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Strong forms of orthogonality for sets of frequency hypercubes

John T. Ethier and Gary L. Mullen

Abstract. For frequency hypercubes of dimension $d \geqslant 2$, we discuss several generalizations of the usual notion of pairwise orthogonality. We provide some constructions for complete sets of orthogonal frequency hypercubes.

1. Strong orthogonality for frequency hypercubes

In this paper we will examine strong forms of orthogonality for frequency hypercubes. The standard definition requires that each ordered pair occurs the same number of times in the superimposition of two hypercubes, but this definition says nothing about the location of the occurrences of these pairs. In [2], the authors examine several different forms of orthogonality which keep track of the positions of ordered pairs for orthogonal hypercubes. In this paper, we extend many of the ideas found in [2] to frequency hypercubes. In [6], the author defines a notion of orthogonality, called equiorthogonality, which requires corresponding subarrays be either orthogonal or isomorphic. We will modify this concept and define more precisely which subarrays must be orthogonal and which can be isomorphic.

2. Definitions

A *frequency square* $F(n; \lambda_1, \ldots, \lambda_m)$ of order n is an $n \times n$ array consisting of m distinct symbols with the property that for each $i = 1, \ldots, m$, the symbol i occurs exactly λ_i times in each row and in each column. Clearly $n = \lambda_1 + \cdots + \lambda_m$ and an $F(n; 1, \ldots, 1)$ frequency square is a latin square. In particular, we are interested in the case where $\lambda_1 = \cdots = \lambda_m$ and we write $F(n; \lambda)$ where $\lambda = n/m$.

Frequency squares can also be generalized to dimensions other than $d = 2$. A *frequency hypercube* of dimension $d \geqslant 2$ and order n, $F^{(d)}(n; \lambda_1, \cdots, \lambda_m)$ is an $n \times \cdots \times n$ (repeated d times) array consisting of m distinct symbols with the property that for each $i = 1, \ldots, m$, the symbol i occurs exactly λ_i times in each 1-subarray. Again, we will focus our attention on only the cases where $\lambda_1 = \cdots = \lambda_m = \lambda$ and denote such hypercubes as $F^{(d)}(n; \lambda)$.

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Two frequency hypercubes $F_1^{(1)}(n;n/m)$ and $F_2^{(1)}(n;n/m)$ are *isomorphic* if each can be obtained by relabeling the m symbols of the other. Note that isomorphism does not allow for the relabeling of coordinates or rearrangement of the subarrays.

We are now ready to introduce Morgan's definition of equiorthogonality for frequency hypercubes [6]. We say that two frequency hypercubes $F_1^{(1)}(n;n/m)$ and $F_2^{(1)}(n;n/m)$ are *equiorthogonal*, denoted $F_1^{(1)}(n;n/m) \perp_e F_2^{(1)}(n;n/m)$, if upon superimposition of the hypercubes, each ordered pair of symbols appears exactly n/m^2 times. Two frequency hypercubes $F_1^{(d)}(n;n/m)$ and $F_2^{(d)}(n;n/m)$ with $d \geqslant 2$ are said to be *equiorthogonal* if:

(a) upon superimposition of the hypercubes, each ordered pair of symbols appears exactly n^d/m^2 times;

(b) corresponding t-subarrays for all $1 \leqslant t < d$ are isomorphic or equiorthogonal; and

(c) if one pair of corresponding t-subarrays is isomorphic (resp. equiorthogonal), then all pairs of corresponding t-subarrays parallel to that pair are isomorphic (resp. equiorthogonal).

We will now modify the definition of equiorthogonality to define orthogonality of strength s or s-strong orthogonality, denoted by $\perp_s$. Two frequency hypercubes $F_1^{(1)}(n;n/m)$ and $F_2^{(1)}(n;n/m)$ are *orthogonal of strength* 1 if upon superimposition of the hypercubes, each ordered pair of symbols appears exactly n/m^2 times. Hypercubes $F_1^{(d)}(n;n/m)$ and $F_2^{(d)}(n;n/m)$ are said to be *orthogonal of strength* s, where $1 \leqslant s \leqslant d$, if

(1) upon superimposition of any corresponding s-subarrays of the hypercubes each ordered pair appears exactly n^s/m^2 times;

(2) corresponding t-subarrays for all $1 \leqslant t < d$ are isomorphic or orthogonal of strength t;

(3) if one pair of corresponding t-subarrays is isomorphic (resp. strongly orthogonal), then all pairs of corresponding t-subarrays parallel to that pair are isomorphic (resp. strongly orthogonal).

Remark 2.1. Theorem 3.3 will show that condition (2) need only be satisfied for all t-subarrays with $1 \leqslant t < s$.

Remark 2.2. In the latin case, that is when $n = m$, orthogonality of strength 1 is not possible.

We say that a set of frequency hypercubes is *mutually equiorthogonal* if any pair of hypercubes from the set is equiorthogonal. Similarly, a set of frequency hypercubes is *mutually s-strong orthogonal* if any pair of hypercubes from the set is s-strong orthogonal.

$$F_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \tag{1}$$

Figure1: Frequency squares illustrating orthogonality of strengths 1 and 2

In Figure 1 we see that F_1, F_2 and F_3 are mutually equiorthogonal. Also, F_1 and F_2 are orthogonal of strength 2, but are not orthogonal of strength 1, since corresponding rows are not orthogonal. However F_3 is orthogonal of strength 1 with both F_1 and F_2.

We will make use of the following result, found in [6], for subarrays of hypercubes in the subsequent sections.

Lemma 2.3. *Let $1 \leqslant t \leqslant d-2$. Suppose S_1 and S_2 are two parallel t-subarrays of $F^{(d)}(n; n/m)$. Then there exist $d-t-1$, t-subarrays $S_3, \dots, S_{d-t+1}$ such that:*

(1) *S_i is parallel to S_1 and S_2 for all i, $3 \leqslant i \leqslant d-t+1$, and*

(2) *S_1 and S_3 lie in a common $(t+1)$-subarray, S_3 and S_4 lie in a common $(t+1)$-subarray, …, S_{d-t+1} and S_2 lie in a common $(t+1)$-subarray.* □

3. Connections between strong orthogonality and equiorthogonality

In this section we will provide some basic results for strong orthogonality as well as provide a link between strong orthogonality and Morgan's definition of equiorthogonality.

Lemma 3.1. *If $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ are orthogonal of strength j, then they are also orthogonal of strength k for all $j < k \leqslant d$.*

Proof. Let $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ be orthogonal of strength j. Let $j < k \leqslant d$. We need only verify that condition (1) for orthogonality of strength k is satisfied. Notice that any corresponding k-subarrays are made up of n^{k-j} corresponding j-subarrays each of which has each ordered pair n^j/m^2 times. Thus each ordered pair appears exactly $n^j/m^2 \cdot n^{k-j}$ or n^k/m^2 times in each corresponding k-subarray and hence condition (1) is satisfied. □

Remark 3.2. As shown in Figure 1, the converse to Lemma 3.1 is not true. The squares F_1 and F_2 are orthogonal of strength 2, but are not orthogonal of strength 1.

Theorem 3.3. *If $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ satisfy condition (1) for orthogonality of strength j, and conditions (2) and (3) are satisfied for all corresponding t-subarrays with $t < j$ then $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ are orthogonal of strength j.*

Proof. If $j = d$ then the result is trivial. Suppose $j < d$ and $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ satisfy condition (1) for orthogonality of strength j, and conditions (2) and (3) are satisfied for all t-subarrays with $t < j$. We will show by induction that all k-subarrays with $j \leqslant k < d$ are orthogonal of strength j, and hence orthogonal of strength k by Lemma 3.1.

