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On multiplicative conjugate loops

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Abstract. The objective of this paper is twofold. Firstly to define MC-loops and show that every conjugate of subloops of such loops also are subloops Secondly to investigate various properties of MC-loops and its relation with numerous other already existing loops, moreover number of examples and counter examples are provided to make these relations more clearer.

1. Introduction

A loop L is an *inverse property loop* [2] if every $x \in L$ has a unique two-sided inverse, denoted by x^{-1}, and if, for all $x, y \in L$ the loop satisfies

$$x^{-1}(xy) = y = (yx)x^{-1}.$$

A loop L is said to be a *conjugate loop* [1] if it satisfies the following identity $x\left(yx^{-1}\right) = (xy)\,x^{-1}$, for all $x, y \in L$. A loop is *IP-conjugate* [1] if it satisfies inverse property and conjugate property. Smallest non-associative IP-conjugate loop is of order 7.

Following [1], flexible C-loops are conjugate IP-loops. Every diassociative loop is a conjugate IP-loop. Conjugate IP-loop L is commutative iff every element in L is self conjugate.

An IP-conjugate loop L is called a *multiplicative conjugate loop* (MC-loop) iff for all $x, y, g \in L$, we have

$$(xy)^g = x^g y^g.$$

Proposition 1.1. *An IP-conjugate loop L is MC-loop iff $T_g(xy) = T_g(x)T_g(y)$ for $T_g \in INN(L)$.*

Proof. Indeed,

$$\begin{aligned}
(xy)^g = x^g y^g &\Leftrightarrow g^{-1}(xy)g = (g^{-1}.xg)(g^{-1}.yg)\\
&\Leftrightarrow (xy)R_gL_{g^{-1}} = (x)R_gL_{g^{-1}}.(y)R_gL_{g^{-1}}\\
&\Leftrightarrow (xy)R_gL_g^{-1} = (x)R_gL_g^{-1}.(y)R_gL_g^{-1} \quad \text{because } L \text{ is an } IP\text{-loop.}\\
&\Leftrightarrow (xy)T_g = (x)T_g.(y)T_g
\end{aligned}$$

□

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2. Counting of multiplicative conjugate loops

In [8] J. Slaney and A. Ali enumerated IP-loops up to order 13 by using finite domain enumerator FINDER. Using that enumeration and our following GAP code we have counted multiplicative conjugate loops.

```
function(L):=IsMCLoop
local x, y, z;
if not IsConjugateIPLoop(L) then return false;
for x in L do
for y in L do
for z in L do
if z^-1*(x*y)*z <> (z^-1*x*z)*(z^-1*y*z) then return false;
fi;
od;od;od;
return true;
end;
```

Size	IP	*Conjugate* IP	MC
7	2	1	1
8	8	0	0
9	7	0	0
10	47	7	6
11	49	3	3
12	2684	27	17
13	10600	16	10

Number of IP, conjugate IP and MC-loops of order $n = 7, \ldots, 13$.

Example 2.1. The smallest non-associative MC-loop has the form.

.	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	6	7	5	4
3	3	1	2	7	6	4	5
4	4	7	6	5	1	2	3
5	5	6	7	1	4	3	2
6	6	4	5	3	2	7	1
7	7	5	4	2	3	1	6

3. Properties of MC-loops

We start with the following obvious lemma.

Lemma 3.1. *In an MC-loop L every $T \in INN(L)$ is pseudo-automorphism with companion* 1. □

Theorem 3.2. *The nucleus of an MC-loop L is a normal subloop.*

Proof. As L is MC-loop so L is also an IP-loop. Moreover let $T : L \to L$ be pseudo-automorphism as described in Lemma 3.1. The restriction of a pseudo-automorphism T from Lemma 3.1 T to the nucleus N of L is an automorphism of N. Hence $aN = Na$ for all $a \in L$ and $N(xy) = (Nx)y$, $(xy)N = x(yN)$ from the definition of a nucleus. □

Theorem 3.3. *A homomorphic image of an MC-loop is an MC-loop.*

Proof. Obvious. □

Proposition 3.4. *If L is an MC-loop, then $[x^y, z^y] = [x, z]^y$ for all $x, y, z \in L$.*

