Clifford congruences on perfect semigroups

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Abstract. A congruence ρ on a semigroup S is called *perfect* if $(a\rho)(b\rho) = (ab)\rho$ for all $a, b \in S$, as sets, and a semigroup S is said to be *η-idempotent-surjective* (respectively *perfect*) if every η-class of S contains an idempotent of S, where η is the least semilattice congruence on S (respectively if each congruence on S is perfect). We describe the least Clifford congruence ξ on an η-idempotent-surjective perfect semigroup S. In addition, a characterization of all Clifford congruences on such a semigroup is given. Furthermore, we find necessary and sufficient conditions for ξ to be idempotent pure or E-unitary. Moreover, we give a full description of all USG-congruences on an η-idempotent-surjective perfect semigroup S. In fact, we show that each USG-congruence ϑ on S is the intersection of a semilattice congruence ε and a group congruence υ (and vice versa), and this expression is unique. Also, $S/\vartheta \cong S/\varepsilon \times S/\upsilon$. Finally, we investigate the lattice of Clifford congruences on a semigroup S which is a semilattice S/η of E-inversive semigroups $e\eta$ ($e \in E_S$).

1. Introduction and preliminaries

The concept of a perfect semigroup was introduced by Vagner [40]. Groups are very well-known examples of perfect semigroups. Another examples of such structures are semigroups having exactly two congruences with the property $S = S^2$ (i.e., S is *globally idempotent*; note that perfect semigroups possess this property). Perfect semigroups were studied first by Fortunatov [9, 10] and then by Hamilton and Tamura [27], Hamilton [26], and by Goberstein [24]. In [3] the authors gave an example of a cancellative simple perfect semigroup without idempotents.

Fortunatov in [9] determined the structure of all perfect *orthogroups* (that is, perfect *semilattices of rectangular groups*; cf. [28]), and then in [10] showed that all *completely* (0)-*simple* semigroups are perfect. He also described the structure of *commutative* perfect semigroups, perfect *bands*, as well as perfect *Clifford* semigroups. Later in [27] the authors generalized some of his results to *finite inverse* perfect semigroups and investigated the lattice of congruences in such semigroups. Goberstein [24] generalized simultaneously Theorem 5 [10] (cf. Result 1.9, below) and some of the principle results of [27]. Finally, in [26] Hamilton determined the structure of *completely regular* perfect semigroups and *finite* perfect semigroups.

Quite-known examples of perfect algebras are: *quasigroups*, *Boolean algebras*, as well as *Cantor's algebras*, cf. [11].

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Recall that a semigroup S is *regular* if S coincides with the set of its *regular* elements

$$\mathrm{Reg}(S) = \{s \in S : s \in sSs\}.$$

Recently, it has been proved in [18] that all eventually regular perfect semigroups are necessarily regular (a semigroup S is *eventually regular* if every element of S has a regular power [7]). Finally, in [19] it has been described the structure of perfect group-bound semigroups (a semigroup S is *group-bound* if every element of S has a power which lies in a subgroup of S).

In this paper we study Clifford congruences on η-idempotent-surjective perfect semigroups. Well-known examples of η-idempotent-surjective semigroups are:

(a) *idempotent-surjective semigroups* (i.e., each idempotent congruence class of such semigroups contains an idempotent [7]);

(b) *regular semigroups*;

(c) *eventually regular semigroups* [7];

(d) *group-bound semigroups* (in particular, *periodic* and *finite semigroups*);

(e) *structurally regular semigroups* (for the definition and numerous examples of such semigroups, see [30]).

On the other hand, recall from [3] that there exists a perfect semigroup which is not η-idempotent-surjective.

Before we start our study, we recall some definitions. For convenience of the reader, we present first general properties of perfect semigroups, and then some facts which will be needed in the sequel.

Denote the set of all idempotents of a semigroup S by E_S, that is,

$$E_S = \{e \in S : e^2 = e\}.$$

The relation $\leq$ defined on E_S by

$$e \leqslant f \Leftrightarrow e = ef = fe$$

is a *natural order relation* on E_S (in fact, $\leq$ is indeed an ordering relation on E_S).

According to Thierrin [38], an element a of a semigroup S is E*-inversive* if there exists $x \in S$ such that $ax \in E_S$, and S is said to be E*-inversive* if every element of S is E-inversive. It is well-known that S is E-inversive if and only if the set

$$W_S(a) = \{x \in S : x = xax\}$$

is non-empty for every $a \in S$. Notice that if $x \in W_S(a)$, then $ax, xa \in E_S$.

For some interesting results on E-inversive semigroups, see e.g. [12, 32].

If A is an *ideal* of a semigroup S, i.e., $AS \cup SA \subseteq A$, then the relation

$$\rho_A = (A \times A) \cup 1_S,$$

where 1_S is the identity relation on S, is a congruence on S (the so-called *Rees congruence*). We shall write S/A instead of S/ρ_A. Obviously, $A \in E_{S/A}$.

The set of all *not* E-inversive elements of S, if non-empty, is an ideal of S.

Let S be a semigroup. Denote by S^1 the semigroup obtained from S by adjoining an identity if necessary. Then S^1aS^1 is the least ideal of S containing $a \in S$. Denote it by $J(a)$. Moreover, we say that the elements a, b of S are *$\mathcal{J}$-related* if $J(a) = J(b)$. An equivalence $\mathcal{J}$-class containing a will be denoted by J_a. We can define an order on $S/\mathcal{J}$ by the rule ($a, b \in S$):

$$J_a \leqslant J_b \iff J(a) \subseteq J(b).$$

We say that a semigroup S without zero is *simple* if and only if it has no proper ideals, that is, if and only if $SaS = S$ for every a of S. Further, a semigroup S with zero is called 0-*simple* if S is *not null* (i.e., $S^2 \neq \{0\}$) and S has exactly two ideals. Obviously, S is 0-simple if and only if $S^2 \neq \{0\}$ and $S/\mathcal{J} = \{\{0\}, S \setminus \{0\}\}$.

By a 0-*minimal* ideal of a semigroup S we shall mean an ideal of S which is a minimal element in the set of all non-zero ideals of S.

The following result seems to belong to folklore of semigroup theory.

Lemma 1.1. [28] *Every* 0-*minimal ideal of a semigroup is either null, or it is a* 0-*simple semigroup.* □

Let a be an element of a semigroup S. Suppose first that J_a is minimal among the $\mathcal{J}$-classes of S. Then $J(a) = J_a$ is the least ideal of S. On the other hand, if J_a is not minimal in $S/\mathcal{J}$, then the set

$$I(a) = \{b \in J(a) : J_b \leqslant J_a,\ J_b \neq J_a\}$$

is an ideal of S such that $J(a) = I(a) \cup J_a$ (and this union is disjoint). Also, if B is a proper ideal of $J(a)$ and $I(a) \subseteq B$, then $I(a) = B$. This implies that $J(a)/I(a)$ is a 0-minimal ideal of $S/I(a)$, that is, $J(a)/I(a)$ is either null, or it is a 0-simple semigroup (Lemma 1.1). For convenience, $J(a)/\emptyset = J(a)$. The semigroups $J(a)/I(a)$ $(a \in S)$ are the so-called *principal factors* of S. Remark that we can think of the principle factor $J(a)/I(a)$ as consisting of the $\mathcal{J}$-class $J_a = J(a) \setminus I(a)$ with zero adjoined (if $I(a) \neq \emptyset$). Evidently, $J(a)/I(a)$ is null if and only if the product of any two elements of J_a always falls into a lower $\mathcal{J}$-class. In particular, if J_a is a subsemigroup of S, then the principal factor $J(a)/I(a)$ is not null.

A semigroup is said to be *semisimple* if each of its principal factors is either 0-simple or simple. Recall that a semigroup is semisimple if and only if all its ideals are globally idempotent (see e.g. [4]).

Lemma 1.2. *Any idempotent congruence class of a perfect semigroup S is globally idempotent. In particular, all ideals of S are globally idempotent, i.e., S is semisimple.*

Proof. Let E be an idempotent congruence class of S. Then clearly $E^2 \subseteq E$. Since S is perfect, then $E^2 = E$. □

Recall that a commutative semigroup in which all elements are idempotents is *semilattice*. Evidently, the least semilattice congruence η on an arbitrary semigroup S exists (note that $\mathcal{J} \subseteq \eta$). This relation induces the greatest semilattice decomposition of S, say $[Y; S_\alpha]$ $(\alpha \in Y)$, where $Y \cong S/\eta$, each S_α is an η-class and $S = \bigcup\{S_\alpha : \alpha \in Y\}$. To indicate this fact we shall always write $S = [Y; S_\alpha]$ $(\alpha \in Y)$ or just $S = [Y; S_\alpha]$. Notice that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, where $\alpha\beta$ is the product of α and β in the semilattice Y.