First let $k = j$. Consider any corresponding k-subarray $S^{(k)}(F_{1/2}^{(d)}(n;\lambda))$ and notice that condition (1) for orthogonality of strength j is satisfied since $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ satisfy condition (1). Also, conditions (2) and (3) are satisfied by our hypothesis. Hence all corresponding j-subarrays are orthogonal of strength j. Now consider $k = j+1$. Once again condition (1) is easily satisfied. Since we have shown that all corresponding j-subarrays are orthogonal of strength j then by our hypothesis conditions (2) and (3) are satisfied for all t-subarrays with $t < j+1$. Hence all corresponding $(j+1)$-subarrays are orthogonal of strength j. We proceed inductively to complete the proof. □

Remark 3.4. The preceding proof not only tells us that conditions (2) and (3) of s-strong orthogonality need only be verified for t-subarrays with $t < s$, but also that for all $s \leqslant t \leqslant d$ any corresponding t-subarrays will be s-strong orthogonal.

Corollary 3.5. *Condition* (1) *is necessary and sufficient for orthogonality of strength* 1.

Corollary 3.6. *In the latin case, that is when $n = m$, condition* (1) *is necessary and sufficient for orthogonality of strength* 2.

Proof. Follows by Corollary 3.5 and Remark 2.2. □

By the previous corollary, we see that for latin squares the definition of orthogonality of strength 2 is equivalent to the standard definition of orthogonality.

The following theorem and proof follows from a similar result in [6].

Theorem 3.7. *If $F_1^{(d)}(n;1)$ and $F_2^{(d)}(n;1)$ satisfy the first two conditions for orthogonality of strength s with $2 \leqslant s \leqslant d$, then they satisfy the third.*

Proof. Fix s with $2 \leqslant s \leqslant d$, and suppose the first two conditions are satisfied. Distinguish an arbitrary $(d-1)$-subarray $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$. By condition (2), this pair of subarrays is isomorphic or orthogonal of strength $d-1$. Suppose the pair is isomorphic. Then only n distinct ordered pairs are in $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$ and each one appears n^{d-2} times. But any ordered pair occurs n^{d-2} times in all of $F_{1/2}^{(d)}(n;1))$ so any ordered pair in $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$ cannot occur elsewhere. Since strongly orthogonal subarrays would have occurrences of every ordered pair we conclude by (2) that all $(d-1)$-subarrays parallel to $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$ are isomorphic.

Now suppose $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$ consists of a pair of $(d-1)$-strong orthogonal subarrays, then each ordered pair occurs $n^{(d-3)}$ times in $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$. If some pair of $(d-1)$-subarrays parallel to $S^{(d-1)}(F_{1/2}^{(d)}(n;1))$ were isomorphic then the ordered pairs in this new subarray would occur n^{d-2} more times. However, each ordered pair can only occur a total of n^{d-2} times in all of $F_{1/2}^{(d)}(n;1))$; hence all of the parallel subarrays must be $(d-1)$-strong orthogonal.

Next we will verify condition (3) for $(d-2)$-subarrays. Distinguish an arbitrary $(d-2)$-subarray $S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$. Let $S_2^{(d-2)}(F_{1/2}^{(d)}(n;1))$ be a pair of corresponding $(d-2)$ subarrays parallel to $S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$. If $S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$ and $S_2^{(d-2)}(F_{1/2}^{(d)}(n;1))$ lie in a common $(d-1)$-subarray, then the corresponding $(d-1)$-subarrays are either isomorphic or orthogonal of strength $d-1$. If the $(d-1)$-subarrays are isomorphic then it is not hard to see that $S_1^{(d-2)}(F_1^{(d)}(n;1))$ and $S_1^{(d-2)}(F_2^{(d)}(n;1))$ are isomorphic as well. If on the other hand, the corresponding $(d-1)$-subarrays are orthogonal of strength $d-1$ then we can use a counting argument similar to that given above to show that $S_2^{(d-2)}(F_1^{(d)}(n;1))$ and $S_2^{(d-2)}(F_2^{(d)}(n;1))$ are orthogonal as well.

If $S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$ and $S_2^{(d-2)}(F_{1/2}^{(d)}(n;1))$ do not lie in a common $(d-1)$-subarray, then by Lemma 2.3 we can find $S_3^{(d-2)}(F_{1/2}^{(d)}(n;1))$ such that
$S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$ and $S_3^{(d-2)}(F_{1/2}^{(d)}(n;1))$ lie in a common $(d-1)$-subarray and
$S_3^{(d-2)}(F_{1/2}^{(d)}(n;1))$ and $S_2^{(d-2)}(F_{1/2}^{(d)}(n;1))$ lie in a common $(d-1)$-subarray.
$S_1^{(d-2)}(F_1^{(d)}(n;1))$ and $S_1^{(d-2)}(F_2^{(d)}(n;1))$ are isomorphic (resp. orthogonal of strength $d-2$), therefore by the previous argument $S_3^{(d-2)}(F_1^{(d)}(n;1))$ and $S_3^{(d-2)}(F_2^{(d)}(n;1))$ are isomorphic (resp. orthogonal of strength $d-2$), therefore $S_2^{(d-2)}(F_1^{(d)}(n;1))$ and $S_2^{(d-2)}(F_2^{(d)}(n;1))$ are isomorphic (resp. orthogonal of strength $d-2$). Thus all pairs of corresponding $(d-2)$-subarrays parallel to $S_1^{(d-2)}(F_{1/2}^{(d)}(n;1))$ are isomorphic (resp. orthogonal of strength $d-2$). We can now proceed similarly for subarrays of all lower dimensions. □

We will now provide a useful link between strong orthogonality and equiorthogonality.

Theorem 3.8. *Two frequency hypercubes $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ are equiorthogonal if and only if they are orthogonal of strength d.*

Proof. By induction on d. For $d=1$, it is easy to see that equiorthogonality and orthogonality of strength 1 are equivalent. Suppose that the result holds for $d \leqslant s$. Now, let $d=s+1$. Since $d=s+1$, there is only one $(s+1)$-subarray, that is the hypercube itself. Hence conditions (1) and (a) are equivalent since $n^{(s+1)}/m^2 = n^d/m^2$. Now, if $t<d=s+1$, then $t \leqslant s$ and hence by our induction hypothesis, for any corresponding t-subarrays equiorthogonality and orthogonality of strength t are equivalent. Thus conditions (2) and (3) are equivalent to (b) and (c) respectively, and hence equiorthogonality and orthogonality of strength $s+1$ are equivalent. □

Corollary 3.9. *If two frequency hypercubes $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ are orthogonal of strength s, $1 \leqslant s \leqslant d$, then they are equiorthogonal.*

Proof. By Lemma 3.1, orthogonality of strength s implies orthogonality of strength d which is equivalent to equiorthogonality. □

The converse to Corollary 3.9 is not necessarily true. Although Theorem 3.8 yields that if $F_1^{(d)}(n;\lambda)$ and $F_2^{(d)}(n;\lambda)$ are equiorthogonal, then they are also orthogonal of strength d, this does not necessarily imply orthogonality of any strength s, for $s<d$ as stated in Remark 3.2.

In [3], Höhler defines two d-dimensional latin hypercubes of order n to be *orthogonal* if in the superimposition of the two hypercubes each ordered pair occurs exactly n^{d-2} times; and furthermore, in any t-subarray, each ordered pair occurs exactly n^s times, $0 \leqslant s < t$, or not at all.

