and so $M = M^{<1>} = [[LTT][LTT][LTT]] = [L[TTL][TTL]TT] \subseteq [[LLL]TT]$. Therefore $L^{<1>} \neq (0)$. □

Theorem 4.13. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T with 0 such that $M^{<1>} \neq (0)$, and assume that M contains at least one 0-minimal left ideal of T. Then every left ideal of M is also a left ideal of T.*

Proof. Let L be a non-zero left ideal of M and $0 \neq a \in L$. By Theorem 4.10, M is 0-t-simple and so $M = [MMaMM]$. Hence $[MMa] \neq (0)$. By Theorem 4.10, there is 0-t-minimal left ideal L_1 of T such that $a \in L_1 \subseteq M$. Since $[MMa]$ is a non-zero left ideal of T contained in L_1, $[MMa] = L_1$. Therefore $a \in [MMa]$. Hence $L = \bigcup\{[MMa] : a \in L\}$ is a left ideal of T. □

Similar results can be proved for right ideals and also for 0-minimal ideals.

5. Completely 0-t-simple ternary semigroups

We recall [13, 14] that a pair of elements (a, b) of a ternary semigroup T is said to be an *idempotent pair* if $[ababt] = [abt]$ and $[tabab] = [tab]$. Two idempotent pairs (a, b) and (c, d) are said to be *equivalent* if $[abt] = [cdt]$ and $[tab] = [tcd]$. $\langle a, b\rangle$ denotes the equivalence class containing the idempotent pair (a, b). If (a, b),

(c,d) are idempotent pairs of T, then $(a,b) \leqslant (c,d)$ if $[abcdt] = [cdabt] = [abt]$ and $[tabcd] = [tcdab] = [tab]$. Then $\leqslant$ is a partial order on the set E of equivalence classes of idempotent pair of T. If S contains 0, then the class $\langle 0,0\rangle$ is the least element of E. An idempotent pair (a,b) is said to be non-zero if (a,b) does not belong to $\langle 0,0\rangle$. If T contains zero, a non-zero idempotent pair (u,v) is called *primitive* if $(a,b) \leqslant (u,v)$ for any idempotent pair (a,b) implies either $(a,b) = \langle 0,0\rangle$ or $\langle a,b\rangle = \langle u,v\rangle$. If T does not contain zero, a primitive idempotent pair is similarly defined. A *completely* 0-*t-simple ternary semigroup* is a 0-t-simple ternary semigroup T containing a primitive idempotent pair.

Lemma 5.1. *If L is a* 0*-minimal left ideal of a ternary semigroup T, then $L\backslash\{0\}$ is an $\mathcal{L}$-class.*

Proof. For every $x \in L$, $[TTx]$ is a left ideal of T contained in L so that $[TTx] = (0)$ or $[TTx] = L$. Suppose $[TTx] = L$ for every $x \in L\backslash\{0\}$. Then $x \cup [TTx] = L = y \cup [TTy]$ for every $x, y \in L\backslash\{0\}$ and so $L\backslash\{0\}$ is contained in the $\mathcal{L}$-class L_x. If $y \in L_x$, then $y \in x \cup [TTx] = L$ so that $L_x \subseteq L\backslash\{0\}$. Therefore $L\backslash\{0\}$ is an $\mathcal{L}$-class of T. Suppose $[TTx] = (0)$ for some $x \in L$. Then $\{0,x\}$ is a non-zero left ideal of T contained in L so that $\{0,x\} = L$. Then $x \cup [TTx] = L$ and $x\mathcal{L}y$ implies $x = y$. Hence in this case also $L\backslash\{0\}$ is a $\mathcal{L}$-class of T. □

A similar result can be proved for 0-minimal right ideals.