We say that a semigroup S is *intra-regular* if every element a of S is $\mathcal{J}$-related to a^2 [4]. It is easily seen that if S is intra-regular, then the relation $\mathcal{J}$ is a semilattice congruence on S, so we have the following well-known result.

Lemma 1.3. [4] *A semigroup S is intra-regular if and only if $\eta = \mathcal{J}$, where every $\mathcal{J}$-class is a simple semigroup.* □

We recall some known results on perfect semigroups in general. For beginning, from the First and the Second Isomorphism Theorem we get the following result.

Lemma 1.4. [10] *Every homomorphic image of a perfect semigroup is a perfect semigroup.* □

An ideal A of a semigroup S is called *completely prime* if $ab \in A$ implies that $a \in A$ or $b \in A$.

From the definition of the Rees congruence follows the following result [10].

Lemma 1.5. *Every non-zero ideal of a perfect semigroup is completely prime.* □

It is not difficult to see that every chain is perfect. Also, if the elements a, b of a semilattice A are incomparable, then the congruence induced by the ideal aA is not perfect. Thus we have the following result.

Lemma 1.6. [10] *A semilattice is perfect if and only if it is a chain.* □

Let $S = [Y; S_\alpha]$. Assume that S is perfect. In the light of Lemmas 1.4 and 1.6, Y is a chain. Moreover, the following results is a little more general than some statements of [10].

Lemma 1.7. *Let $S = [Y; S_\alpha]$ be a perfect semigroup. Then Y is a chain and the following statements hold*:

(*a*) *if S does not have a zero, then each S_α is simple and $Y \cong S/\mathcal{J}$*;

(*b*) *if S contains a zero 0, then Y has a least element 0_Y, S_α is a simple semigroup for $\alpha \neq 0_Y$, and either $S_{0_Y} = \{0\}$ (then $Y \cong S/\mathcal{J}$) or S_{0_Y} is a 0-simple semigroup whose zero is not adjoined (and $J_a = a\eta \setminus \{0\}$ if $a \neq 0$).*

Proof. (*a*). Suppose first that S has no a zero element. Since $a^2 \in S^1a^2S^1$, then $a \in S^1a^2S^1$ (Lemma 1.5), so S is intra-regular. Thus every S_α is a simple semigroup and $Y \cong S/\mathcal{J}$ (Lemma 1.3).

(*b*). Let now S contains a zero 0, say $0 \in S_{0_Y}$. Since $S_{0_Y} S_\alpha \subseteq S_{0_Y}$ for all $\alpha \in Y$, then $S_{0_Y} S_\alpha = S_{0_Y}$ for all $\alpha \in Y$ (since S is perfect). This implies that Y has least element 0_Y.

Since Y is a chain and every S_α is a semigroup, then the condition $a^2 = 0$ implies that $a \in S_{0_Y}$. Thus S_α is a simple semigroup for all $\alpha \neq 0_Y$.

If $S_{0_Y} \neq \{0\}$, then $S_{0_Y}^2 = S_{0_Y} \neq \{0\}$ (Lemma 1.2), since it is clear that S_{0_Y} is an ideal of S, that is, S_{0_Y} is *not null*. Suppose that $A \subseteq S_{0_Y}$ is a non-zero ideal of S. Then A is completely prime (by Lemma 1.5). It follows that A is a non-zero completely prime ideal of S_{0_Y}. Hence the partition $\{A, S_{0_Y} \setminus A\}$ of S_{0_Y} induces a semilattice congruence on S_{0_Y}. On the other hand, it is well-known that every η-class of S has no semilattice congruences except the universal relation (cf. [37]). In particular, S_{0_Y} possesses this property. It follows that $A = S_{0_Y}$, i.e., S_{0_Y} is a 0-minimal ideal of S. Finally, observe that if 0 is adjoined to S_{0_Y}, then the partition

$$\{S_\alpha\,(\alpha \neq 0_Y), S_{0_Y} \setminus \{0\}, \{0\}\}$$

of S induces a semilattice congruence on S which is properly contained in the least semilattice congruence η, a contradiction, so S_{0_Y} is a 0-minimal ideal of S whose zero is not adjoined. Consequently, S_{0_Y} is a 0-simple semigroup whose zero is not adjoined (Lemma 1.1). Clearly, $J_a = a\eta \setminus \{0\}$ if $a \neq 0$. □

The following result will be very crucial in our further studies.

Proposition 1.8. *Let $S = [Y; S_\alpha]$ be an η-idempotent-surjective perfect semigroup. Then each semigroup S_α is E-inversive, therefore, S is E-inversive.*

Proof. It is sufficient to give a proof in the case when S has a zero element. Clearly, S_{0_Y} is E-inversive. Consider now a semigroup S_α, where $\alpha \neq 0_Y$. By assumption S_α contains some idempotent e of S. Since S_α is simple (Lemma 1.7), then the set A of all elements of S_α which are *not* E-inversive must be empty (otherwise, $A \neq S_\alpha$ is an ideal of S_α, since $e \notin A$, a contradiction), i.e., S_α is E-inversive. □

Let $\mathcal{C}$ be a class of semigroups (call its elements *$\mathcal{C}$-semigroups*). Recall that a semigroup is a *semilattice of $\mathcal{C}$-semigroups* if there exists a semilattice congruence ρ on S (that is, S/ρ is a semilattice) such that each ρ-class of S is a $\mathcal{C}$-semigroup. In particular, if every ρ-class of S is a group, then S is a *semilattice of groups*.

Let Y be a semilattice and $\mathcal{F} = \{G_\alpha : \alpha \in Y\}$ be a family of disjoint groups, indexed by the set Y. Suppose also that for each pair $(\alpha, \beta) \in Y \times Y$ such that $\alpha \geqslant \beta$ there is an associated homomorphism $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ such that

(*a*) $\phi_{\alpha,\alpha}$ is the identical automorphism of G_α for every $\alpha \in Y$, and

(*b*) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geqslant \beta \geqslant \gamma$.

Put $S = \bigcup\{G_\alpha : \alpha \in Y\}$, and define a binary operation $\cdot$ on S by the rule that if $a_\alpha \in G_\alpha$ and $a_\beta \in G_\beta$, then

$$a_\alpha \cdot a_\beta = (a_\alpha \phi_{\alpha,\alpha\beta})(a_\beta \phi_{\beta,\alpha\beta}),$$

where the multiplication on the right side takes place in the group $G_{\alpha\beta}$.

It is a matter of routine to check that $(S, \cdot)$ is a semigroup. Finally, in the light of the condition (*a*), the new multiplication coincides with the given of each G_α, so S is certainly a semilattice Y of groups G_α. We usually denote the product in S also by juxtaposition and write $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$.

We call the semigroup $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$ a *strong semilattice of groups.* In the case when Y is a chain and each homomorphism $\phi_{\alpha,\beta}$ is surjective, we say that S is *chain-surjective.*

By a *Clifford* semigroup we mean a regular semigroup in which the idempotents are central. It is well-known that a semigroup is a Clifford semigroup if and only if it is a (strong) semilattice of groups [28]. In fact, S is a Clifford semigroup if and only if $S = [E_S; H_e; \phi_{e,f}]$, where H_e is a maximal subgroup of S having the identity e ($e \in E_S$) and for all $e, f \in E_S$ such that $e \geqslant f$, the homomorphism $\phi_{e,f} : H_e \to H_f$ is given by $a\phi_{e,f} = af$ for every $a \in H_e$.

The following result is due to Fortunatov [10].

Result 1.9. *A Clifford semigroup $S = [E_S; H_e; \phi_{e,f}]$ is perfect if and only if it is chain-surjective.* □

An equivalence relation ρ on a semigroup S is called *idempotent pure* if $e\rho \subseteq E_S$ for all $e \in E_S$. Recall from [28] that in an arbitrary semigroup S the relation

$$\tau = \{(a,b) \in S \times S : (\forall x, y \in S^1)\ xay \in E_S \Leftrightarrow xby \in E_S\}$$

is the largest idempotent pure congruence on S.

Recall from [28] that a semigroup S with $E_S \neq \emptyset$ is *left E-unitary* if for all $a \in S$ and $e \in E_S$, the condition $ea \in E_S$ implies $a \in E_S$. The notion of a *right E-unitary* semigroup is defined dually. Finally, S is *E-unitary* if it is both left and right unitary. In [14] it has been shown that an E-inversive semigroup is E-unitary if and only if it is left (right) unitary.

