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On sheaf spaces of partially ordered quasigroups

Ján Brajerčík and Milan Demko

Abstract. The conditions under which a partially ordered quasigroup can be represented as sections of a sheaf space of partially ordered quasigroups are investigated.

1. Introduction

There are known some characterizations of representable lattice ordered groups, i.e., lattice ordered groups, shortly l-groups, which are l-isomorphic to a subdirect product of totally ordered groups; see, e.g., [2]. One of these characterizations is based on the theory of sheaf spaces of l-groups. The central theorem used for this purpose gives the conditions (using ideals of l-groups) under which an l-group can be represented as sections of a sheaf space of l-groups (see [2, Theorem 49.4]). In this paper we generalize this result for partially ordered quasigroups.

2. Preliminaries

A *quasigroup* is an algebra $(Q, \cdot, \backslash, /)$ with three binary operations $\cdot, \backslash, /$ satisfying the following identities

$$y\backslash(y \cdot x) = x; \qquad (x \cdot y)/y = x; \qquad y \cdot (y\backslash x) = x; \qquad (x/y) \cdot y = x. \tag{1}$$

It is easy to see that

$$x/(y\backslash x) = y; \qquad (x/y)\backslash x = y \tag{2}$$

follow from (1). Further, the identities (1) imply that, given $a, b \in Q$, the equations $b \cdot x = a$ and $y \cdot b = a$ have unique solutions $x = b\backslash a$ and $y = a/b$, respectively. Conversely, if G is a groupoid such that the equations $b \cdot x = a$ and $y \cdot b = a$ have unique solutions x, $y \in G$, then G is a quasigroup, where $b\backslash a$ and a/b are defined as the solution of the equation $b \cdot x = a$ or $x \cdot b = a$, respectively. Clearly, every group is a quasigroup with $x/y = x \cdot y^{-1}$ and $y\backslash x = y^{-1} \cdot x$. General information concerning the properties of quasigroups can be found, e.g., in [1], [5].

A quasigroup $(Q, \cdot, \backslash, /)$ with a binary relation $\leqslant$ is called a *partially ordered quasigroup* (*po-quasigroup*) if

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(i) $(Q, \leqslant)$ is a partially ordered set,

(ii) for all $x, y, a \in Q$, $x \leqslant y$ implies

$$ax \leqslant ay,\ xa \leqslant ya,\ x/a \leqslant y/a,\ a\backslash x \leqslant a\backslash y,\ a/y \leqslant a/x,\ y\backslash a \leqslant x\backslash a.$$

For a po-quasigroup we will use the notation $\mathcal{Q} = (Q, \cdot, \backslash, /, \leqslant)$. Clearly, every partially ordered group is a po-quasigroup.

A partially ordered quasigroup $\mathcal{Q}$ is called a *lattice ordered quasigroup* (shortly *l-quasigroup*), if $\leqslant$ is a lattice order. Analogously to the case of the lattice ordered groups it can be proved that for l-quasigroups the following identities, determining the relationship between the quasigroup operations and the lattice operations $\vee$, $\wedge$, hold

(L1) $a(b \vee c) = ab \vee ac$; $(b \vee c)a = ba \vee ca$,
$a(b \wedge c) = ab \wedge ac$; $(b \wedge c)a = ba \wedge ca$.

(L2) $(b \vee c)/a = (b/a) \vee (c/a)$; $a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c)$,
$(b \wedge c)/a = (b/a) \wedge (c/a)$; $a\backslash(b \wedge c) = (a\backslash b) \wedge (a\backslash c)$.

(L3) $a/(b \vee c) = (a/b) \wedge (a/c)$; $(b \vee c)\backslash a = (b\backslash a) \wedge (c\backslash a)$,
$a/(b \wedge c) = (a/b) \vee (a/c)$; $(b \wedge c)\backslash a = (b\backslash a) \vee (c\backslash a)$.

Here we prove only the first identity from (L3); the proofs of remaining identities are analogous. Since $b, c \leqslant b \vee c$, we have $a/(b \vee c) \leqslant a/b, a/c$, and therefore $a/(b \vee c) \leqslant (a/b) \wedge (a/c)$. On the other hand, $(a/b) \wedge (a/c) \leqslant a/b, a/c$. Using (2) we obtain $c, b \leqslant ((a/b) \wedge (a/c))\backslash a$, which implies $b \vee c \leqslant ((a/b) \wedge (a/c))\backslash a$. Hence $(a/b) \wedge (a/c) \leqslant a/(b \vee c)$. Therefore we can conclude that $a/(b \vee c) = (a/b) \wedge (a/c)$.