Theorem 3.10. *The definition for orthogonality of strength d is equivalent to Höhler's definition of orthogonality for latin hypercubes $F^{(d)}(n;1)$ for $d \geqslant 2$.*

Proof. Orthogonality of strength d is equivalent to equiorthogonality for latin hypercubes $F^{(d)}(n;1)$ which is equivalent to Höhler's definition by Theorem 3.1 of [6]. □

If we let $s < d$, then orthogonality of strength s satisfies Höhler's definition, but strengthens the second condition. For example, if two d-dimensional latin hypercubes of order n are orthogonal of strength 2, then in any t-subarray, $t > 1$, each ordered pair occurs exactly n^{t-2} times; whereas if they were Höhler orthogonal then each ordered pair would occur n^s times, where s is some value $0 \leqslant s < t$, or not at all.

4. s-Strong orthogonal frequency hypercubes

In this section we will focus on sets of mutually s-strong orthogonal frequency hypercubes (MS_sOFH). We begin by determining a bound for the maximum number of MS_sOFH. The following theorem will be useful to us and can be found in [6].

Theorem 4.1. *The maximum number of mutually equiorthogonal frequency hypercubes $F^{(d)}(n;n/m)$ is at most $(n-1)^d/(m-1)$.* □

Using the equivalence found in Theorem 3.8, this immediately leads to the following corollary.

Corollary 4.2. *The maximum possible number of MS_dOFH, $F^{(d)}(n;n/m)$ is at most $(n-1)^d/(m-1)$.* □

We will now extend this to find a bound when $s < d$.

Corollary 4.3. *Let $s \leqslant d$. Then an upper bound for the number of MS_sOFH $F^{(d)}(n;n/m)$ is $(n-1)^s/(m-1)$.*

Proof. Let S be a set of MS_sOFH $F^{(d)}(n;n/m)$. Consider the set, R, of any corresponding s-subarrays of all of the members of S. Then R is a set of MS_sOFH $F^{(s)}(n;n/m)$ and thus has maximal size $(n-1)^s/(m-1)$ by Corollary 4.3. Our result follows since R and S have the same cardinality. □

A set which reaches this bound is called *complete*. We will denote a complete set of $(n-1)^s/(m-1)$, MS_sOFH $F^{(d)}(n;n/m)$ by $M_s^{(d)}(n;n/m)$. In Section , we will see that if m is a prime power and n is a power of m, then we can construct complete sets for all $d \geqslant 1$ when $s = 1$ or when $s = d$.

Corollary 4.4. *The maximum possible number of MS_1OFH $F^{(d)}(n;n/m)$ is at most $(n-1)/(m-1)$ for all $d \geqslant 1$.* □

We will now list some results which follow immediately from Theorem 3.8 and the known results for sets of mutually equiorthogonal frequency hypercubes given in [4] and [6].

Corollary 4.5. *There are at most $m-1$ MS_2OFS with isomorphic corresponding rows and isomorphic corresponding columns.* □

Corollary 4.6. *The maximum possible number of MS_jOFH $F^{(d)}(n;n/m)$, $j > 1$, with isomorphic corresponding $(j-1)$-subarrays is at most $n-1$.* □

Remark 4.7. By definition, there does not exist a pair of MS_1OFS with isomorphic corresponding rows and isomorphic corresponding columns.

Corollary 4.8. *For $d \geqslant 2$ the existence of a complete set of $(n-1)^d/(m-1)$ $MS_d OFH$ $F^{(d)}(n;n/m)$ implies the existence of a complete set of $(n-1)^t/(m-1)$ $MS_t OFH$ $F^{(t)}(n;n/m)$ for all $1 \leqslant t < d$.* □

Corollary 4.9. *Let $M_2^{(2)}(n;n/m)$ be a complete set of $(n-1)^2/(m-1)$ $MS_2 OFS$ $F^{(2)}(n;n/m)$. Then every square in the set has isomorphic rows and isomorphic columns.* □

Remark 4.10. The result of Corollary 4.9 does not extend to orthogonality of strength 1. If S is a complete set of $(n-1)/(m-1)$ MS_1OFS $F^{(2)}(n;n/m)$, then it is not necessarily true that every square in the set has isomorphic rows and isomorphic columns, as seen in Figure 2.

$$F_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{2}$$

Figure 2: A complete set of MS_1OFS $F^{(2)}(4;2)$

Theorem 4.11. *For $d > s$ the existence of a complete set of $(n-1)^s/(m-1)$ $MS_s OFH$ $F^{(d)}(n;n/m)$ implies the existence of a complete set of $(n-1)^s/(m-1)$ $MS_s OFH$ $F^{(t)}(n;n/m)$ for all $d > t \geqslant s$.*

Proof. Suppose we have a complete set of $(n-1)^s/(m-1)$ MS_sOFH $F^{(d)}(n;n/m)$. Take any set of corresponding t-subarrays with $d > t \geqslant s$. By Remark 3.4, these subarrays are also orthogonal of strength s and hence form a complete set of $(n-1)^s/(m-1)$ MS_sOFH $F^{(t)}(n;n/m)$. □

Lemma 4.12. *Let $M_2^{(2)}(n;n/m)$ be a complete set of $(n-1)^2/(m-1)$ $MS_2 OFS$ $F^{(2)}(n;n/m)$. Then $M_2^{(2)}(n;n/m)$ can be partitioned into $n-1$ classes in which the $(n-1)/(m-1)$ squares in any one class form a complete set of $MS_1 OFS$ $F^{(2)}(n;n/m)$.*

Proof. First, recall that we can partition MS_2OFS $F^{(2)}(n;n/m)$ hypercubes into $(n-1)/(m-1)$ column classes $C_1, \ldots, C_{(n-1)/(m-1)}$, each of which contains $n-1$ squares with isomorphic columns. Note that squares from different column classes have 1-strong orthogonal columns. Similarly we can partition MS_2OFS $F^{(2)}(n;n/m)$ into $(n-1)/(m-1)$ row classes $R_1, \ldots, R_{(n-1)/(m-1)}$.

Theorem 4.3 of [5] shows that the intersection of a column class and a row class, say C_i/R_j, contains exactly $m-1$ squares. For each k with $1 \leqslant k \leqslant (n-1)/(m-1)$ take one square from each of the intersections $C_1/R_{k+1(\mathrm{mod}(n-1)/(m-1))}$, $C_2/R_{k+2(\mathrm{mod}(n-1)/(m-1))}$, $\ldots$, $C_{(n-1)/(m-1)}/R_{k(\mathrm{mod}(n-1)/(m-1))}$.

Notice that each square is in a different column and row class and hence have 1-strong orthogonal columns and 1-strong orthogonal rows making these squares orthogonal of strength 1. Since there are $(n-1)/(m-1)$ squares we have formed a complete set of MS_1OFS $F^{(2)}(n;n/m)$. Since each intersection C_i/R_j contains $m-1$ squares we can form $m-1$ complete sets of MS_1OFS $F^{(2)}(n;n/m)$ in this manner for each of the $(n-1)/(m-1)$ values of k and hence $n-1$ complete sets as desired. □

Theorem 4.13. *Let $C_2^{(2)}(n;n/m)$ be a complete set of $(n-1)^2/(m-1)$ MS_2OFS $F^{(2)}(n;n/m)$ hypercubes. Then there exists a complete set of $(n-1)/(m-1)$ MS_1OFS $F^{(2)}(n;n/m)$ hypercubes such that every square in the set has isomorphic rows and isomorphic columns.*

Proof. We know that if a complete set exists then every square in the set has isomorphic rows and isomorphic columns. From Lemma 4.12, we know that $M_2^{(2)}(n;n/m)$ contains a subset of $(n-1)/(m-1)$ $\mathrm{M}S_1\mathrm{OFS}$ $F^{(2)}(n;n/m)$. □

Lemma 4.14. *Let $1 \leqslant s \leqslant d$ and let $M_s^{(d)}(n;n/m)$ be a complete set of $(n-1)^s/(m-1)$ MS_sOFH $F^{(d)}(n;n/m)$. Then there exists a complete set of $(n-1)/(m-1)$ MS_1OFS $F^{(2)}(n;n/m)$.*