Lemma 5.2. *Let T be a* 0*-t-simple ternary semigroup containing a* 0*-minimal left ideal and a* 0*-minimal right ideal. Then to each* 0*-minimal left ideal L of T there exists a* 0*-minimal right ideal R of T such that $[LRT] \neq (0)$ and $[LRT] = T$. Also $[LTR] \neq (0)$ and $[LTR] = T$.*

Proof. $[LTT]$ is a two-sided ideal of T so that $[LTT] = (0)$ or $[LTT] = L$. If $[LTT] = (0)$, then $L^{<1>} = (0)$ and L is a two-sided ideal of T so that $T = L$ and $T^{<1>} = L^{<1>} = (0)$ contrary to the hypothesis. Therefore $[LTT] = L$. Then for some $x \in T$ $[LxT] \neq (0)$. Since T is the union of all the 0-minimal right ideals of T (by the dual of Theorem 4.11), $x \in R$ for some 0-minimal right ideal R of T. Hence $[LRT] \neq (0)$, $[LRT]$ is a non-zero two sided ideal of T and so $[LRT] = T$. Similarly it can be shown that $[LTR] = T$. □

Lemma 5.3. *Let L and R be* 0*-minimal left and right ideals respectively of a* 0*-t-simple ternary semigroup T. Then $[LRT] \neq (0)$ if and only if $[TLR] \neq (0)$. In this case $[LRT] = T = [TLR]$.*

Proof. By Lemma 5.2, if $[LRT] \neq (0)$, then $[LRT] = T$. Then $T = T^{<1>} = [LRTTLRT]$, whence $[TLR] \neq (0)$. Then $[TLR] = T$. Conversely, if $[TLR] \neq (0)$, then we can show that $[TLR] = T$. Further, $T = T^{<1>} = [TLRTT]$. Therefore, $[LRT] \neq (0)$ and $[LRT] = T$. □

Lemma 5.4. *Let L (resp. R) be a* 0*-minimal left (right) ideal of a* 0*-t-simple ternary semigroup and $a \in L\backslash\{0\}$ (resp. $R\backslash\{0\}$). Then $[TTa] = L$ (resp. $[aTT] = R$).*

Proof. Since T is a 0-t-simple, by Lemma 2.9, $T = [TTaTT]$, so $[TTa] \neq (0)$. Since $[TTa]$ is a non-zero left ideal contained in L, $[TTa] = L$. Similarly we can show that $[aTT] = R$ for $a \in R\backslash\{0\}$. $\square$

Let T be a 0-t-simple ternary semigroup and L and R are 0-minimal left and right ideals of T such that $[LRT] \neq (0)$. Then we have the following result.

Lemma 5.5. *$[RTL]$ is a ternary group with* 0.

Proof. Since $[LRT] \neq (0)$ by Lemma 5.3 $[LRT] = T = [TLR]$. Then $T = T^{<1>} = [LRTLRTT]$ and so $[RTL] \neq (0)$. Choose $a \in [RTL]$, $a \neq 0$. Then $a \in R \cap L$. Then by Lemma 2.9, $T = [TTaTT]$ and so $[aTT] \neq (0)$. Therefore $[aTT] = R$. Similarly $[TTa] = L$ and $T = [TLR] = [TLaTT]$. Therefore $[TLa] \neq (0)$, so $[TLa] = L$. $[RTL] = [RTTLa] = [RT[LRT]La] = [[RTL][RTL]a]$ proving that $[RTL]$ is left simple. Similarly $[aRT] = R$ and $[a[RTL][RTL]] = [aR[LRT]TL] = [aRTTL] = [RTL]$. Therefore $[RTL]$ is right simple. Hence by Theorem 1.1 in [13], $[RTL]$ is a ternary group with 0. $\square$