Moreover, some preliminaries about group congruences on a semigroup S are needed. A subset A of S is called (respectively) *full*; *reflexive* and *dense* if $E_S \subseteq A$; $(\forall a, b \in S)(ab \in A \Rightarrow ba \in A)$ and $(\forall s \in S)(\exists\, x, y \in S)\ sx, ys \in A$. Also, we define the *closure operator* ω on S by $A\omega = \{s \in S : (\exists\, a \in A)\ as \in A\}$ (where $A \subseteq S$). We shall say that $A \subseteq S$ is *closed* (in S) if $A\omega = A$. Further, a subsemigroup N of a semigroup S is said to be *normal* if it is full, dense, reflexive and closed (if N is normal, then we shall write $N \lhd S$). Finally, if a subsemigroup of S is dense and reflexive, then it is called *quasi-normal.*

By the *kernel* $\ker(\rho)$ of a congruence ρ on a semigroup S we shall mean the set $\{x \in S : (x, x^2) \in \rho\}$. Also, S is an *E-semigroup* if $E_S E_S \subseteq E_S$.

Result 1.10. [14] *Let B be a quasi-normal subsemigroup of a semigroup S. Then the relation*

$$\rho_B = \{(a,b) \in S \times S : (\exists\ x, y \in B)\, ax = yb\}$$

is a group congruence on S. Furthermore, $B \subseteq B\omega = \ker(\rho_B)$, and if $B \lhd S$, then $B = \ker(\rho_B)$.

Conversely, if ρ is a group congruence on S, then there is a normal subsemigroup N of S such that $\rho = \rho_N$ (in fact, $N = \ker(\rho)$).

Moreover, the least group congruence on an E-inversive E-semigroup S is given by

$$\sigma = \{(a,b) \in S \times S : (\exists\, e, f \in E_S)\ ea = bf\}.$$

□

Remark 1.11. [14] Let B be a quasi-normal subsemigroup of S. Then:

$$(a,b) \in \rho_B \Leftrightarrow (\exists\, x \in S)\ xa, xb \in B.$$

It is easily seen that if S is an E-inversive semigroup (and so E_S is dense), then there exists the least normal subsemigroup of S. In the light of Result 1.10, every E-inversive semigroup possesses the least group congruence σ.

Note that if ρ is a group congruence on an E-inversive semigroup S, then $a\,\rho\,b$ if and only if $ab^* \in \ker(\rho)$ for some (all) $b^* \in W_S(b)$.

Result 1.12. [14] *The following conditions concerning an E-inversive semigroup S are equivalent*:

(a) *S is E-unitary*;

(b) $\tau = \sigma$;

(c) $\ker(\sigma) = E_S$.

In particular, every E-unitary E-inversive semigroup is an E-semigroup. □

The next result will be very useful (for the definition of Green's relations and undefined terms the reader is referred to the books [28, 36]).

Theorem 1.13. *Let S be an E-unitary perfect Clifford semigroup. Then*:

(a) $\eta \cap \sigma = 1_S$;

(b) $\eta\sigma = \sigma\eta = S \times S = \eta \vee \sigma$.

Consequently, $S \cong (S/\eta \times S/\sigma)$, where $S/\eta \cong E_S$ and $S/\sigma \cong H_e$ for all $e \in E_S$.

Conversely, let S be the direct product of a chain E and a group G. Then S is an E-unitary perfect Clifford semigroup, $E_S \cong E$ and $H_e \cong G$ for every $e \in E_S$.

Proof. Remark first that S is intra-regular, therefore, $\mathcal{J}$ is the least semilattice congruence on S. On the other hand, it is well-known that $\mathcal{H} = \eta$ in an arbitrary Clifford semigroup. It follows that $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$ in S.

(a). This follows from Proposition III.7.2 [36] (p. 152).

(b). By Proposition VII.5.22 [36] (p. 343), $\eta\sigma = S \times S$, so $(\eta\sigma)^{-1} = \sigma\eta = S \times S$. Thus $\eta\sigma = \sigma\eta = S \times S = \eta \vee \sigma$.

Consequently, $S \cong (S/\eta \times S/\sigma)$. It is evident that $S/\eta \cong E_S$.

Finally, it is well-known that a Clifford semigroup $[E_S; H_e; \phi_{e,f}]$ is E-unitary if and only if each $\phi_{e,f}$ is injective. This implies that $H_e \cong H_f$ for all $e, f \in E_S$, since E_S is chain (see also Result 1.9). Let $e \in E_S$. Then the restriction of a natural morphism $\sigma^\natural : S \to S/\sigma$ to H_e is an isomorphism of H_e onto S/σ. Indeed, denote this restriction by φ. Clearly, φ is a homomorphism. Also, if $a\sigma = b\sigma$ $(a, b \in H_e)$, then $(ab^{-1}, bb^{-1}) \in \sigma$. Hence $ab^{-1} \in E_S \cap H_e$, since $\sigma = \tau$ (Result 1.12), so $a = b$. Thus φ is an injective homomorphism. Furthermore, take any $a \in S$, say $a \in H_f$ (where $f \in E_S$). If $e \geqslant f$, then there is $b \in H_e$ such that $b\phi_{e,f} = a$, that is, $a = bf$. Consequently, $b \in a\sigma \cap H_e$. On the other hand, if $e < f$, then $ae \in a\sigma \cap H_e$, so φ is the required isomorphism between H_e and S/σ.

It is a matter of routine to verify the converse of the theorem. □

Note that in Proposition VII.5.22 [36] the condition *"σ is perfect"* implies the condition *"$\mathcal{L}\sigma = S \times S$"*, therefore, we get the following corollary.

Corollary 1.14. *An E-unitary Clifford semigroup S is perfect if and only if the least group congruence σ on S is perfect.* □

Recall from [25] that any full quasi-normal subsemigroup of a semigroup S is called *seminormal*. Clearly, an arbitrary E-inversive semigroup contains the least seminormal subsemigroup, say B.

Finally, we have need the following two results. The first of them is clear.

Lemma 1.15. *Let T be a seminormal subsemigroup of a semigroup S which is a semilattice of E-inversive semigroups S_α $(\alpha \in A)$. Then $T \cap S_\alpha$ is a seminormal subsemigroup of S_α $(\alpha \in A)$.* □

Lemma 1.16. *If B is the least seminormal subsemigroup of an E-inversive semigroup S and φ is an epimorphism of S onto a Clifford semigroup T, then $B\phi \subseteq E_T$.*

Proof. Put $A = (E_T)\varphi^{-1}$. Clearly, A is a full subsemigroup of S. Thus A is dense. Moreover, if $xy \in A$, then $E_T \ni (xy)\varphi = x\varphi \cdot y\varphi = y\varphi \cdot x\varphi = (yx)\varphi$ (since E_T is reflexive), so $yx \in A$. Hence $B \subseteq A$. Thus $B\varphi \subseteq ((E_T)\varphi^{-1})\varphi \subseteq E_T$. □

2. Clifford congruences

Let X be a semilattice and let $a, b \in X$ be such that $a \leqslant b$. Then the sets $\{a\}$ (if $a = b$), (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ are called the *intervals* of X. Recall that if ρ is a semilattice congruence on a semigroup $S = [Y, S_\alpha]$, where Y is a chain, then a typical element A of S/ρ is of the form $\bigcup\{S_\alpha : \alpha \in Z\}$, where Z is a non-empty interval of Y. In particular, S/ρ is a chain.

Suppose that S is an η-idempotent-surjective perfect semigroup.

Let ε be a semilattice congruence on S. Denote the ε-classes of S by S_α, where α's are elements of some set Z, and define on Z a binary operation $\circ$, as follows: if $a \in S_\alpha, b \in S_\beta$, then
$$\alpha \circ \beta = \gamma \Leftrightarrow ab \in S_\gamma.$$
Clearly, $(Z, \circ)$ is a semilattice (isomorphic to S/ϵ), so $S = \bigcup\{S_\alpha : \alpha \in Z\}$ is a semilattice Z of E-inversive S_α (Proposition 1.8). For any seminormal subsemigroup A of S, put $A_\alpha = A \cap S_\alpha$ $(\alpha \in Z)$. Then by Lemma 1.15 and Remark 1.11, for every α, the relation
$$\rho_{A_\alpha} = \{(a, b) \in S_\alpha \times S_\alpha : (\exists\, x \in S_\alpha)\ xa, xb \in A_\alpha\}$$
is a group congruence on S_α. Put $\rho = \bigcup\{\rho_{A_\alpha} : \alpha \in Z\}$. In a similar way as in [13], we can show that ρ is a congruence on S. Moreover, $a\rho = a\rho_{A_\alpha}$ if $a \in S_\alpha$. Put $G_\alpha = S_\alpha/\rho_{A_\alpha}$. Then $S/\rho = \bigcup\{G_\alpha : \alpha \in Z\}$ is a semilattice Z of groups G_α.