Proof. We know that the existence of $M_s^{(d)}(n;n/m)$ implies the existence of $M_s^{(s)}(n;n/m)$ which implies the existence of $M_2^{(2)}(n;n/m)$. Our result follows from Lemma 4.12. □

Theorem 4.15. *Suppose $d \geqslant s \geqslant 3$. If $M_s^{(d)}(n;n/m)$ exists then m is a prime power.*

Proof. We know that the existence of $M_s^{(d)}(n;n/m)$ implies the existence of $M_s^{(s)}(n;n/m)$ which is equivalent to a complete set of MEFH. The result follows by Theorem 4.8 of [5]. □

We conclude this section by revisiting the bound given in Corollary 4.3. In the next section, we will see that this bound is attainable in certain cases. Specifically, if m is a prime power and n a power of m, we can construct complete sets of $(n-1)/(m-1)$ $\mathrm{M}S_1\mathrm{OFH}$ $F^{(d)}(n;n/m)$ for any d. Furthermore, in these prime power orders we will also be able to construct complete sets when $s = d$. The following results show that when $m = 2$ and $1 < s < d$ then the bound given in Corollary 4.3 is too large.

In the proofs of the following theorems we will use some basic graph theory. A *graph* G is comprised of a finite set of elements called *vertices* and a set of pairs of distinct vertices called *edges*. A graph with n vertices is called *complete* if each pair of distinct vertices forms an edge. It is not hard to see that such a graph has $\binom{n}{2}$ edges. A *multigraph* is a graph which allows a pair of vertices to form more than one edge.

Theorem 4.16. *When $d > \binom{n}{2}$ there are at most $n-1$, MS_2OFH $F^{(d)}(n;n/2)$.*

Proof. First, it is not hard to see that we cannot have more than one $\mathrm{M}S_2\mathrm{OFH}$ $F^{(2)}(n;n/2)$ with isomorphic rows and columns. Thus if we have two frequency hypercubes with isomorphic 1-subarrays in two coordinate directions, say x_i and x_j, then the corresponding squares formed by these 1-subarrays could not be 2-strong orthogonal and hence the frequency hypercubes themselves are not 2-strong orthogonal. Suppose there were n such frequency hypercubes. By Corollary 4.4 we know that we can have at most $n-1$ frequency hypercubes with 1-strong orthogonal corresponding 1-subarrays in any coordinate direction x_i with $1 \leqslant i \leqslant d$. Thus by the pigeonhole principle, we have at least two frequency hypercubes with isomorphic x_i 1-subarrays for each i with $1 \leqslant i \leqslant d$.

Next, consider each of the n frequency hypercubes as a vertex of a graph G, and consider the edges in the graph as representing that two frequency hypercubes have an isomorphic 1-subarray in a coordinate direction. For each i with $1 \leqslant i \leqslant d$, we must have at least two frequency hypercubes with an isomorphic 1-subarray in the x_i

direction, and hence we must add an edge to our graph. A complete graph K_n has exactly $\binom{n}{2}$ edges. Hence if $d > \binom{n}{2}$, then G has more than $\binom{n}{2}$ edges and thus at least two frequency hypercubes have isomorphic 1-subarrays in at least 2 coordinate directions, a contradiction. $\square$

Theorem 4.17. *When $d > n$, there exist at most $(n-1)^2 - 1$, MS_2OFH $F^{(d)}(n; n/2)$ hypercubes.*

Proof. Suppose that we have $(n-1)^2$, $\mathrm{MS}_2\mathrm{OFH}$ $F^{(d)}(n; n/2)$. As in the preceding proof, consider the $(n-1)^2$ frequency hypercubes to be vertices in a graph G. In order to obtain this bound, we know that for the corresponding 1-subarrays in each coordinate direction x_i, $1 \leqslant i \leqslant d$, we must have $n-1$ classes each containing $n-1$ hypercubes with isomorphic corresponding 1-subarrays in that direction. Each class represents a complete graph on $n-1$ vertices and hence has $\binom{n-1}{2}$ edges. Thus for each coordinate direction x_i we must add $(n-1)\binom{n-1}{2}$ edges to G. As in the preceding proof, G cannot have multiple edges between vertices and hence the most edges that G can have is $\binom{(n-1)^2}{2}$. Thus, we have at most $\frac{\binom{(n-1)^2}{2}}{(n-1)\binom{n-1}{2}} = n$ coordinate directions. $\square$

We will now generalize Theorem 4.16 to orthogonality of any strength.

Lemma 4.18. *If $F_1^{(d)}(n; n/2)$ and $F_2^{(d)}(n; n/2)$ have isomorphic corresponding 1-subarrays in each coordinate direction $x_1, x_2, \ldots, x_d$, then $F_1^{(d)}(n; n/2)$ and $F_2^{(d)}(n; n/2)$ are not orthogonal of strength d.*

Proof. Obvious. $\square$

Theorem 4.19. *When $d > \binom{n}{2} \times (s-1)$ there are at most $n-1$, MS_sOFH $F^{(d)}(n; n/2)$.*

Proof. First, notice if two frequency hypercubes $F_1^{(d)}(n; n/2)$ and $F_2^{(d)}(n; n/2)$ have isomorphic corresponding 1-subarrays in s different coordinate directions, then Lemma 4.18 implies that the corresponding s-subarrays formed by those 1-subarrays could not be orthogonal of strength s and hence $F_1^{(d)}(n; n/2)$ and $F_2^{(d)}(n; n/2)$ are not orthogonal of strength s. Suppose, for a contradiction, that we have n, $\mathrm{MS}_s\mathrm{OFH}$ $F^{(d)}(n; n/2)$ with $d > \binom{n}{2} \times (s-1)$. By Corollary 4.4 we know that we can have at most $n-1$ frequency hypercubes with 1-strong orthogonal corresponding 1-subarrays in any coordinate direction x_i with $1 \leqslant i \leqslant d$. Thus by the pigeonhole principle, we have at least two frequency hypercubes with isomorphic x_i 1-subarrays for each i with $1 \leqslant i \leqslant d$. Now, consider the n frequency hypercubes as the n vertices in a multi-graph G, and as before consider the edges in the graphs as representing that two frequency hypercubes have an isomorphic 1-subarray in a coordinate direction. By the above argument, there can be at most $s-1$ edges between any two vertices. Thus, at most, G can have $\binom{n}{2} \times (s-1)$ edges. Hence, if $d > \binom{n}{2} \times (s-1)$, then G has more than $\binom{n}{2} \times (s-1)$ edges and we have a contradiction. Therefore, there are at most $n-1$ $\mathrm{MS}_s\mathrm{OFH}$ $F^{(d)}(n; n/2)$ hypercubes. $\square$

Remark 4.20. The bound of $n-1$ $\mathrm{MS}_s\mathrm{OFH}$ $F^{(d)}(n; n/2)$ in Theorem 4.19 is attainable if n is a power of 2. This follows by Corollary 5.6, which tells us that there is a complete set of $n-1$ $\mathrm{MS}_1\mathrm{OFH}$ $F^{(d)}(n; n/2)$, and the fact that strength 1 orthogonality implies all higher strengths.

We can also generalize Theorem 4.17 to higher strengths by using similar arguments as found in the proofs of Theorem 4.17 and Theorem 4.19 to obtain the following result:

Theorem 4.21. *When* $d > \frac{[(n-1)^s-1]\times(s-1)}{[(n-1)^{s-1}-1]}$, *there exist at most* $(n-1)^s - 1$ $MS_s OFH$ $F^{(d)}(n; n/2)$. □

5. Constructions

If m is a prime power and n a power of m, we can construct complete sets of $(n-1)/(m-1)$ $\mathrm{M}S_1\mathrm{OFH}$ $F^{(d)}(n; n/m)$. The construction uses techniques similar to those found in [2], [5] and [7].