Proof. Indeed,

$$\begin{aligned}[x^y, z^y] &= (x^y)^{-1}(z^y)^{-1} \cdot x^y z^y = (x^{-1})^y (z^{-1})^y \cdot x^y z^y \\ &= (x^{-1}z^{-1})^y \cdot (x^y z^y) = (x^{-1}z^{-1} \cdot xz)^y = [x, z]^y.\end{aligned}$$ □

Theorem 3.5. *Let L be an MC-loop, then $[L, L] = \langle [x, y]; x, y \in L\rangle$ is a weak normal subloop of L.*

Proof. In fact, we have $[L, L]^l = [L^l, L^l] = [L, L]$ for every $l \in L$. □

Theorem 3.6. *If L is an MC-loop and $H \leqslant L$, then $H^x = \{x^{-1}hx : \ \forall h \in H\}$ is a subloop of L.*

Proof. For $x \in L$ and $a, b \in H^x$, there exists $h_1, h_2 \in H$ such that $a = x^{-1}h_1x$ and $b = x^{-1}h_2x$. Thus, $ab = (x^{-1}h_1x)(x^{-1}h_2x) = h_1^x h_2^x = (h_1h_2)^x \in H^x$. Analogously, $a^{-1} = (x^{-1}hx)^{-1} = x^{-1}h^{-1}x = (h^{-1})^x \in H^x$. Thus, $H^x \leqslant L$. □

Theorem 3.7. *In an MC-loop the conjugate of a maximal subloop is also maximal.*

Proof. Let M be a maximal subloop of an MC-loop L. Then M^g is its conjugate subloop. If there is a subloop H such that $M^g \leqslant H \leqslant L$, then $M \leqslant H^{g^{-1}} \leqslant L^{g^{-1}}$. Hence, $M \leqslant H^{g^{-1}} \leqslant L$ which is a contradiction. So, M is maximal. □

Recall that an intersection of all maximal subloops is again a subloop. It is known as the *Frattini subloop*. For a loop L, the Frattini subloop is denoted by $\Phi(L)$.

Theorem 3.8. *If L is an MC-loop, then $\Phi(L)$ is a weak normal in L.*

Proof. Let $\{M_i : i \in I\}$ be the family of all maximal subloops of L and $\Phi(L) = \cap_{i\in I} M_i$. Then $x \in \Phi(L)$ implies $x^g \in \Phi(L)$ for all $g \in L$. Hence, $\Phi(L)$ is weakly normal in L. □

The subloop generated by all the nilpotent normal subloops of L is called the *Fitting subloop* of L and is denoted by $Fit(L)$. Below we prove that in MC-loops it is normal.

Lemma 3.9. *If M and N be normal subloops of an MC-loop L, then the product $MN = \{mn : m \in M, n \in N\}$ is also a normal subloop of L.*

Proof. Let L be an MC-loop and M, N be its two normal subloops. Then for any $m \in M$, $n \in N$ and $l \in L$ we have $(mn)^l = m^l n^l \in MN$. Moreover,

$$(mn \cdot y)z = (m(n_1 y))z = m_1(n_1 y \cdot z) = m_1(n_2 \cdot yz) = m_2 n_2(yz).$$

Similarly, we can prove that $(yz)(MN) = y(z(MN)$. Hence, MN is normal. □

Remark 3.10. It can be shown by induction that the product of a finite family of normal subloops of any MC-loop is its normal subloop.

Theorem 3.11. *If L be an MC-loop, then $Fit(L)$ is normal in L.*

Proof. Let $Fit(L) = \langle N_1, N_2, N_3, \ldots, N_m \rangle$, where all $N_1, N_2, \ldots, N_m$ are nilpotent normal subloops of L. Since, all subloops are normal therefore we can express $Fit(L)$ alternatively as, $Fit(L) = N_1 N_2 \cdots N_m$. This completes the proof. □

Theorem 3.12. *In an MC-loop the centralizer of any its non-empty subset is a subloop.*

Proof. The centralizer of X has the form $C_L(X) = \{a \in L : ax = xa \ , \forall\ x \in X\}$.
Let $a, b \in C_L(X)$ and $x \in X$,then

$$(ab)x = x(x^{-1}(ab.x)) = x(ab)^x = x(a^x b^x) = x(ab),$$

which implies $ab \in C_L(X)$. Now, for $b \in C_L(X)$ we have $bx = xb$. Thus, $b^{-1}xb = x$. Hence, $x = b(b^{-1}xb)b^{-1} = bxb^{-1}$, i.e., $b^{-1}x = xb^{-1}$. So, $b^{-1} \in C_L(X)$. □

Corollary 3.13. *The commutant $C(L)$ of an MC-loop L is its subloop.* □

Corollary 3.14. *Let L_1, L_2 be a subloop of a MC-loop L. If $L = L_1 \times L_2$, then $C(L) = C(L_1) \times C(L_2)$.* □

The following fact is obvious.