Applying the above construction (of ρ) to the least semilattice congruence η on S and to the least seminormal subsemigroup B of S, we obtain some semilattice of groups congruence on S, say ξ.

Let S be an η-idempotent-surjective perfect E-semigroup. Then each η-class of S is an E-semigroup. Define on every S_α the least group congruence σ_α (see Result 1.10). Then the relation ξ^*, induced by this partition of S, is a congruence on S (for the proof, see [13]).

Using Lemma 1.16, one can show (in a very similar way as in Section 2 of [13]) the following result (denote by $B_{a\eta}$ the intersection of $a\eta$ and B ($a \in S$)).

Theorem 2.1. *The least Clifford congruence on an η-idempotent-surjective perfect semigroup S is given by*

$$\xi = \{(a,b) \in \eta : (\exists\, x, y \in B_{a\eta})\ xa = by\}.$$

The least Clifford congruence on an η-idempotent-surjective perfect E-semigroup S is given by

$$\xi^* = \{(a,b) \in \eta : (\exists\, e, f \in E_{a\eta})\ ea = bf\}. \qquad \square$$

Remark 2.2. In the light of Remark 1.11,

$$\xi = \{(a,b) \in \eta : (\exists\, x \in a\eta)\ xa, xb \in B_{a\eta}\}.$$

Theorem 2.3. *Let ε be an arbitrary semilattice congruence on an η-idempotent-surjective perfect semigroup S and let A be a seminormal subsemigroup of S. Then the relation*

$$\rho_{A,\varepsilon} = \{(a,b) \in \varepsilon : (\exists\, x, y \in a\varepsilon \cap A)\ xa = by\}$$

is a Clifford congruence on S.

Conversely, if ρ is a Clifford congruence on S, then there exists a semilattice congruence ε on S and a seminormal subsemigroup A of S such that $\rho = \rho_{A,\varepsilon}$.

Proof. The proof is closely similar to the proof of Theorem 2.4 [13]. $\square$

Using Result 1.12 and Remark 2.2, one can prove in a similar way as in [13] the following (if (b) below holds, then S is an E-semigroup by Remark 3(b) in [19]).

Theorem 2.4. *The following conditions concerning an η-idempotent-surjective perfect semigroup S are equivalent*:

(a) *ξ is idempotent pure*;

(b) *each η-class of S is an E-unitary E-inversive subsemigroup of S*;

(c) $\xi = \eta \cap \tau$. $\square$

A semigroup S is called *strongly E-reflexive* if for all $a, b \in S, e \in E_{S^1}$, the condition $eab \in E_S$ implies $eba \in E_S$ [31].

Corollary 2.5. *Let S be an η-idempotent-surjective perfect semigroup. Then ξ is idempotent pure if and only if S is a semilattice of E-unitary E-inversive semigroups.*

Moreover, if it is the case, then S is a strongly E-reflexive E-semigroup and

$$\xi = \{(a,b) \in \eta : E(a) = E(b)\},$$

where $E(s) = \{x \in S : sx, xs \in E_S\}$ ($s \in S$).

Proof. The first part is clear. We show that S is strongly E-reflexive. Let $eab \in E_S$, where $a, b \in S$ and $e \in E_{S^1}$. Then $(eab)\xi$ is an idempotent of the Clifford semigroup S/ξ. Since S/ξ is strongly E-reflexive, then $(eba)\xi \in E_{S/\xi}$. On the other hand, ξ is idempotent pure. Thus $eba \in E_S$, as required. Hence the relation

$$\chi = \{(a, b) \in S \times S : E(a) = E(b)\}$$

is a congruence on S (Proposition 3.1 [16]). Moreover, S is an E-semigroup. Indeed, if e, f are idempotents of the same η-class of S, then clearly $ef \in E_S$. Otherwise, $e < f$ or $f < e$ (cf. Remark 3(b) in [19]). Hence $ef \in E_S$, as exactly required.

Finally, suppose that $E(a) = E(e)$ ($a \in S$ and $e \in E_S$). Then $f \in E(e) = E(a)$ for some $f \in E_{a\eta}$ (where $a\eta$ is E-unitary). Thus $fa \in E_{a\eta}$, so $a \in E_{a\eta} \subseteq E_S$, i.e., χ is idempotent pure. Consequently, $\tau = \chi$ (since $\tau \subseteq \chi$ by Proposition 3.3 [16]). This implies the thesis of the corollary (cf. Theorem 2.4(c)). □

Moreover, we have the following theorem.

Theorem 2.6. *Let S be an idempotent-surjective perfect semigroup. The following conditions are equivalent*:

(a) *S is E-unitary*;
(b) *ξ is an idempotent pure E-unitary congruence on S*;
(c) *for every $a \in S$, $a\eta$ is E-unitary and $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. The proof is closely similar to the proof of Theorem 2.8 [13]. □

The next result gives some equivalent conditions for ξ to be E-unitary, when ξ is idempotent pure.

Corollary 2.7. *If an idempotent-surjective perfect semigroup S is a semilattice of an E-unitary E-inversive semigroups, then the following conditions are equivalent*:

(a) *S is E-unitary*;
(b) *$\xi = \eta \cap \sigma$*;
(c) *ξ is E-unitary*;
(d) *for every $a \in S$, $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. The proof is closely similar to the proof of Corollary 2.9 [13]. □

Finally, we have the following corollary.

Corollary 2.8. *In an E-unitary η-idempotent-surjective perfect semigroup S,*

$$\xi \cap \mathcal{H} = 1_S.$$

If in addition E_S forms a semilattice, then

$$\xi \cap \mathcal{L} = \xi \cap \mathcal{R} = 1_S.$$

Proof. This follows from Theorem 5.5 [14], since $\xi \subseteq \sigma$. □

3. USG-congruences

A semigroup S is said to be a USG-*semigroup* if it is an E-unitary Clifford semigroup. Remark that if S is a USG-semigroup, then $\sigma \cap \eta = 1_S$ [29].

Theorem 3.1. *In any η-idempotent-surjective perfect semigroup S, $\sigma \cap \eta = 1_S$ if and only if S is a* USG-*semigroup.*

Proof. This follows easily from Theorem 2.1 and the above remark (see the proof of Theorem 3.1 [13]). □

The following result describes all USG-congruences on η-idempotent-surjective perfect semigroups.

Theorem 3.2. *The intersection ϑ of a semilattice congruence ε and a group congruence υ on an arbitrary η-idempotent-surjective perfect semigroup S is a* USG-*congruence. Conversely, any* USG-*congruence on S can be expressed uniquely in this way. Moreover,*

$$S/\vartheta \cong S/\varepsilon \times S/\upsilon.$$

Proof. Let ρ be any congruence on S. Using the Second Isomorphism Theorem, one can prove without difficulty that S/ρ is an η-idempotent-surjective semigroup, therefore, the class of η-idempotent-surjective perfect semigroups is closed under taking homomorphic images (by the First Isomorphism Theorem and Lemma 1.4).

All assertions of the theorem except a uniqueness follows from Theorem 3.1 and Theorem 3.2 [13]. The proof of the uniqueness is very similar to the corresponding proof of Theorem 3.4 [13].

We show the last part of the theorem. If S has zero, then the thesis is obvious. Notice that every idempotent ϑ-class of S contains an idempotent. Suppose next that S has no zero and observe that $(a, b \in S)$

$$(a\vartheta, b\vartheta) \in \eta = \mathcal{H} \Leftrightarrow (aa^*, bb^*) \in \vartheta \Leftrightarrow (aa^*, bb^*) \in \varepsilon \Leftrightarrow (a, b) \in \varepsilon,$$

where $c^* \in W(c) \cap c\eta$ for $c \in \{a, b\}$, therefore we get $(S/\vartheta)/\eta \cong S/\varepsilon$. Moreover (below $b^* \in W(b) \cap b\eta$),

$$(a\vartheta, b\vartheta) \in \sigma \Leftrightarrow ab^* \in \ker(\vartheta) \Leftrightarrow ab^* \in \ker(\upsilon) \Leftrightarrow (a, b) \in \upsilon,$$

since S/ϑ is E-unitary (and so $\sigma = \tau$ in S/ϑ). Hence $(S/\vartheta)/\sigma \cong S/\upsilon$. In the light of Theorem 1.13, $S/\vartheta \cong S/\varepsilon \times S/\upsilon$. □

Corollary 3.3. *The relation $\sigma \cap \eta$ is the least* USG-*congruence on an arbitrary η-idempotent-surjective perfect semigroup S.* □

Corollary 3.4. *An arbitrary η-idempotent-surjective perfect semigroup S is a subdirect product of a group and a semilattice if and only if it is a* USG-*semigroup, that is, if and only if $S \cong E_S \times S/\sigma$, where E_S is a chain* (*see Theorem* 1.13).