From [8], a polynomial $f(x_1,\ldots,x_n) \in \mathbb{F}_q[x_1,\ldots,x_n]$ is a permutation polynomial if the equation $f(x_1,\ldots,x_n) = \alpha$ has exactly q^{n-1} solutions in $\mathbb{F}_q^n$ for every $\alpha \in \mathbb{F}_q$. Two permutation polynomials $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$ are orthogonal if the system

$$\begin{aligned} f(x_1,\ldots,x_n) &= \alpha \\ g(x_1,\ldots,x_n) &= \beta \end{aligned} \tag{3}$$

has exactly q^{n-2} solutions for each $(\alpha, \beta) \in \mathbb{F}_q^2$. Also it is easy to check that a linear polynomial is a permutation polynomial over $\mathbb{F}_q$ if and only if it has at least one non-zero coefficient. Also, we know that two non-constant linear polynomials form an orthogonal system if and only if neither is a scalar multiple of the other. Finally, addition of a constant to either or both polynomials does not affect orthogonality or the property of being a permutation polynomial.

We begin by considering the case where $d = 1$. For m a prime power, we can construct a $F^{(1)}(m^i; m^{(i-1)})$ hypercube from $a_1x_1 + \cdots + a_ix_i$ over $\mathbb{F}_m$ as follows. We use the elements of $\mathbb{F}_m$ as the m symbols and the m^i, i-tuples in $\mathbb{F}_m^i$ as the coordinate labels. Then the symbol in entry $(x_1,\ldots,x_i)$ is given by $a_1x_1 + \cdots + a_ix_i$.

Lemma 5.1. *Let m be a prime power and suppose that $a_k \in \mathbb{F}_m$ for $1 \leqslant k \leqslant i$ and $(a_1,\ldots,a_i) \neq (0,\ldots,0)$. Then the hypercube constructed from $a_1x_1 + \cdots + a_ix_i$ is a $F^{(1)}(m^i; m^{(i-1)})$ hypercube.*

Proof. It is only necessary to show that each symbol appears m^{i-1} times in the constructed cube. Let $\alpha \in \mathbb{F}_m$ then according to our construction α appears when $a_1x_1 + \cdots + a_ix_i = \alpha$ for some i-tuple $(x_1,\ldots,x_i)$. Now, $a_1x_1 + \cdots + a_ix_i = \alpha$ is a linear polynomial with at least one non-zero coefficient, hence it is a permutation polynomial. Thus, we know that there are m^{i-1} solutions and therefore α appears m^{i-1} times. □

Lemma 5.2. *Let m be a prime power and suppose that $a_k \in \mathbb{F}_m$ for $1 \leqslant k \leqslant i$ and $(a_1,\ldots,a_i) \neq (0,\ldots,0)$. Also, let $b_k \in \mathbb{F}_m$ for $1 \leqslant k \leqslant i$ and $(b_1,\ldots,b_i) \neq (0,\ldots,0)$ with $(a_1,\ldots,a_i) \neq u(b_1,\ldots,b_i)$ where $u \in \mathbb{F}_m^*$. Then the hypercubes constructed from $a_1x_1 + \cdots + a_ix_i$ and $b_1x_1 + \cdots + b_ix_i$ are orthogonal of strength 1.*

Proof. Since $(a_1,\ldots,a_i) \neq u(b_1,\ldots,b_i)$, we know that $a_1x_1+\cdots+a_ix_i$ and $b_1x_1+\cdots+b_ix_i$ form an orthogonal system. Hence, we have that the system:

$$\begin{aligned} a_1x_1 + \cdots + a_ix_i &= \alpha \\ b_1x_1 + \cdots + b_ix_i &= \beta \end{aligned} \tag{4}$$

has exactly m^{i-2} solutions for each $(\alpha, \beta) \in \mathbb{F}_m^2$. Thus, upon superimposition of the two hypercubes each ordered pair (α, β) will appear exactly m^i/m^2 times and therefore the two hypercubes are orthogonal of strength 1. □

Theorem 5.3. *If m is a prime power and n is a power of m, then there exists a complete set of $(n-1)/(m-1)$, MS_1OFH $F^{(1)}(n; n/m)$.*

Proof. Let $n = m^i$. We need to find a set of $(n-1)/(m-1)$ linear polynomials $a_1x_1 + \cdots + a_ix_i$ over $\mathbb{F}_m$ such that each polynomial has at least one non-zero coefficient and that no two sets of coefficients are scalar multiples of each other. First, we include the polynomials, P_1, such that $a_1 = 1$ where 1 represents the multiplicative identity of $\mathbb{F}_m$. Notice that there are m choices for each of the $i-1$ coefficients $a_2, \ldots, a_i$, and hence m^{i-1} such polynomials overall. Also, no two different polynomials can be scalar multiples of one another since $a_1 = 1$ is fixed and hence the scalar would have to be 1, making the polynomials identical. Next, we include the polynomials, P_2, where $a_1 = 0$ and $a_2 = 1$. Here we have m^{i-2} polynomials, and by the same argument as above the elements of P_2 are not scalar multiples of one another. Furthermore, by examining a_1 it is easy to see that no element of P_2 is a non-zero scalar multiple of P_1. Continuing in this manner, we obtain a set of polynomials $P_1, P_2, \ldots, P_i$ such that each polynomial has at least one non-zero coefficient and that no two sets of coefficients are scalar multiples of each other. Furthermore, we have $m^{i-1} + m^{i-2} + \cdots + 1 = (m^i - 1)/(m-1) = (n-1)/(m-1)$ polynomials which generate our complete set of $\mathrm{M}S_1$OFH $F^{(1)}(n; n/m)$ hypercubes. □

We can use similar methods to create strongly orthogonal frequency hypercubes of all dimensions $d \geqslant 1$. For m a prime power, we can construct a $F^{(d)}(m^i; n/m)$ from the polynomial $a_1x_1 + \cdots + a_{di}x_{di}$ over $\mathbb{F}_m$ similar to before. Once again, we use the elements of $\mathbb{F}_m$ as symbols and the m^i, i-tuples in $\mathbb{F}_m^i$ as the coordinate labels. We now associate each block of variables $(x_{(k-1)i+1}, \ldots, x_{ki})$ to the k-th coordinate direction for $1 \leqslant k \leqslant d$. Then the symbol in cell $((x_1, \ldots, x_i), \ldots, (x_{(d-1)i+1}, \ldots, x_{di}))$ is given by $a_1x_1 + \cdots + a_{di}x_{di}$. The proof of the following lemma is similar to the proof of Lemma 5.1.