Lemma 5.6. *$[RTL] = R \cap L$.*

Proof. Clearly $[RTL] \subset R \cap L$. By Lemma 5.1, $L\backslash\{0\}$ is a $\mathcal{L}$-class of T. Similarly $R\backslash\{0\}$ is a $\mathcal{R}$-class of T. Therefore $H = R\backslash\{0\} \cap L\backslash\{0\}$ is a $\mathcal{H}$-class of T. Since $[RTL]$ is a ternary group with 0, for every $a \in [RTL]$, $a \neq 0$ there exists the ternary group inverse a^{-1} of a in $[RTL]$. Thus (a, a^{-1}) is an idempotent pair in $[RTL]$ and for every $z \in [RTL]$, $z \neq 0$ $z = [za^{-1}a]$. Since $a, a^{-1}, z \in [RTL]$, $a, a^{-1}, z \in H$ and $z = [za^{-1}a] \in H^{<1>} \cap H$. Hence, by Theorem 3.9, H is a ternary group. Therefore $R \cap L$ is a ternary group with 0. If $z \in R \cap L$, $z \neq 0$, then, by Lemma 5.1, z and a are in some $\mathcal{L}$-class and so for some $u, v \in T$, we have $z = [uva] = [uvaa^{-1}a] = [za^{-1}a] \in [RTL]$. Therefore $[RTL] = R \cap L$. $\square$

Lemma 5.7. *For every non-zero idempotent pair (a, b) in $[RTL]$, $R = [abT]$, $L = [Tab]$ and $[RTL] = [abTab]$.*

Proof. Let (a, b) be a non-zero idempotent pair in $[RTL]$. If $[aba] = 0$, then $[abx] = [ababx] = 0$ for every $x \in [RTL]$. Similarly $[xab] = 0$. Therefore (a, b) is equivalent to the zero idempotent pair, contrary to the hypothesis that (a, b) is a non-zero idempotent pair. Therefore $[aba] \neq 0$ and $[bab] \neq 0$. Then $[Tab] \neq (0)$ and $[abT] \neq (0)$. If $L = [Tab]$ and $R = [abT]$, then $[RTL] = [abTTTab] = [abTab]$. In particular for every $a \in [RTL]$, $a \neq 0$, $[RTL] = [aa^{-1}Taa^{-1}]$. $\square$

Lemma 5.8. *Every idempotent pair in $[RTL]$ is primitive in T.*

Proof. Let (a, b) be an idempotent pair in $[RTL]$. Then $[aba]$ is regular with $[bab]$ as the inverse in $[RTL]$ and (a, b) and $([aba], [bab])$ are equivalent to $[RTL]$. Therefore $[abz] = [abababz] = z$ for all $z \in [RTL]$. Similarly $[zab] = z$. Since $[aba]$ is regular, $([aba], [bab])$ is an idempotent pair in T. Therefore for any $t \in T$, $[abt] = [[abababa]bt] = [ababt]$. Similarly $[tab] = [tabab]$. Thus (a, b) is an idempotent pair in T and $(a,b) \sim ([aba],[bab])$. Hence without loss of generality we can take an idempotent pair (a,a^{-1}) in $[RTL]$. $\square$

Lemma 5.9. *Let T be a completely 0-t-simple ternary semigroup and (a,b) a primitive idempotent pair in T. Then $[Tab]$ and $[abT]$ are 0-minimal left and right ideals of T, respectively.*