Proof. The proof is very similar to the proof of Corollary 3.6 [13]. $\square$

One can show without difficulty that on an arbitrary (E-inversive) semigroup S the least *E-unitary* congruence π (that is, S/π is E-unitary) exists.

Theorem 3.5. *In an η-idempotent-surjective perfect semigroup S,*

$$\sigma \cap \eta = \xi \vee \pi.$$

Proof. The proof is closely similar to the proof of Theorem 3.9 [13]. $\square$

Corollary 3.6. *In an E-unitary η-idempotent-surjective perfect semigroup S,*

$$\xi = \sigma \cap \eta$$

and

$$S/\xi \cong S/\eta \times S/\sigma.$$

4. The condition $\pi \cap \xi = 1_S$

In this section we characterize those idempotent-surjective perfect semigroups S which are a subdirect product of an E-unitary semigroup and a Clifford semigroup, that is, those semigroups S for which $\pi \cap \xi = 1_S$. Since E-unitary semigroups and Clifford semigroups are both E-semigroups, then S are E-semigroups, too.

In [7] Edwards defined the relation μ on a semigroup S by

$$(a, b) \in \mu \iff \begin{cases} (x \mathcal{L} ax \text{ or } x \mathcal{L} bx) \Longrightarrow ax \mathcal{H} bx, \\ (x \mathcal{R} xa \text{ or } x \mathcal{R} xb) \Longrightarrow xa \mathcal{H} xb, \end{cases}$$

where x is an arbitrary element of $\mathrm{Reg}(S)$. He proved in [8] that μ is the maximum *idempotent-separating* (that is, $\mu \cap (E_S \times E_S) = 1_S$) congruence on an arbitrary idempotent-surjective semigroup S.

Recall that a semigroup S is:

- *fundamental* if $\mu = 1_S$ [6];
- *η-simple* if $\eta = S \times S$ [37].

Proposition 4.1. *If S is an η-idempotent-surjective perfect semigroup such that $\pi \cap \xi = 1_S$, then S is a semilattice of (η-simple) E-unitary E-inversive semigroups.*

Proof. The proof is very similar to the proof of Proposition 4.2 [13]. $\square$

Theorem 4.2. *Let S be a fundamental η-idempotent-surjective perfect semigroup. Then $\pi \cap \xi = 1_S$ if and only if S is E-unitary.*

Proof. Let $\pi \cap \xi = 1_S$; $e, f \in E_S$. If $(e, f) \in \pi$, then $(e, f) \in \eta$. Hence $(e, f) \in \xi$. Thus $e = f$, so $\pi \subseteq \mu = 1_S$. Consequently, S is E-unitary.

The converse implication is trivial. $\square$

Remark 4.3. The above theorem is valid for any $\mathcal{C}$-congruence ρ instead of π (that is, $S/\rho \in \mathcal{C}$, where $\mathcal{C}$ is some fixed class of $\mathcal{C}$-semigroups) contained in η (i.e., if we replace in the theorem π by ρ, then we must replace "E-unitary" with "$\mathcal{C}$-semigroup").

Recall from [17] that (for idempotent-surjective semigroups) every congruence of the interval $[\pi, \sigma]$ is E-unitary. Also, $\ker(\rho) = \ker(\pi)$ for every $\rho \in [\pi, \sigma]$.

Recall that in regular semigroups S, $\mu \cap \tau = 1_S$. The next theorem gives necessary and sufficient conditions for $\pi \cap \xi$ to be the identity relation on an idempotent-surjective perfect semigroup S such that $\mu \cap \tau = 1_S$.

Theorem 4.4. *Let S be an idempotent-surjective perfect semigroup, $\mu \cap \tau = 1_S$. Then the following conditions are equivalent*:

(a) $\pi \cap \xi = 1_S$;

(b) *S is a semilattice of E-unitary E-inversive semigroups and $\pi \subseteq \mu$*;

(c) *S is a semilattice of E-unitary E-inversive semigroups and $\pi \subseteq \mu \cap \sigma \subseteq \sigma$*;

(d) *S is a semilattice of E-unitary E-inversive semigroups and the congruence $\mu \cap \sigma$ is E-unitary*;

(e) *S is a semilattice of E-unitary E-inversive semigroups and at least one idempotent-separating congruence on S is E-unitary*;

(f) *S is a subdirect product of an E-unitary idempotent-surjective semigroup and a Clifford semigroup*;

(g) *S is a semilattice of E-unitary E-inversive semigroups and the relation $\mathcal{H} \cap \sigma$ is E-unitary congruence on S.*

Proof. The proof is closely similar to the proof of Theorem 4.6 [13]. □

5. Lattice of Clifford congruences

Throughout the entire section S denotes an arbitrary semigroup which is a semilattice S/η of E-inversive semigroups $e\eta$ ($e \in E_S$).

Firstly, we shall indicate which of the above results are valid for S.

Result 5.1. *Let ε be a semilattice congruence on S and let A be a seminormal subsemigroup of S. Then the relation*

$$\rho_{A,\varepsilon} = \{(a, b) \in \varepsilon : (\exists\, x, y \in a\varepsilon \cap A)\ xa = by\}$$

is a Clifford congruence on S. □

Conversely, if ρ is a Clifford congruence on S, then there exists a semilattice congruence ε on S and a seminormal subsemigroup A of S such that $\rho = \rho_{A,\varepsilon}$.

Result 5.2. *If S is also an E-semigroup, then the least Clifford congruence on S is given by*

$$\xi = \{(a, b) \in \eta : (\exists\, e, f \in E_{a\eta})\ ea = bf\}. \quad \square$$

Result 5.3. *The following conditions concerning the least Clifford congruence ξ on S are equivalent*:

(a) ξ *is idempotent pure*;

(b) S *is an E-semigroup and every η-class of S is E-unitary*;

(c) $\xi = \eta \cap \tau$. □

Proof. Suppose that (a) holds and take any $e, f \in E_S$. Then clearly $(ef)\xi \in E_{S/\xi}$. Hence $ef \in E_S$. Thus S is an E-semigroup. The rest of the proof is closely similar to the corresponding proof of Theorem 2.6 in [13]. □

Result 5.4. *The following conditions concerning S are equivalent*:

(a) S *is E-unitary*;

(b) ξ *is an idempotent pure E-unitary congruence on S*;

(c) *for every $a \in S$, $a\eta$ is E-unitary and $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. (a) $\Rightarrow$ (b). Indeed, S is an E-semigroup by [14]. As every η-class of S is an E-unitary E-inversive semigroup, ξ is idempotent pure. Further, if $(a\xi)(e\xi) \in E_{S/\xi}$ ($a \in S, e \in E_S$), then $ae \in E_S$ (since ξ is idempotent pure). Hence $a \in E_S$ by (a). Thus $a\xi \in E_{S/\xi}$, so S/ξ is E-unitary, that is, ξ is E-unitary.

(b) $\Rightarrow$ (a). Let $ae \in E_S$, where $a \in S$ and $e \in E_S$. Then $(a\xi)(e\xi) \in E_{S/\xi}$, so $a\xi \in E_{S/\xi}$ (since ξ is E-unitary). Hence $a \in E_S$ (because ξ is idempotent pure). Thus S is E-unitary.

(a) $\Leftrightarrow$ (c). The proof is similar to the corresponding proof of Theorem 2.8 in the paper [13]. □

Result 5.5. *If every η-class of S is E-unitary, then the following conditions are equivalent*:

(a) S *is E-unitary*;

(b) $\xi = \eta \cap \sigma$;

(c) ξ *is E-unitary*;

(d) *for every $a \in S$, $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. (a) $\Rightarrow$ (b). Firstly, S is an E-semigroup by (a). Hence ξ is idempotent pure (Result 5.3). Thus $\xi = \eta \cap \tau$ (Result 5.3). Further, as S is E-unitary, $\sigma = \tau$ [14]. Consequently, $\xi = \eta \cap \sigma$.

(b) $\Rightarrow$ (c). It follows from the fact that η and σ are both E-unitary.

(c) $\Rightarrow$ (a) & (a) $\Leftrightarrow$ (d). This follows from Result 5.4. □

Result 5.6. *If S is E-unitary, then*

$$\xi \cap \mathcal{H} = 1_S.$$

If in addition E_S forms a semilattice, then

$$\xi \cap \mathcal{L} = \xi \cap \mathcal{R} = 1_S. \quad \square$$

Result 5.7. *If S is a fundamental and $\pi \cap \xi = 1_S$, then S is E-unitary.* □

Result 5.8. *S is a* USG*-semigroup if and only if $\sigma \cap \eta = 1_S$.* □

Let ρ be a congruence on S. Using the Second Isomorphism Theorem, one can prove that $T \cong S/\rho$ is a semilattice T/η of E-inversive semigroups $e\eta$ $(e \in E_T)$.