Lemma 5.4. *Let m be a prime power and suppose that $a_k \in \mathbb{F}_m$ for $1 \leqslant k \leqslant di$ and $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq (0, \ldots, 0)$ for $1 \leqslant k \leqslant d$. Then the hypercube constructed from $a_1x_1 + \cdots + a_{di}x_{di}$ is a $F^{(d)}(m^i; m^{(i-1)})$ hypercube.*

Lemma 5.5. *Let m be a prime power and suppose that $a_j \in \mathbb{F}_m$ for $1 \leqslant j \leqslant di$ and $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq (0, \ldots, 0)$ for $1 \leqslant k \leqslant d$. Also, let $b_j \in \mathbb{F}_m$ for $1 \leqslant j \leqslant di$ and $(b_{(k-1)i+1}, \ldots, b_{ki}) \neq (0, \ldots, 0)$ for $1 \leqslant k \leqslant d$. Also, let $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq u(b_{(k-1)i+1}, \ldots, b_{ki})$ for $1 \leqslant k \leqslant d$ where $u \in \mathbb{F}_m^*$. Then the hypercubes constructed from $a_1x_1 + \cdots + a_{di}x_{di}$ and $b_1x_1 + \cdots + b_{di}x_{di}$ are orthogonal of strength-1 frequency hypercubes of dimension d.*

Proof. We need to show that in the superimposition of any corresponding 1-subarrays each ordered pair (α, β) appears exactly m^i/m^2 times. Consider any subarray which varies in the k-th coordinate direction. To find the symbols of each hypercube in this subarray, we can substitute the fixed coordinates of the subarray to obtain the polynomials $p_1 = a_{(k-1)i+1}x_{(k-1)i+1} + \cdots + a_{ki}x_{ki} + c_1$ and $p_2 = b_{(k-1)i+1}x_{(k-1)i+1} + \cdots + b_{ki}x_{ki} + c_2$. Since $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq (0, \ldots, 0)$ and $(b_{(k-1)i+1}, \ldots, b_{ki}) \neq (0, \ldots, 0)$ we know that

p_1 and p_2 are permutation polynomials. Also, $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq u(b_{(k-1)i+1}, \ldots, b_{ki})$ implies that p_1 and p_2 form an orthogonal system. Hence, the system:

$$\begin{aligned} a_{(k-1)i+1}x_{(k-1)i+1} + \cdots + a_{ki}x_{ki} + c_1 &= \alpha \\ b_{(k-1)i+1}x_{(k-1)i+1} + \cdots + b_{ki}x_{ki} + c_2 &= \beta \end{aligned} \tag{5}$$

has exactly m^{i-2} solutions for each $(\alpha, \beta) \in \mathbb{F}_m^2$. Thus, upon superimposition of the two corresponding 1-subarrays each ordered pair (α, β) will appear exactly $m^i/m^2 = m^{i-2}$ times. □

Theorem 5.6. *If m is a prime power and n is a power of m, then there exists a complete set of $(n-1)/(m-1)$, MS_1OFH $F^{(d)}(n; n/m)$.*

Proof. From Theorem 5.3, we know that we can find a set of $(n-1)/(m-1)$ linear polynomials $p_1, \ldots, p_{(n-1)/(m-1)}$ over $\mathbb{F}_m$ such that each polynomial has at least one non-zero coefficient and that no two sets of coefficients are scalar multiples of each other. Now, for each polynomial $p_j = a_1x_1 + \cdots + a_ix_i$ create a new polynomial $p_j' = a_1x_1 + \cdots + a_{di}x_{di}$ by letting $(a_{(k-1)i+1}, \ldots, a_{ki}) = (a_1, \ldots, a_i)$ for $2 \leqslant k \leqslant d$. Then the conditions of Lemma 5.5 are met and the d-dimensional frequency hypercubes generated by the polynomials $p_1', \ldots, p_{(n-1)/(m-1)}'$ form a complete set of MS$_1$OFH $F^{(d)}(n; n/m)$ hypercubes. □

Theorem 5.7. *Fix s, let m be a prime power, and suppose that:*

(*a*) *$a_j \in \mathbb{F}_m$ and $b_j \in \mathbb{F}_m$ for $1 \leqslant j \leqslant di$ with the property that $(a_{(k-1)i+1}, \ldots, a_{ki}) \neq (0, \ldots, 0)$ and $(b_{(k-1)i+1}, \ldots, b_{ki}) \neq (0, \ldots, 0)$ for $1 \leqslant k \leqslant d$.*

(*b*) *For all possible indices $k_1, k_2, \ldots, k_s$, $1 \leqslant k_1 < k_2 < \ldots < k_s \leqslant d$ $(a_{(k_1-1)i+1}, \ldots, a_{k_1 i}, a_{(k_2-1)i+1}, \ldots, a_{k_2 i}, \ldots, a_{(k_s-1)i+1}, \ldots, a_{k_s i}) \neq$ $u(b_{(k_1-1)i+1}, \ldots, b_{k_1 i}, b_{(k_2-1)i+1}, \ldots, b_{k_2 i}, \ldots, b_{(k_s-1)i+1}, \ldots, b_{k_s i})$ where $u \in \mathbb{F}_m$.*

Then the hypercubes constructed from $a_1x_1 + \cdots + a_{di}x_{di}$ and $b_1x_1 + \cdots + b_{di}x_{di}$ are s-strong orthogonal frequency hypercubes of order m^i and dimension d.

Proof. Sketch. Condition (a) ensures that the construction yields frequency hypercubes. Condition (b) first ensures that the two sets of coefficients are not scalar multiples and hence the two hypercubes are at least orthogonal of strength d and therefore conditions (2) and (3) for strong orthogonality of any type are satisfied. To see that condition (1) is satisfied, consider that any s-subarray is constructed essentially with the coefficients $(a_{(k_1-1)i+1}, \ldots, a_{k_1 i}, a_{(k_2-1)i+1}, \ldots, a_{k_2 i}, \ldots, a_{(k_s-1)i+1}, \ldots, a_{k_s i})$ for some indices $k_1, k_2, \ldots, k_s$. Condition (b) ensures that these two sets of coefficients are not scalar multiples and hence the constructed s-subarrays are orthogonal of strength s. Thus condition (1) is satisfied. □

If $s = d$, then the construction in Theorem 5.7 reduces to that given in [5] for the construction of mutually equiorthogonal frequency hypercubes. In this case, we can construct a complete set.

Theorem 5.8. *If m is a prime power and n is a power of m, then there exists a complete set of $(n-1)^s/(m-1)$ MS_sOFH $F^{(s)}(n; n/m)$.* □

6. Frequency hyperrectangles

In this section we will generalize from hypercubes to hyperrectangles. A *Youden F-hyperrectangle* $YF(n_1, \ldots, n_d; m)$, where $m \mid \prod_{j \neq i} n_j$ for each $1 \leqslant i \leqslant d$, is an $n_1 \times \cdots \times n_d$ array consisting of $m \geqslant 2$ symbols with the property that for each i, $1 \leqslant i \leqslant d$, each symbol appears exactly $(\prod\limits_{j \neq i} n_j)/m$ times in each $(d-1)$-subarray obtained by fixing the i-th coordinate. Two such F-hyperrectangles are *orthogonal* if upon superimposition every ordered pair appears the same number of times. A set of F-hyperrectangles are called *mutually orthogonal* if every pair of distinct F-hyperrectangles from the set is orthogonal. Cheng [1] proved the following result:

Theorem 6.1. *The maximum number of mutually orthogonal* $F(n_1, \ldots, n_d; m)$ *is at most* $(\prod n_i - \sum(n_i - 1) - 1)/(m-1)$. □

In [9], Suchower was able to construct complete sets of F-hyperrectangles when m is a prime power and each n_i, $1 \leqslant i \leqslant d$, is a power of m.

We would like to turn our attention to a more structured definition of frequency hyperrectangles. Notice that a Youden F-hyperrectangle only requires that the symbols appear the same number of times in each $(d-1)$-subarray; whereas much of our previous work in frequency objects required that each symbol appears the same number of times in each 1-subarray. In the rest of this work we will define a *frequency hyperrectangle* $F(n_1, \ldots, n_d; m)$, where $m \mid n_i$ for each $1 \leqslant i \leqslant d$, as an $n_1 \times \cdots \times n_d$ array consisting of $m \geqslant 2$ symbols with the property that each symbol appears the same number of times in each 1-subarray. Notice that this definition of a frequency hyperrectangle is a generalization of frequency hypercubes, since a frequency hypercube of dimension d is a frequency hyperrectangle with $n_1 = n_2 = \cdots = n_d$. We will now work to extend the results for equiorthogonality and strong orthogonality to frequency hyperrectangles.