Proof. Since (a,b) is a primitive idempotent pair, as in Lemma 5.7 we see that $L = [Tab]$ is a non-zero left ideal of T and $R = [abT]$ is a non-zero right ideal. Let A be a non-zero right ideal of T contained in R. Let $x \neq 0$, $x \in A$. Then $x \in R$ and $[abx] = x$. Since T is 0-t-simple, $T = [TTxTT]$ (Lemma 2.9). Hence for some $u_i, v_i, w_i, z_i \in T$, $i = 1,2$, $a = [u_1v_1xw_1z_1]$, $b = [u_2v_2xw_2z_2]$. Put $c_1 = [abau_2v_2ab]$, $d_1 = [w_2z_2aba]$, $c_2 = [babu_1v_1ab]$, $d_2 = [w_1z_1bab]$. We can easily show that $[c_1xd_1] = [aba]$ and $[c_2xd_2] = [bab]$. Clearly $c_i, d_i \neq 0$, $i = 1,2$. So, $([c_1xd_1],[c_2xd_2])$ is an idempotent pair equivalent to (a,b). Also, $[abc_1] = c_1 = [c_1ab]$, $[bac_2] = c_2 = [c_2ab]$. Put $f_1 = [xd_1c_2]$, $f_2 = [xd_2c_1]$, $[c_1f_1xd_2a] = [aba]$ and $[c_2f_2xd_1b] = [bab]$. Therefore $f_1 \neq 0, f_2 \neq 0$. Further $[f_1f_2f_1] = f_1$ and $[f_2f_1f_2] = f_2$. Therefore (f_1,f_2) is a non-zero idempotent pair in T. Moreover, also $(f_1,f_2) \leqslant (a,b)$. Since (a,b) is a primitive idempotent pair, $(f_1,f_2) \sim (a,b)$. Therefore $R = [abT] = [f_1f_2T] = [xd_1c_2xd_2c_1T] = [xTT] \subseteq A$. Thus $R = A$ and R is 0-minimal. Let B be a non-zero left ideal of T contained in L. Let $x \in B$, $x \neq 0$. Since T is 0-t-simple, $T = [TTxTT]$. Hence we can find elements $u_i, v_i, w_i, z_i \in T$, $i = 1,2$ such that $a = [u_1v_1xw_1z_1]$ and $b = [u_2v_2xw_2z_2]$. Put $c_1 = [abau_2v_2]$, $d_1 = [abw_2z_2aba]$, $c_2 = [babu_1v_1]$, $d_2 = [abw_1z_1bab]$. Then $[c_1xd_1] = [aba]$ and $[c_2xd_2] = [bab]$. Put $f_1 = [d_1c_2x]$, $f_2 = [d_2c_1x]$. As before we can show that (f_1,f_2) is a non-zero idempotent pair such that $(f_1,f_2) \leqslant (a,b)$. Therefore $(f_1,f_2) \sim (a,b)$. Hence $L = [Tab] = [Tf_1f_2] = [Td_1c_2xd_2c_1x] \subseteq [TTx] \subseteq B$. Therefore $L = B$ and L is 0-minimal. $\square$

Theorem 5.10. *Let T be 0-t-simple. T is completely 0-t-simple if and only if T contains at least one 0-minimal left ideal and at least one 0-minimal right ideal.*

Proof. If T is completely 0-t-simple, then T contains a primitive idempotent pair (a,b). By Lemma 5.9, $[Tab]$ and $[abT]$ are 0-minimal left ideal and 0-minimal right ideal, respectively. Conversely, assume that T contains at least one 0-minimal left ideal and one 0-minimal right ideal. Let L be a 0-minimal left ideal of T. Then by Lemma 5.2, there exists a 0-minimal right ideal R of T such that $[LTR] \neq (0)$. Then by Lemma 5.4, T contains a primitive idempotent pair and so T is completely 0-t-simple. $\square$

Corollary 5.11. *A completely 0-t-simple ternary semigroup is union of its 0-minimal left (right) ideals.*

Proof. Follows from the above Theorem and Lemma 5.4. $\square$

Corollary 5.12. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T such that $M^{<1>} \neq (0)$. If M contains at least one 0-minimal left ideal and at least one 0-minimal right ideal, then M is a completely 0-t-simple ternary subsemigroup of T.* □

Theorem 5.13. *Let T be a completely 0-t-simple ternary semigroup. Then non-zero elements of T form a $\mathcal{D}$-class and T is regular.*