Let $\mathcal{Q}$ and $\mathcal{H}$ be the partially ordered quasigroups. We say that a mapping $\Phi : Q \to H$ is an *o-embedding* of $\mathcal{Q}$ into $\mathcal{H}$ if Φ is a quasigroup homomorphism and

$$\Phi(x) \leqslant \Phi(y) \Longleftrightarrow x \leqslant y.$$

In that case we say that $\mathcal{Q}$ is o-embedded into $\mathcal{H}$.

Let $\mathcal{Q} = (Q, \cdot, \backslash, /, \leqslant)$ be a partially ordered quasigroup. Let θ be a congruence relation on $(Q, \cdot, \backslash, /)$. The congruence class of θ containing $a \in Q$ will be denoted by $[a]\theta$, i.e., $[a]\theta = \{x \in Q |\, x\theta a\}$. Clearly, every congruence class $[a]\theta$ is a partially ordered set under the relation induced by $\leqslant$. We say that θ is a *convex congruence relation* on $\mathcal{Q}$ if θ is a congruence relation on $(Q, \cdot, \backslash, /)$ and there exists $a \in Q$ such that the congruence class $[a]\theta$ is a convex subset of $\mathcal{Q}$. We say that θ is a *directed congruence relation* on $\mathcal{Q}$ if θ is a congruence relation on $(Q, \cdot, \backslash, /)$ and there exists $a \in Q$ such that the congruence class $[a]\theta$ is a directed subset of $\mathcal{Q}$ (i.e., for each $x, y \in [a]\theta$ there exist $u, v \in [a]\theta$ such that $u \leqslant x, y$ and $x, y \leqslant v$).

Let $\mathcal{Q}$ be a po-quasigroup and let θ be a convex congruence relation on $\mathcal{Q}$. Let us put

$$[x]\theta \leqslant [y]\theta \text{ if and only if there exist } x_0 \in [x]\theta, y_0 \in [y]\theta \text{ such that } x_0 \leqslant y_0. \quad (3)$$

A quotient-quasigroup $(Q,\cdot,\backslash,/)/\theta$ with the relation defined by (3) is a partially ordered quasigroup; it will be denoted by $\mathcal{Q}/\theta$ (see [3, Theorem 2.6]). If $\mathcal{Q}$ is an l-quasigroup and θ is a convex directed congruence relation on $\mathcal{Q}$, then $\mathcal{Q}/\theta$ is an l-quasigroup with the lattice operations $\vee$ and $\wedge$ defined by (see [3])

$$[x]\theta \vee [y]\theta = [x \vee y]\theta;\ \ [x]\theta \wedge [y]\theta = [x \wedge y]\theta.$$

3. Sheaf spaces of po-quasigroups

Let E and X be topological spaces. A continuous mapping $\sigma : E \to X$ is called a *local homeomorphism*, if each point $s \in E$ has a neighborhood V such that $\sigma(V)$ is an open set in X and the restricted mapping $\sigma|_V : V \to \sigma(V)$ is a homeomorphism. If $x \in X$ is a point, the set $E_x = \sigma^{-1}(x)$ is called the *fibre* over x. Let U be an open set in X. A continuous mapping $f : U \to E$ such that $f(x) \in \sigma^{-1}(x)$ for all $x \in U$ is called a *continuous local section* of σ *over* U. If σ is surjective and $U = X$, f is called a *continuous global section*. The basic facts on sections of a local homeomorphism can be find, e.g., in [4]. For the sake of convenience, we summarize here some results which will be frequently used.