Result 5.9. *The intersection ϑ of any semilattice congruence ε and any group congruence υ on S is a* USG*-congruence on S. Conversely, each* USG*-congruence on S can be expressed uniquely in this way.*

In particular, $\eta \cap \sigma$ is the least USG*-congruence on S.* □

Result 5.10. *S is a subdirect product of a group and a semilattice if and only if S is a* USG*-semigroup.* □

Result 5.11. *The following equality holds in the lattice of congruences on S:*

$$\sigma \cap \eta = \xi \vee \pi.$$

In particular, if S is E-unitary, then $\xi = \sigma \cap \eta$. □

Secondly, we investigate the lattice of Clifford congruences on S. Remark that the interval $[\xi, S \times S]$ consists of all Clifford congruences on S (since the class of Clifford semigroups is closed under taking homomorphic images), so it is a complete sublattice of $\mathcal{C}(S)$ (the lattice of congruences on S). Denote it by $\mathcal{CC}(S)$. Moreover, the lattice of all semilattice congruences on S is denoted by $\mathcal{SC}(S)$. Clearly, $\mathcal{SC}(S) = [\eta, S \times S]$ is a complete sublattice of $\mathcal{CC}(S)$.

The following concepts will be useful. A congruence ρ on a semigroup A is called *idempotent-surjective* (resp. *regular-surjective*) if each idempotent (resp. regular) ρ-class of A contains some idempotent of A [7] (resp. regular element of A [15]).

Proposition 5.12. *Each Clifford congruence ρ on S is regular-surjective and every idempotent ρ-class of S is an E-inversive subsemigroup of S (in particular, ρ is idempotent-surjective).*

Proof. Indeed, every ρ-class of S is an equivalence class of some group congruence on a certain E-inversive semigroup, therefore, it contains a regular element of S. This also implies that any idempotent ρ-class of S is an E-inversive semigroup (for details, see [21]). □

By the *trace* $\mathrm{tr}(\rho)$ of a congruence ρ on an arbitrary semigroup A we shall mean the restriction of ρ to the set E_A, and by the *kernel* $\ker(\rho)$, the set $\{a \in A : a\,\rho\,a^2\}$. Observe that if ρ is a Clifford congruence on S, then $\ker(\rho) = \bigcup_{e \in E_S} e\rho$.

Using Proposition 5.12, we can show in a similar way, as in [15], the following theorems.

Theorem 5.13. *The following conditions concerning Clifford congruences ρ_1, ρ_2 on S are equivalent:*

(a) $e\rho_1 \subseteq e\rho_2$ *for every* $e \in E_S$;

(b) $\rho_1 \subseteq \rho_2$.

Thus $\rho_1 = \rho_2$ if and only if $e\rho_1 = e\rho_2$ for every $e \in E_S$. □

Theorem 5.14. *The following conditions concerning Clifford congruences* ρ_1, ρ_2 *on* S *are equivalent*:

(*a*) $\ker(\rho_1) \subseteq \ker(\rho_2)$ & $\operatorname{tr}(\rho_1) \subseteq \operatorname{tr}(\rho_2)$;

(*b*) $\rho_1 \subseteq \rho_2$.

Thus $\rho_1 = \rho_2$ *if and only if* $\ker(\rho_1) = \ker(\rho_2)$ *and* $\operatorname{tr}(\rho_1) = \operatorname{tr}(\rho_2)$. □

Define an equivalence relation on $\mathcal{CC}(S)$, as follows:

$$\theta = \{(\rho_1, \rho_2) \in \mathcal{CC}(S) \times \mathcal{CC}(S) : \operatorname{tr}(\rho_1) = \operatorname{tr}(\rho_2)\}.$$

Take further $\rho \in \mathcal{CC}(S)$ and note that $\mathcal{H}^{S/\rho} = \eta_{S/\rho}$. Also, the relation

$$\rho^\theta = \{(a, b) \in S \times S : a\rho\, \mathcal{H}^{S/\rho}\, b\rho\}$$

is a semilattice congruence on S. Clearly, $\rho \subseteq \rho^\theta$ and $\operatorname{tr}(\rho) = \operatorname{tr}(\rho^\theta)$. Moreover, if $\varrho \in \mathcal{CC}(S)$ with $\operatorname{tr}(\varrho) = \operatorname{tr}(\rho)$, then $\operatorname{tr}(\varrho^\theta) = \operatorname{tr}(\rho^\theta)$, so $\varrho^\theta = \rho^\theta$ (Theorem 5.14). We have just proved that ρ^θ is a greatest element in $\rho\theta$. Further, $\rho\theta$ has a least element $\rho_\theta = \bigcap\{\alpha \in \mathcal{CC}(S) : \operatorname{tr}(\rho) \subseteq \operatorname{tr}(\alpha)\}$, therefore, we get $\rho\theta = [\rho_\theta, \rho^\theta]$ for all $\rho \in \mathcal{CC}(S)$. Finally, note that if $\rho, \varrho \in \mathcal{CC}(S)$ with $\rho \subseteq \varrho$, then clearly $\rho_\theta \subseteq \varrho_\theta$ and $\rho^\theta \subseteq \varrho^\theta$ (again by Theorem 5.14).

From the above consideration follows that θ is a complete congruence on $\mathcal{CC}(S)$ (see Lemma 4.13 [35]). As every θ-class of $\mathcal{CC}(S)$ contains exactly one semilattice congruence on S, $\mathcal{CC}(S)/\theta \cong \mathcal{SC}(S)$.

We shall summarize the above consideration in the following theorem.

Theorem 5.15. *Let* S *be a semilattice* S/η *of* E*-inversive semigroups* $e\eta$ $(e \in E_S)$. *Put*

$$\theta = \{(\rho_1, \rho_2) \in \mathcal{CC}(S) \times \mathcal{CC}(S) : \operatorname{tr}(\rho_1) = \operatorname{tr}(\rho_2)\}.$$

Then the following statements hold:

(*a*) θ *is a complete congruence on* $\mathcal{CC}(S)$;

(*b*) *for every* $\rho \in \mathcal{CC}(S)$, $\rho\theta = [\rho_\theta, \rho^\theta]$ *is a complete sublattice of* $\mathcal{CC}(S)$, *where*

$$\rho_\theta = \bigcap\{\alpha \in \mathcal{CC}(S) : \operatorname{tr}(\rho) \subseteq \operatorname{tr}(\alpha)\}, \quad \rho^\theta = \{(a, b) \in S \times S : a\rho\, \mathcal{H}^{S/\rho}\, b\rho\}.$$

Moreover, ρ^θ *is a unique semilattice congruence in* $\rho\theta$.

(*c*) $\mathcal{CC}(S)/\theta \cong \mathcal{SC}(S)$. □

Some background material on *biordered sets* will be useful. For a definition of a biordered set, its related axioms and concepts see [33, 6]. Let S be a semigroup with $E = E_S \neq \emptyset$. Define

$$\omega^l = \{(e, f) \in E \times E :\ ef = e\}, \qquad \omega^r = \{(e, f) \in E \times E :\ fe = e\}$$

$$\leqslant = \omega^l \cap \omega^r, \qquad L = \omega^l \cap (\omega^l)^{-1}, \qquad R = \omega^r \cap (\omega^r)^{-1}$$

$$D_E = \{(e, f) \in E \times E : ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

Then the partial algebra E with domain D_E is a *biordered set* (Theorem 1.1 [33]). Observe that $\leqslant$ is a *natural partial order* on E; and if $e, f \in E$, then $(e, f) \in L$ (R) if and only if $(e, f) \in \mathcal{L}$ $(\mathcal{R})$ (in a semigroup S). Further, the relations ω^l and ω^r are quasi-orders on E. For $\rho = \omega^l$ or $\rho = \omega^r$ and any $e \in E$, we put

$$\rho(e) = \{g \in E : (g, e) \in \rho\}.$$

Moreover, for any two elements e, f of a biordered set E, we define the *M-set* $M(e, f)$ of e and f by (cf. [33])

$$M(e, f) = \omega^l(e) \cap \omega^r(f) = \{g \in E : g = ge = fg\}.$$

If $M(e, f) \neq \emptyset$ for all $e, f \in E$, then we say that E is an *M-biordered set*.

For an arbitrary semigroup S, the M-set $M(e, f)$ $(e, f \in E_S)$ will be at times denoted by $M_S(e, f)$ if absolutely necessary.