Proposition 3.15. *For an MC-loop L the map $\delta_x : L \to L$ defined by $(a)\delta_x = x^{-1}ax$ is its automorphism.* □

4. Relation of MC-loops with other loops

In this section we describe connections of MC-loops with other types of loops.
The following fact is well known but we give a short proof of this fact.

Theorem 4.1. *Every commutative IP-loop L is an MC-loop.*

Proof. Let L be an arbitrary commutative IP-loop. Then for all $x, x^{-1}, y \in L$ we have $x^{-1} \cdot yx = x^{-1} \cdot xy = x^{-1}x \cdot y = y$. On the other hand, $x^{-1}y \cdot x = yx^{-1} \cdot x = y \cdot x^{-1}x = y$. Hence, we get $x^{-1} \cdot yx = x^{-1}y \cdot x$. So, L is an IP-conjugate loop.

Moreover, $x^g y^g = (g^{-1} \cdot xg)(g^{-1} \cdot yg) = (g^{-1} \cdot gx)(g^{-1} \cdot gy) = (g^{-1}g \cdot x)(g^{-1}g \cdot y) = xy$ and $(xy)^g = g^{-1}.(xy)g = g^{-1}.g(xy) = (g^{-1}g)(xy) = xy$. So, $(xy)^g = x^g y^g$.

Hence, L is an MC-loop. □

Corollary 4.2. *Every Steiner loop, every commutative C-loop and every commutative Moufang loop are MC-loops but the converse is not true.* □

Example 4.3. The following loop

.	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	7	12	11	10	9
3	3	6	5	2	1	4	9	10	11	12	7	8
4	4	5	6	1	2	3	10	9	8	7	12	11
5	5	4	1	6	3	2	11	12	7	8	9	10
6	6	3	2	5	4	1	12	11	10	9	8	7
7	7	8	11	10	9	12	1	2	5	4	3	6
8	8	7	12	9	10	11	2	1	4	5	6	3
9	9	12	7	8	11	10	3	4	1	6	5	2
10	10	11	8	7	12	9	4	3	6	1	2	5
11	11	10	9	12	7	8	5	6	3	2	1	4
12	12	9	10	11	8	7	6	5	2	3	4	1

is a noncommutative Moufang loop which is not an MC-loop since $(xy)^g = x^g y^g$ is not true for $x = 2$, $y = 3$ and $g = 7$. □

Example 4.4. This is a non-commutative C-loop which is not an MC-loop.

.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	5	6	3	4	8	7	10	9	16	14	15	12	13	11
3	3	8	1	7	6	5	4	2	11	13	9	15	10	16	12	14
4	4	6	7	1	8	2	3	5	12	14	15	9	16	10	11	13
5	5	7	2	8	4	3	6	1	13	11	14	16	12	15	10	9
6	6	4	8	2	7	1	5	3	14	12	13	10	11	9	16	15
7	7	5	4	3	2	8	1	6	15	16	12	11	14	13	9	10
8	8	3	6	5	1	7	2	4	16	15	10	13	9	11	14	12
9	9	10	11	12	16	14	15	13	1	2	3	4	8	6	7	5
10	10	9	13	14	15	12	16	11	2	1	8	6	3	4	5	7
11	11	16	9	15	10	13	12	14	3	5	1	7	6	8	4	2
12	12	14	15	9	13	10	11	16	4	6	7	1	5	2	3	8
13	13	15	10	16	9	11	14	12	5	3	6	8	1	7	2	4
14	14	12	16	10	11	9	13	15	6	4	5	2	7	1	8	3
15	15	13	12	11	14	16	9	10	7	8	4	3	2	5	1	6
16	16	11	14	13	12	15	10	9	8	7	2	5	4	3	6	1

It is not an MC-loop because $(2.3)^9 \neq 2^9 3^9$. □

Example 4.5. Consider the following commutative loop.