Two one-dimensional frequency hyperrectangles $F_1(n; m)$ and $F_2(n; m)$ are *equiorthogonal* if upon superimposition of the hyperrectangles, each ordered pair of symbols appears exactly n/m^2 times. Two frequency hyperrectangles $F_1(n_1, \ldots, n_d; m)$ and $F_2(n_1, \ldots, n_d; m)$ with $d \geqslant 2$ are said to be *equiorthogonal* if:

(a) upon superimposition of the hyperrectangles, each ordered pair of symbols appears exactly $n_1 n_2 \ldots n_d/m^2$ times;
(b) corresponding t-subarrays for all $1 \leqslant t < d$ are isomorphic or equiorthogonal; and
(c) if one pair of corresponding t-subarrays is isomorphic (resp. equiorthogonal), then all pairs of corresponding t-subarrays parallel to that pair are isomorphic (resp. equiorthogonal).

We will begin by finding an upper bound on the number of mutually equiorthogonal frequency hyperrectangles $F(n_1, \ldots, n_d; m)$. This will be a generalization of the bound and proof [6] for equiorthogonal frequency hypercubes.

Lemma 6.2. *There are at most* $m-1$ *equiorthogonal frequency rectangles,* $F(n_1, n_2; m)$, *with isomorphic corresponding rows and isomorphic corresponding columns.*

Proof. If $m = 2$ the result is trivial. Now, let $m > 2$, and permute the n_1 rows and n_2 columns of one rectangle, R_1, so that the first row and first column are $0 \ldots 0 1 \ldots 1 \ldots (m-1) \ldots (m-1)$. By the hypothesis of isomorphic corresponding rows the first row of each rectangle must consist of m blocks, each consisting of the same symbol repeated n_2/m

times; similarly the first column of each rectangle must consist of m blocks, each consisting of the same symbol repeated n_1/m times. Now, permute the symbols of each rectangle so that the first row of each rectangle is $0 \dots 0 1 \dots 1 \dots (m-1) \dots (m-1)$.

Notice that for any two rectangles, when corresponding rows are superimposed, since they are isomorphic, a given ordered pair can either occur 0 or n_2/m times. Similarly when corresponding columns are superimposed a given ordered pair can either occur 0 or n_1/m times. Therefore, by condition (a) for equiorthogonality, a given ordered pair occurs in exactly n_1/m rows and in exactly n_2/m columns. Notice that in the superimposition of any two rectangles the ordered pair $(0,0)$ must occur in the first n_1/m rows and the first n_2/m columns. Thus every rectangle has an $n_1/m \times n_2/m$ block of zeros in the upper left corner.

Now, consider that since the first row of every rectangle is identical, then for every $0 \leqslant j \leqslant m-1$, the ordered pair (j,j) occurs in the n_2/m columns $jn_2/m+1, jn_2/m+2, \dots (j+1)n_2/m$ and therefore in no other columns. Thus for $j \neq 0$, the ordered pair (j,j) cannot occur in the first n_2/m columns. If we then consider the entry in row $n_1/m+1$ of the first column, we know that the entries for each rectangle must be distinct by the above argument, and that they are non-zero. Therefore, we have at most $m-1$ rectangles. □

Lemma 6.3. *There are at most $(n-1)/(m-1)$ mutually equiorthogonal frequency hyperrectangles $F(n;m)$.*

Proof. This follows from the bound for MEFH and the fact that a hyperrectangle of dimension 1 and hypercube of dimension 1 are by definition equivalent as is the definition of equiorthogonality. □

Lemma 6.4. *The maximum number of mutually equiorthogonal rectangles with n_1 rows, n_2 columns, and based on m symbols with isomorphic corresponding columns is at most $n_2 - 1$.*

Proof. By definition of equiorthogonality, the first rows of all the rectangles must be either isomorphic or equiorthogonal. Since isomorphism is an equivalence relation, we can partition the rectangles into classes with the property that rectangles in the same class have isomorphic first rows and thus rectangles from different classes have equiorthogonal first rows. Therefore, condition (c) implies that rectangles from the same class will have every pair of corresponding rows isomorphic, and rectangles from distinct classes will have all corresponding rows equiorthogonal.

Since rectangles from the same class will have isomorphic corresponding rows and columns, Lemma 6.2 asserts that there are at most $m-1$ rectangles in each class. Also, since rectangles from distinct classes will have equiorthogonal rows, Lemma 6.3 tells us that there are at most $(n_2-1)/(m-1)$ distinct classes. Therefore we have at most $(m-1) \times (n_2-1)/(m-1)$ or $n_2 - 1$ such rectangles. □

Proofs of the following lemma and theorem are similar to those found in [6] for frequency hypercubes.

Lemma 6.5. *The maximum number of mutually equiorthogonal frequency hyperrectangles $F(n_1, \dots, n_d; m)$ with isomorphic corresponding $(d-1)$-subarrays with fixed coordinate x_i, $1 \leqslant i \leqslant d$, is at most $n_i - 1$.* □

Theorem 6.6. *The maximum number of mutually equiorthogonal frequency hyperrectangles* $F(n_1, \ldots, n_d; m)$ *is at most* $(\prod_{i=1}^{d}(n_i - 1))/(m-1)$. □

For m a prime power and n_i a power of m, for each $1 \leqslant i \leqslant d$ we can construct complete sets of MEFR $F(n_1, \ldots, n_d; m)$ using permutation polynomials over finite fields $\mathbb{F}_q$ where $q = m$. For m a prime power, we can construct a $F(m^{i_1}, \ldots, m^{i_d}; m)$ from the polynomial $a_1x_1 + \cdots + a_lx_l$, where $l = \sum_{j=1}^{d} i_j$, over $\mathbb{F}_m$ similar to before. Once again, we use the elements of $\mathbb{F}_m$ as symbols. Next we use the m^{i_j}, i_j-tuples in $\mathbb{F}_m^{i_j}$ as the coordinate labels for coordinate x_{i_j}. We now associate the first i_1 block of variables to the first coordinate, the next i_2 variables to the second coordinate, and so forth. Then the symbol in cell $((x_1, \ldots, x_{i_1}), \ldots, (x_{l-i_d+1}, \ldots, x_l))$ is given by $a_1x_1 + \cdots + a_lx_l$.

Lemma 6.7. *Let* m *be a prime power and suppose that* $a_j \in \mathbb{F}_m$ *for* $1 \leqslant j \leqslant l$, *where* $l = \sum_{j=1}^{d} i_j$, *and* $(a_{l_{k-1}+1}, \ldots, a_{l_k}) \neq (0, \ldots, 0)$ *for* $1 \leqslant k \leqslant d$, *where* $l_k = \sum_{j \leqslant k} i_j$. *Also, let* $b_j \in \mathbb{F}_m$ *for* $1 \leqslant j \leqslant l$, *where* $l = \sum_{j=1}^{d} i_j$, *and* $(b_{l_{k-1}+1}, \ldots, b_{l_k}) \neq (0, \ldots, 0)$ *for* $1 \leqslant k \leqslant d$, *where* $l_k = \sum_{j \leqslant k} i_j$. *Also, let* $(a_1, \ldots, a_l) \neq u(b_1, \ldots, b_l)$ *where* $u \in \mathbb{F}_m^*$. *Then the hyperrectangles constructed from* $a_1x_1 + \cdots + a_lx_l$ *and* $b_1x_1 + \cdots + b_lx_l$ *are equiorthogonal.* □

Theorem 6.8. *If* m *is a prime power and* n_i *is a power of* m *for each* $1 \leqslant i \leqslant d$, *then we can construct a complete set of* $(\prod_{i=1}^{d}(n_i - 1))/(m-1)$ *MEFR* $F(n_1, \ldots, n_d; m)$.