Proof. Let T be a completely 0-t-simple ternary semigroup. Let a, b be non-zero elements of T. Then a lies in some 0-minimal left ideal L and b lies in some 0-minimal right ideal R of T. Thus $L = [TTa]$ and $R = [bTT]$. By Lemma 5.1 $L\backslash\{0\}$ is the $\mathcal{L}$-class containing a, $R\backslash\{0\}$ is the $\mathcal{R}$-class containing b and $[bTa] \subseteq R \cap L$. Since T is 0-t-simple, $T = [TTaTT]$ and $T = [TTbTT]$. Hence $T = [TTT] = [TTbTTTTTaTT] = [TTbTaTT]$. Therefore $[bTa] \neq (0)$ and, by Lemma 5.1, its dual $[bTa] \subset \mathrm{R_b} \cap \mathrm{L_a}$. If $c \in \mathrm{R_b} \cap \mathrm{L_a}$, then $a\mathcal{L}c$, $c\mathcal{R}b$ so, $a\mathcal{D}b$. Since a completely 0-t-simple ternary semigroup T containing a primitive idempotent pair (u, v) then (u, v) and $[uvu]$ both belongs to $\mathcal{D}$, and $[uvu]$ is a regular element in $\mathcal{D}$. Therefore by Proposition 3.4 the $\mathcal{D}$-class $T\backslash\{0\}$ is regular. Hence T is regular. □

6. $\mathcal{M}$-ternary semigroups

Below we introduce the concept of $\mathcal{M}$-ternary semigroups generalizing the notion of Rees matrix ternary semigroups.

Let G be a ternary group. We consider $G \cup \{0\}$, where we extend the ternary multiplication in G to $G \cup \{0\}$ by putting $[abc] = 0$ whenever any of a, b, c is zero. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. P is said to be *regular* if for every $i \in I$ there exists $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$ and for every $\lambda \in \Lambda$ there exists $i \in I$ such that $p_{\lambda i} \neq 0$. Consider the set

$$\mathcal{M}^0(G; I, \Lambda; P) = \{(a)_{i\lambda} : a \in G \cup \{0\},\ i \in I,\ \lambda \in \Lambda\},$$

where $(a)_{i\lambda}$ denotes the $\Lambda \times I$ matrix with entries a in (i, λ) position and 0 in other places. The $(0)_{i\lambda}$ is written as 0 and is independent of i and λ. We see that $(a)_{i\lambda} = (b)_{j\mu}$ if and only if $a = b$, $i = j$, $\lambda = \mu$. A ternary multiplication is introduced on this set as follows:

$$[(a)_{i\lambda}(b)_{j\mu}(c)_{k\nu}] = ([ap_{\lambda j}bp_{\mu k}c])_{i\nu}.$$

Lemma 6.1. *$\mathcal{M}^0(G; I, \Lambda; P)$ is a ternary semigroup.* □

Definition 6.2. The ternary semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ is called a *$\mathcal{M}$-ternary semigroup* (Matrix ternary semigroup).

Lemma 6.3. *If P is regular, then $\mathcal{M}^0(G; I, \Lambda; P)$ is a regular ternary semigroup.*

Proof. For given $(a)_{i\lambda}$ consider the elements $([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu}$ for every (j,μ). The set $\{([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu}\}$ of non-zero element is the set $I((a)_{i\lambda})$ of all inverses of $(a)_{i\lambda}$. □

Corollary 6.4. *If P is regular, then the pair $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ is an idempotent pair.* □

Lemma 6.5. *If P is regular, then the idempotent pairs $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are equivalent if and only if $k=i$ and $\mu=\omega$.*

Proof. Suppose that $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are idempotent pairs. Then for all $z=(z)_{m\delta}$ we have $[xx_1z]=[(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(z)_{m\delta}]$ $=([p_{\mu i}^{-1}p_{\mu m}z])_{i\delta}$ and $[yy_1z]=[(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(z)_{m\delta}]=([p_{\omega k}^{-1}p_{\omega m}z])_{k\delta}$. They are equivalent if and only if $i=k$ and $\omega=\mu$. In the same manner we obtain $[zxx_1]=[(z)_{m\delta}(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})]=([zp_{\delta i}p_{\mu i}^{-1}])_{m\mu}$, and analogously, $[zyy_1]=[(z)_{m\delta}(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})]=([zp_{\delta k}p_{\omega k}^{-1}])_{m\omega}$. Therefore, $[zxx_1]=[zyy_1]$ if and only if $k=i$ and $\omega=\mu$. □

Theorem 6.6. *If P is regular, then $\mathcal{M}^0(G;I,\Lambda;P)$ is a 0-t-simple ternary semigroup.*