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	1	4	3	6	5	8	7	10	9
3	3	4	1	2	7	9	5	10	6	8
4	4	3	2	1	10	8	9	6	7	5
5	5	6	7	10	1	2	3	9	8	4
6	6	5	9	8	2	1	10	4	3	7
7	7	8	5	9	3	10	1	2	4	6
8	8	7	10	6	9	4	2	1	5	3
9	9	10	6	7	8	3	4	5	1	2
10	10	9	8	5	4	7	6	3	2	1

It is a commutative MC-loop but not C-loop. □

Since in MC-loops the inverses are unique, we will use unique inverses instead of right or left inverses.

Theorem 4.6. *An MC-loop is a group iff it is conjugacy closed loop (CC loop).*

Proof. If L is a CC-loop, then

$$x(yz) = (x \cdot yz)(x^{-1}x) = ((x \cdot yz)x^{-1})x = (yz)^{x^{-1}} \cdot x = (y^{x^{-1}} \cdot z^{x^{-1}})x$$
$$= (y^{x^{-1}} \cdot x)(x^{-1}(z^{x^{-1}} \cdot x)) = (xyx^{-1} \cdot x)(x^{-1}(xzx^{-1} \cdot x)) = (xy)z.$$

Hence, L is a group. The converse statement is obvious. □

Corollary 4.7. *An MC-loop is a group iff it is an extra loop.*

Proof. Since every extra loop is a conjugacy closed loop so the corollary follows from the last theorem. □

Theorem 4.8. *Every MC-loop is three power associative.*

Proof. Every MC-loop is conjugate IP-loop. Every conjugate IP loop is flexible. Flexible loops are always three power associative. Hence, MC-loop is three power associative. □

Example 4.9. This loop

·	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

is three power associative but it is not an MC-loop. □

Example 4.10. Consider the following multiplicative conjugate loop.

$\cdot$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	7	11	12	9	10
3	3	4	1	2	9	11	10	12	5	7	6	8
4	4	3	2	1	11	9	12	10	6	8	5	7
5	5	6	10	12	1	2	9	11	7	3	8	4
6	6	5	12	10	2	1	11	9	8	4	7	3
7	7	8	9	11	10	12	1	2	3	5	4	6
8	8	7	11	9	12	10	2	1	4	6	3	5
9	9	11	7	8	3	4	5	6	12	1	10	2
10	10	12	5	6	7	8	3	4	1	11	2	9
11	11	9	8	7	4	3	6	5	10	2	12	1
12	12	10	6	5	8	7	4	3	2	9	1	11

It is neither diassociative nor alternative loop. □

The above example shows that "Moufang theorem" is not always applicable in MC-loops. Indeed, in the above loop

$$11(6.12) = (11.6)12.$$

But the subloop $< 11, 6, 12 >$ is a loop which is not associative. From this, we can conclude that in MC-loops three elements associate with each other generata a subloop which is not a group, in general.

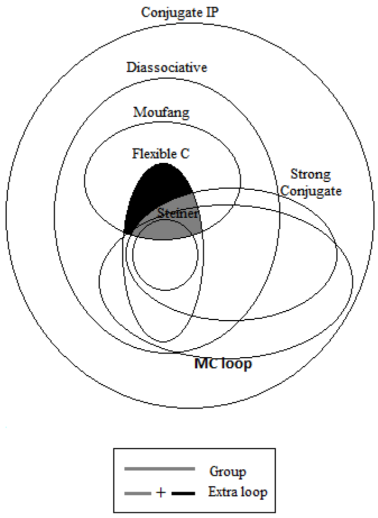
Example 4.11. This loop is a multiplicative conjugate loop but it is not power associative.

$\cdot$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	1	5	6	3	4	9	10	7	8
3	3	5	7	1	9	2	10	4	8	6
4	4	6	1	8	2	10	3	9	5	7
5	5	3	9	2	8	1	6	7	10	4
6	6	4	2	10	1	7	8	5	3	9
7	7	9	10	3	6	8	5	1	4	2
8	8	10	4	9	7	5	1	6	2	3
9	9	7	8	5	10	3	4	2	6	1
10	10	8	6	7	4	9	2	3	1	5

Indeed, the subloop $\langle 3 \rangle = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is not associative. □

Power associative loops are not MC-loop because Moufang loops are power associative but not MC-loop.

The relationship of MC-loops with other loops is illustrated by the following diagram.



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