Proof. Let $n_i = m^{j_i}$ for each $1 \leqslant i \leqslant d$. Then there are exactly $n_i - 1$ non-zero j_i-tuples $(a_1, \ldots, a_{i_j})$ with $a_k \in \mathbb{F}_m$. Hence we have $(\prod_{i=1}^{d}(n_i - 1))$ polynomials such that $(a_{l_{k-1}+1}, \ldots, a_{l_k}) \neq (0, \ldots, 0)$ for $1 \leqslant k \leqslant d$, where $l_k = \sum_{j \leqslant k} i_j$ as in Lemma 6.7. We then divide by the $m-1$ non-zero scalar multiples of $\mathbb{F}_m$ and obtain a set of $(\prod_{i=1}^{d}(n_i-1))/(m-1)$ polynomials which satisfy Lemma 6.7, generating a complete set. □

We will now look how strong orthogonality extends from hypercubes to hyperrectangles. Two frequency hyperrectangles $F_1(n; m)$ and $F_2(n; m)$ are *strongly orthogonal*, denoted $F_1(n; m) \perp_s F_2(n; m)$, if upon superimposition of the hyperrectangles, each ordered pair of symbols appears exactly n/m^2 times. Two frequency hyperrectangles $F_1(n_1, \ldots, n_d; m)$ and $F_2(n_1, \ldots, n_d; m)$ with $d \geqslant 2$ are said to be *orthogonal of strength* s if:

(1) upon superimposition of any corresponding s-subarrays of the hyperrectangles, each ordered pair of symbols appears exactly $n_{i_1}n_{i_2}\ldots n_{i_s}/m^2$ times, where $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ are the free coordinates of the s-subarray;
(2) corresponding t-subarrays for all $1 \leqslant t < d$ are isomorphic or strongly orthogonal;
(3) if one pair of corresponding t-subarrays is isomorphic (resp. strongly orthogonal), then all pairs of corresponding t-subarrays parallel to that pair are isomorphic (resp. strongly orthogonal).

Lemma 6.9. *If $F_1(n_1,\ldots,n_d;m)$ and $F_2(n_1,\ldots,n_d;m)$ are orthogonal of strength j, then they are also orthogonal of strength k for all $j \leqslant k \leqslant d$.*

Proof. Let $F_1(n_1,\ldots,n_d;m)$ and $F_2(n_1,\ldots,n_d;m)$ be orthogonal of strength j. Let $j \leqslant k \leqslant d$. We need only verify that condition (1) for orthogonality of strength k is satisfied. Consider corresponding k-subarrays, without loss of generality suppose that the corresponding k-subarrays have free coordinates $x_1,\ldots,x_k$. Notice that this k-subarray is made up of corresponding j-subarrays with free coordinates $x_1,\ldots x_j$ each of which has each ordered pair $(\prod_{i=1}^{j}(n_i))/m^2$ times. Since these k-subarrays are formed by exactly $\prod_{i=j+1}^{k} n_i$ such j-subarrays, it follows that each ordered pair appears exactly $(\prod_{i=1}^{j}(n_i))/m^2 \cdot (\prod_{i=j+1}^{k} n_i$ or $(\prod_{i=1}^{k}(n_i))/m^2$ times in each corresponding k-subarrays and hence condition (1) is satisfied. □

The proof of the following theorem mirrors the proof of Theorem 3.8.

Theorem 6.10. *Two frequency hyperrectangles $F_1(n_1,\ldots,n_d;m)$ and $F_2(n_1,\ldots,n_d;m)$ are equiorthogonal if and only if they are orthogonal of strength d.* □

Corollary 6.11. *If $F_1(n_1,\ldots,n_d;m)$ and $F_2(n_1,\ldots,n_d;m)$ are orthogonal of strength s, $s \leqslant d$, then they are also equiorthogonal.*

Proof. Follows by the preceding theorem and Lemma 6.9 □

Corollary 6.12. *The maximum number of mutually d-strong orthogonal frequency hyperrectangles $F(n_1,\ldots,n_d;m)$ is at most $(\prod_{i=1}^{d}(n_i-1))/(m-1)$.*

Proof. Follows by Corollary 6.11 and Theorem 6.6 □

The following corollary can be proved in a similar manner as Corollary 4.3

Corollary 6.13. *The maximum number of mutually s-strong orthogonal frequency hyperrectangles $F(n_1,\ldots,n_d;m)$ is at most $\min(\prod_{i_1,\ldots i_s}(n_i-1))/(m-1)$ where $1 \leqslant i_1 < \ldots < i_s \leqslant d$.*

Corollary 6.14. *For $n = \min\{n_1,\ldots,n_d\}$ the maximum number of mutually 1-strong orthogonal frequency hyperrectangles $F(n_1,\ldots,n_d;m)$ is at most $(n-1)/(m-1)$.* □

From Theorems 6.8 and 6.10, we know that the bound found in Corollary 6.12 is attainable in prime power orders. We will now show that the bound in Corollary 6.14 is attainable in prime power orders as well.

Lemma 6.15. *Let m be a prime power and suppose that $a_j \in \mathbb{F}_m$ for $1 \leqslant j \leqslant l$, where $l = \sum_{j=1}^{d} i_j$, and $(a_{l_{k-1}+1},\ldots,a_{l_k}) \neq (0,\ldots,0)$ for $1 \leqslant k \leqslant d$, where $l_k = \sum_{j \leqslant k} i_j$. Also, let $b_j \in \mathbb{F}_m$ for $1 \leqslant j \leqslant l$, where $l = \sum_{j=1}^{d} i_j$, and $(b_{l_{k-1}+1},\ldots,b_{l_k}) \neq (0,\ldots,0)$ for $1 \leqslant k \leqslant d$, where $l_k = \sum_{j \leqslant k} i_j$. Also, for all $1 \leqslant k \leqslant d$, let $(a_{l_{k-1}+1},\ldots,a_{l_k}) \neq u(b_{l_{k-1}+1},\ldots,b_{l_k})$*

where $u \in \mathbb{F}_m^*$. Then the hyperrectangles constructed from $a_1x_1 + \cdots + a_lx_l$ and $b_1x_1 + \cdots + b_lx_l$ are orthogonal of strength 1. □

Theorem 6.16. *Let m be a prime power and n_i be a power of m for each $1 \leqslant i \leqslant d$. Furthermore, for $n = \min\{n_1, \ldots, n_d\}$ we can construct a complete set of $(n-1)/(m-1)$ mutually 1-strong orthogonal frequency hyperrectangles $F(n_1, \ldots, n_d; m)$.* □

Proof. Let $n_i = m^{j_i}$ for each $1 \leqslant i \leqslant d$ and $n = \min\{n_1, \ldots, n_d\}$. Then there are at least $n-1$ non-zero j_i-tuples $(a_1, \ldots, a_{i_j})$ with $a_k \in \mathbb{F}_m$ for each i. Dividing by the $m-1$ non-zero scalar multiples of $\mathbb{F}_m$, we have that there are at least $(n-1)/(m-1)$ j_i-tuples for each coordinate direction which are not scalar multiples. Hence, we can construct $(n-1)/(m-1)$ polynomials which satisfy the conditions of Lemma 6.15. □

As with s-strongly orthogonal hypercubes, if $1 < s < d$, then constructing complete sets of s-strongly orthogonal hyperrectangles becomes much more difficult. Although, we will not provide results at this time, it should be feasible to use the methods similar to those found in Theorems 4.17, 4.19, and 5.7 to generate an equivalent refinement of Corollary 6.13.

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J.T. Ethier
Department of Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO 80217, USA
E-mail: jethier@msudenver.edu

G.L. Mullen
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail: mullen@math.psu.edu