A major result in the theory of biordered sets is that every biordered set is the biordered set of a semigroup [6]. Also, each M-biordered set is the biordered set of some E-inversive semigroup [12].

In 1996 Auinger and Hall [2] introduced the concept of a *congruence* for *biordered sets*. Recall that if E is a *partial groupoid* (that is, the multiplication is defined for some non-empty subset D_E of the Cartesian product $E \times E$), then an equivalence relation ρ on E is called a *congruence* if and only if the following condition is satisfied:

(C1) $\forall e, f, g, h \in E$

$$\big((e, f), (g, h) \in \rho \ \&\ eg, fh \in D_E\big) \Longrightarrow (eg, fh) \in \rho.$$

In that case, the product $(e\rho)(f\rho)$ is defined in S/ρ if and only if $gh \in D_E$ for some $g \in e\rho, h \in f\rho$, and then $(e\rho)(f\rho) = (gh)\rho$ (by $(C1)$, this partial binary operation on E/ρ is well-defined).

By a *congruence on a biordered set* E [2] we shall mean an equivalence relation ρ on E satisfying, at the same time, the above condition $(C1)$ and the following three conditions:

(C2) $\big((\forall e, f \in E)\, ef = e\big)\, \big(\forall g \in e\rho\big)\, \big((\exists\, h \in f\rho)\, gh = g\big)$;

(C3) $\big((\forall e, f \in E)\, fe = f\big)\, \big(\forall g \in e\rho\big)\, \big((\exists\, h \in f\rho)\, hg = h\big)$;

(C4) $\big(\forall (e, f) \in \rho\big)\big(\exists\, g \in e\rho\big)\, g \in M(e, f)$.

Result 5.16. [2] *For every congruence ρ on a biordered set E, the quotient partial groupoid E/ρ is a biordered set.* □

Theorem 5.17. *If ρ is a Clifford congruence on S, then* $\mathrm{tr}(\rho)$ *is a congruence on the biordered set E_S.*

Proof. Clearly, $\mathrm{tr}(\rho)$ satisfies (C1). By Proposition 5.12, $\mathrm{tr}(\rho)$ fulfills (C4).

(C2), (C3). Let $e, f \in E_S$ with $ef = f$ and let $g \in E_{e\rho}$. Then $gf \,\rho\, ef \,\rho\, f$. Take any $a \in W_{f\rho}((gf)^2)$ and put $h = gfagf$. Then

$$h^2 = gf(a(gf)^2a)gf = gfagf = h \in E_S,$$

$$gh = h = gfagf \,\rho\, efaef = faf \,\rho\, f,$$

that is, $h \in E_{f\rho}$ and $gh = h$. By duality, (C3) is also satisfied. □

Remark 5.18. $E_S/\mathrm{tr}(\rho)$ is the biordered set of the Clifford semigroup S/ρ. Hence $E_S/\mathrm{tr}(\rho)$ is a *regular* biordered set [33]. Also, the quasi-orders ω^l and ω^r coincide in $E_S/\mathrm{tr}(\rho)$, so $E_S/\mathrm{tr}(\rho)$ is a *semilattice* biordered set, cf. Remark 1.3 in [33].

Corollary 5.19. *The following statements concerning a Clifford congruence ρ on S are valid:*

(*a*) *for all $e, f \in E_S$ such that $ef = f$ and every $g \in e\rho$ there exists $h \in f\rho$ such that $gh = h$;*

(*b*) *for all $e, f \in E_S$ such that $fe = f$ and every $g \in e\rho$ there exists $h \in f\rho$ such that $hg = h$;*

(*c*) *for all $e, f \in E_S$ such that $e \,\rho\, f$ there is $g \in M_{e\rho}(e, f)$.* □

Recall that $H_a \leqslant H_b$ if and only if $aS^1 \subseteq bS^1$ and $S^1a \subseteq S^1b$.

We know that any idempotent congruence class of a Clifford congruence on S contains an idempotent. Also, the following analogous of the famous Lallement's Lemma is valid.

Proposition 5.20. *Let ρ be a Clifford congruence on S, $a \in S$. If $a\rho \in E_{S/\rho}$, then there is $e \in E_{a\rho}$ such that $H_e \leqslant H_a$.*

Proof. Take any $x \in W_{a\rho}(a^2)$ and put $e = axa$. Then evidently $e \,\rho\, a$ and $H_e \leqslant H_a$. Also, $e^2 = (axa)(axa) = a(xa^2x)a = axa = e$, as required. □

The following three results follows directly from Theorem 5.17 and from [2], see Lemmas 3.3, 3.4 and 4.1(i) in [2].

Result 5.21. *Let ρ be a Clifford congruence on S, $e, f \in E_S$. If $(e\rho)(f\rho) = f\rho$ $\big((f\rho)(e\rho) = f\rho\big)$, then for every $g \in E_{e\rho}$ there exists $h \in E_{f\rho}$ such that $gh = h$ $(hg = h)$.* □

Result 5.22. *Let ρ be a Clifford congruence on S. If $g\rho \in M_{S/\rho}(e\rho, f\rho)$, where $e, f, g \in E_S$, then for all $h \in E_{e\rho}$, $i \in E_{f\rho}$ there exists $j \in E_{g\rho} \cap M_S(h, i)$.* □

Result 5.23. *Let ρ be a Clifford congruence on S, $e, f \in E_S$. If $(e\rho, f\rho) \in \mathcal{L}$ $(\mathcal{R})$, then $e \geqslant g \,\mathcal{L}\, h = fh$ $(e \geqslant g \,\mathcal{R}\, h = hf)$ for some $g \in E_{e\rho}$ and $h \in E_{f\rho}$.* □

Corollary 5.24. *Let ρ be a Clifford congruence on S, $e, f \in E_S$. If $f\rho \leqslant e\rho$, then for every $g \in E_{e\rho}$ there exists $h \in E_{f\rho}$ such that $h \leqslant g$.*

Proof. Let $f\rho \leqslant e\rho$, $g \in E_{e\rho}$. Then $f\rho = (e\rho)(f\rho) = (f\rho)(e\rho)$. By Result 5.21, there exist $h_1, h_2 \in E_{f\rho}$ such that $gh_1 = h_1$ and $h_2g = h_2$. Fix $h \in M_{f\rho}(h_2, h_1)$ (Corollary 5.19(c)) and observe that

$$gh = g(h_1h) = (gh_1)h = h_1h = h, \quad hg = (hh_2)g = h(h_2g) = hh_2 = h.$$

Thus $h \in E_{f\rho}$ and $h \leqslant g$. □

Let A be a semigroup and $b \in A$. The set $V_A(b) = \{a \in A : a = aba \;\&\; b = bab\}$ is called the set of all *inverses* of b in the semigroup A.

Theorem 5.25. *If ρ is a Clifford congruence on S, $x\rho \in V_{S/\rho}(y\rho)$, then there exist $a \in x\rho, b \in y\rho$ such that $a = xbx, b = ycy$ $(c \in S)$ and $a \in V_S(b)$. Also, if $x \in E_S$, then the element a can be chosen to be an idempotent, and if $x, y, xy \in E_S$, then both elements a and b can be chosen to be idempotents.*

Proof. Let $x\rho \in V_{S/\rho}(y\rho)$. Then the class $(xy)\rho$ is an idempotent of S/ρ. Take any $z \in W_{(xy)\rho}\big((xy)^2\big)$ and put $a = x(yzxy)x$ and $b = yzxy$. Then

$$aba = (xyzxyx)yzxy(xyzxyx) = (xy)\big(z(xy)^2(z(xy)^2z)\big)xyx = xyzxyx = a,$$

$$bab = (yzxy)xyzxyx(yzxy) = y\big((z(xy)^2z)(xy)^2z\big)xy = yzxy = b,$$

so $a \in V_S(b)$. Furthermore,

$$b\rho = (y(zxy))\rho = (yxy)\rho = y\rho, \qquad a\rho = (xbx)\rho = (xyx)\rho = x\rho.$$

Suppose now that $x \in E_S$. Then

$$a^2 = xyzxy(xx)yzxyx = xy(z(xy)^2z)xyx = xyzxyx = a.$$

Finally, if $x, y, xy \in E_S$, then $z \in W_{(xy)\rho}\big((xy)^2\big) = W_{(xy)\rho}(xy)$. Hence

$$b^2 = yzx(yy)zxy = y(zxyz)xy = yzxy = b,$$

as required. □

Remark 5.26. Notice that in the above theorem $H_a \leqslant H_x$ and $H_b \leqslant H_y$.