Proof. For $(a)_{i\lambda}$, $(b)_{j\mu}$ we have $[(a^{-1})_{j\gamma}([p_{\gamma i}^{-1}ap_{\gamma i}^{-1}])_{i\gamma}(a)_{i\lambda}([p_{\lambda k}^{-1}a^{-1}p_{\lambda k}^{-1}])_{k\lambda}(b)_{k\mu}]$ $=([a^{-1}p_{\gamma i}p_{\gamma i}^{-1}ap_{\gamma i}^{-1}p_{\gamma i}ap_{\lambda k}p_{\lambda k}^{-1}a^{-1}p_{\lambda k}^{-1}bp_{\lambda k}])_{j\mu}=[(b)_{j\mu}]$. Hence $\mathcal{M}^0(G;I,\Lambda;P)$ is a 0-t-simple ternary semigroup by Lemma 2.9. □

Theorem 6.7. *If P is regular, then in $\mathcal{M}^0(G;I,\Lambda;P)$ every idempotent pair is primitive.*

Proof. Suppose that $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are idempotent pairs. If $(x,x_1)\leqslant(y,y_1)$ for some $x=(a)_{i\lambda}$, $x_1=([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$, $y=(b)_{k\nu}$ and $y_1=([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega}$, then for any $z=(t)_{m\alpha}\in\mathcal{M}^0(G;I,\Lambda;P)$ we have $[xx_1yy_1z]=[(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(t)_{m\alpha}]$ and $[yy_1xx_1z]=$ $[(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(t)_{m\alpha}]$, which obviously implies that $[ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega\alpha}t]_{i\alpha}=[bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu\alpha}t]_{k\alpha}$ $=([p_{\mu i}^{-1}p_{\mu\alpha}])_{i\alpha}$ Therefore $i=k$. Using the same method we can see that $[zxx_1yy_1]$ $=[(t)_{m\alpha}(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})]$. Analogously, $[zyy_1xx_1]=$ $[(t)_{m\alpha}(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})]$. From the above we obtain that $[tp_{\alpha i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}]_{m\omega}=[tp_{\alpha k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}]_{m\mu}$ Therefore, (x,x_1) is primitive if and only if $k=i$, $\omega=\mu$. This, by Lemma 6.5, means that (x,x_1) and (y,y_1) are equivalent. Thus every idempotent pair is primitive. □

As a consequence of Theorem 6.6 and Theorem 6.7 we obtain the following corollary.

Corollary 6.8. *If P is regular, then $\mathcal{M}^0(G; I, \Lambda; P)$ is a completely 0-t-simple semigroup.* $\square$

Lemma 6.9. *If P is regular, then in $\mathcal{M}^0(G; I, \Lambda; P)$*

$$(a)_{i\lambda}\mathcal{L}(b)_{j\mu} \Longleftrightarrow \lambda = \mu,$$

$$(a)_{i\lambda}\mathcal{R}(b)_{j\mu} \Longleftrightarrow i = j.$$

Corollary 6.10. *If P is regular, then non-zero elements of $\mathcal{M}^0(G; I, \Lambda; P)$ form a single $\mathcal{D}$-class in G.*

Proof. Indeed, $(a)_{i\lambda}\mathcal{L}(c)_{j\lambda}\mathcal{R}(b)_{j\mu}$ for any $c \in G$. $\square$

It is clear that the set of non-zero $\mathcal{L}$-classes in $\mathcal{M}^0(G; I, \Lambda; P)$ is $\{L_\lambda; \lambda \in \Lambda\}$, where $L_\lambda = \{(a)_{i\lambda} : a \in G,\ i \in I\}$. Similarly, the set of non-zero $\mathcal{R}$-classes is $\{R_i : i \in I\}$, where $R_i = \{(a)_{i\lambda} : a \in G, \lambda \in \Lambda\}$.

Corollary 6.11. *If P is regular, then $H_{i\lambda} = L_\lambda \cap R_i = \{(a)_{i\lambda} : a \in G\}$.* $\square$

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