Theorem 5.27. *If ρ is a Clifford congruence on S and $x\rho \in W_{S/\rho}(y\rho)$, then there is $z \in x\rho$ such that $z \in W_S(y)$ and $H_z \leqslant H_x$.*

Proof. Let $x\rho \in W_{S/\rho}(y\rho)$ and $a \in W_{(yx)\rho}\big((yx)^3\big)$. Put $z = xyxayx$. Then

$$zyz = (xyxayx)y(xyxayx) = xyx(a(yx)^3a)yx = xyxayx = z,$$

$$z\rho = (x(yxayx))\rho = (xyx)\rho = x\rho,$$

that is, $z \in x\rho \cap W_S(y)$. Clearly, $H_z \leqslant H_x$. □

Denote the complete lattice of all equivalence relations on the set X by $\mathcal{E}(X)$. In the light of Proposition 5.12, using Corollary 5.19(c) and Theorem 5.27 instead of (respectively) Proposition 2.4(c) [20] and Theorem 2.11 [20], we can prove in a similar way, as in Section 4 of [20], the following theorem.

Theorem 5.28. *If S is a semilattice S/η of E-inversive semigroups $e\eta$ $(e \in E_S)$, then the map $\bar{\theta} : \mathcal{CC}(S) \to \mathcal{E}(E_S)$, where $\rho\bar{\theta} = \mathrm{tr}(\rho)$ for every $\rho \in \mathcal{CC}(S)$, is a complete lattice homomorphism* (*between the complete lattices $\mathcal{CC}(S)$ and $\mathcal{E}(E_S)$*) *which induces the complete congruence θ* (*cf. Theorem* 5.15). □

Finally, we have the following remark.

Remark 5.29. In [22] the author introduced the concept of a *fruitful semigroup* (a semigroup A is defined to be *fruitful* if each idempotent congruence class of A is an E-inversive subsemigroup of A). Note that the class of fruitful semigroups is very large. For example, all *structurally eventually regular semigroups* [30], *compact semigroups* are fruitful. The papers [22, 23] contain a great number of interesting results. In particular, in [23] it has been shown that in any fruitful semigroup A, the map $\bar{\theta} : \mathcal{C}(A) \to \mathcal{E}(E_A)$, where

$$\rho\bar{\theta} = \mathrm{tr}(\rho)$$

for every $\rho \in \mathcal{C}(A)$, is a complete lattice homomorphism (between the complete lattices $\mathcal{C}(A)$ and $\mathcal{E}(E_A)$) which induces the complete congruence θ. This result has been proved for certain classes of regular semigroups and for group-bound semigroups by Pastijn and Petrich in [34]. Moreover, Pastijn and Petrich asked whether the above result is valid for regular semigroups in general. In 1986 Trotter [39] solved a famous problem of Pastijn and Petrich. Finally, in 1996 Auinger and Hall proved the above result for a special class of eventually regular semigroups, and they asked whether the result is true for *all* eventually regular semigroups, see [1]. As all eventually regular semigroups are structurally eventually regular (by the definition), their problem has been solved in [23].

References

[1] **K. Auinger and T. E. Hall**, *Representations of semigroups by transformations and the congruence lattice of an eventually regular semigroup*, Int. J. Algebra Comput. **6** (1996), 655–685.

[2] **K. Auinger and T. E. Hall**, *Concepts of congruence, morphic image and substructure for biordered sets*, Commun. Algebra **24** (1996), 3933–3968.

[3] **S. Bulman-Fleming and K. McDowell**, *A non-regular perfect semigroup*, Semigroup Forum **35** (1987), 119–121.

[4] **A. H. Clifford and G. B. Preston**, *The algebraic theory of semigroups*, Vol. I, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I. (1961).

[5] **D. Easdown**, *Biordered sets come from semigroups*, J. Algebra **96** (1985), 581–591.

[6] **D. Easdown**, *Biorder-preserving coextensions of fundamental semigroups*, Proc. Edinburgh Math. Soc. **31** (1988), 463–467.

[7] **P. M. Edwards**, *Eventually regular semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23–38.

[8] **P. M. Edwards**, *Maximizing a congruence with respect to its partition of idempotents*, Semigroup Forum **39** (1985), 313–317.

[9] **V. A. Fortunatov**, *Perfect semigroups decomposable in a semi-lattice of rectangular groups*, (Russian), Studies in Algebra **2** (1970), 67–78 (Saratov Univ. Press).

[10] **V. A. Fortunatov**, *Perfect semigroups*, (Russian), Izv. Vysš. Učebn. Zaved. Matematika **3** (1972), 80–90.

[11] **V. A. Fortunatov**, *Varieties of perfect algebras*, (Russian), Studies in Algebra **4** (1974), 110–114 (Saratov Univ. Press).

[12] **R. S. Gigoń**, *Some results on E-inversive semigroups*, Quasigroups Relat. Syst. **20** (2012), 53–60.

[13] **R. S. Gigoń**, *Clifford congruences on an idempotent-surjective R-semigroup*, Quasigroups Relat. Syst. **20** (2012), 219–232.

[14] **R. S. Gigoń**, *Congruences and group congruences on a semigroup*, Semigroup Forum **86** (2013), 431–450.

[15] **R. S. Gigoń**, *Regular congruences on an idempotent-regular-surjective semigroup*, Bull. Austral. Math. Soc. **88** (2013), 190–196.

[16] **R. S. Gigoń**, *E-disjunctive semigroups and idempotent pure congruences*, Quasigroups Relat. Syst. **21** (2013), 201–210.

[17] **R. S. Gigoń**, *On the lattice of congruences on idempotent-regular-surjective semigroups*, Commun. Algebra **42** (2014), 306–324.

[18] **R. S. Gigoń**, *Eventually regular perfect semigroups* (submitted to Filomat).

[19] **R. S. Gigoń**, *The structure of perfect group-bound semigroups* (submitted to Commun. Algebra).

[20] **R. S. Gigoń**, *Matrix congruences on E-inversive semigroups* (submitted to J. Austral. Math. Soc.).

[21] **R. S. Gigoń**, *Completely simple congruences on E-inversive semigroups* (submitted to J. Algebra Appl.).

[22] **R. S. Gigoń**, *The concept of a fruitful congruence and a fruitful semigroup* (submitted to Semigroup Forum).

[23] **R. S. Gigoń**, *Representations of semigroups by transformations and the congruence lattice of a fruitful semigroup* (in preparation).

[24] **F. S. Goberstein**, *Perfect completely semisimple inverse semigroups*, Semigroup Forum **45** (1992), 395–397.

[25] **G. Gomes**, *A characterization of the group congruences on a semigroup*, Semigroup Forum **46** (1993), 48–53.

[26] **H. Hamilton**, *Perfect completely regular semigroups*, Math. Nachr. **123** (1984), 169–176.

[27] **H. Hamilton and T. Tamura**, *Finite inverse perfect semigroups and their congruences*, J. Austral. Math. Soc. **32A** (1982), 114–128.

[28] **J. M. Howie**, *Fundamentals of semigroup theory*, Oxford University Press, New York (1995).

[29] **J. M. Howie and G. Lallement**, *Certain fundamental congruences on a regular semigroup*, Proc. Glasgow Math. Soc. **7** (1966), 145–159.

[30] **S. J. L. Kopamu**, *The concept of structural regularity*, Portugaliae Math. **53** (1996), 435–456.

[31] **D. R. LaTorre**, *The least semilattice of groups congruence on a regular semigroup*, Semigroup Forum **27** (1983), 319–329.

[32] **H. Mitsch and M. Petrich**, *Restricting idempotents in E-inversive semigroups*, Acta Sci. Math. (Szeged) **67** (2001), 555–570.

[33] **K. S. S. Nambooripad**, *Structure of regular semigroups. I*, Memoirs Amer. Math. Soc. **22** (1979).

[34] **F. Pastijn and M. Petrich**, *Congruences on regular semigroups*, Trans. Amer. Math. Soc. **295** (1986), 607–633.

[35] **F. Pastijn and M. Petrich**, *The congruence lattice of a regular semigroup*, J. Pure Appl. Algebra **53** (1988), 93–123.

[36] **M. Petrich**, *Inverse semigroups*, Wiley, New York (1984).

[37] **T. Tamura**, *Note of the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261.

[38] **G. Thierrin**, *Demi-groupes inversés et rectangulaires*, Bull. Cl. Sci. Acad. Roy. Belgique **41** (1955), 83–92.

[39] **P. G. Trotter**, *On a problem of Pastijn and Petrich*, Semigroup Forum **34** (1986), 249–252.

[40] **V. V. Vagner**, *Algebraic topics of the general theory of partial connections in fiber bundles*, (Russian), Izv. Vysš. Učebn. Zaved. Matematika **11** (1968), 26–32.

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