Proposition 3.1. (cf. [4, Lemma 1])

(i) *A local homeomorphism is an open mapping.*

(ii) *The restriction of a local homeomorphism to a topological subspace is a local homeomorphism.*

Proposition 3.2. (cf. [4, Lemma 2]) *Let $\sigma : E \to X$ be a local homeomorphism.*

(i) *To each point $s \in E$ there exist a neighborhood U of $x = \sigma(s)$ and a continuous section $f : U \to E$ such that $f(x) = s$.*

(ii) *Let f be a continuous section of E over an open subset U of X. To each point $x \in U$ and each neighborhood V of $f(x)$ such that $\sigma(V)$ is open and $\sigma|_V$ is a homeomorphism, there exists a neighborhood U_0 of x such that $f(U_0) \subseteq V$ and $f|_{U_0} = (\sigma|_V)^{-1}|_{U_0}$.*

(iii) *If U, V are open sets in X, and $f : U \to E$, $g : V \to E$ are continuous sections, then the set $\{x \in U \cap V \mid f(x) = g(x)\}$ is open.*

(iv) *Every continuous section of E defined on an open set is an open mapping.*

Proposition 3.3. (cf. [4, Lemma 3]) *Let $\sigma : E \to X$ be a local homeomorphism.*

(i) *The open sets $V \subseteq E$ such that $\sigma|_V : V \to \sigma(V)$ is a homeomorphism form a basis of the topology of E.*

(ii) *The topology of E coincides with the final topology associated with the set of all continuous sections of E.*

Let $\sigma : E \to X$ be a local homeomorphism. For any $U \subseteq X$ we denote

$$E_U = \bigcup_{x \in U} E_x.$$

Immediately from the definition of a local homeomorphism we obtain

Lemma 3.4. *If $U \subseteq X$ is open in X, then E_U is an open set in E.*

By $E\Delta E$ we denote the set $\bigcup_{x\in X}(E_x \times E_x)$ with the induced topology from $E \times E$.

Definition 3.5. Let E and X be topological spaces and let $\sigma : E \to X$ be a surjective local homeomorphism. We say that a triplet (E, X, σ) is a *sheaf space of po-quasigroups* if

(i) each fibre E_x is a po-quasigroup,

(ii) the mappings $(s,t) \mapsto s \cdot t$, $(s,t) \mapsto t\backslash s$ and $(s,t) \mapsto s/t$ from $E\Delta E$ to E are continuous.

Definition 3.6. A sheaf space of po-quasigroups (E, X, σ) is said to be a *sheaf space of l-quasigroups* if each fibre E_x is an l-quasigroup and the mappings

$$(s,t) \mapsto s \vee t, \qquad (s,t) \mapsto s \wedge t$$

from $E\Delta E$ to E are continuous.

Let (E, X, σ) be a sheaf space of po-quasigroups. Let f, g be continuous sections defined over the same open set $U \subseteq X$. Define fg, $g\backslash f$ and f/g by

$$(fg)(x) = f(x) \cdot g(x); \quad (g\backslash f)(x) = g(x)\backslash f(x); \quad (f/g)(x) = f(x)/g(x).$$

Since $\cdot, \backslash, /$ are continuous mappings from $E\Delta E$ to E, fg, $g\backslash f$ and f/g are continuous sections over U.

Lemma 3.7. *Let (E, X, σ) be a sheaf space of po-quasigroups and let $f : U \to E$ be a continuous local section over an open set $U \subseteq X$. Then the mapping $\varphi_f : E_U \to E_U;\ E_x \ni s \mapsto f(x)/s$ is a homeomorphism.*

Proof. By Lemma 3.4, E_U is an open set in E. Clearly, $\varphi_f : E_U \to E_U;\ E_x \ni s \mapsto f(x)/s$ is a bijection. Using (2) it is easy to verify that the inverse mapping $\varphi_f^{-1} : E_U \to E_U$ is defined by $E_x \ni s \mapsto s\backslash f(x)$.

Let $s \in E_U$, $\sigma(s) = x \in U$. Let $W \subseteq E_U$ be an open set, $f(x)/s \in W$. In view of Proposition 3.3(i) for the proof of the continuity of φ_f we may suppose that $\sigma|_W$ is a homeomorphism. Denote $(\sigma|_W)^{-1} = g$. Clearly, g is a continuous local section over $U_0 = \sigma(W)$ and $g(x) = f(x)/s$. Put $V = (g\backslash f)(U_0)$. Since $g\backslash f$ is a continuous local section, by Proposition 3.2(iv), V is open in E_U. Moreover, since $(g\backslash f)(x) = g(x)\backslash f(x) = (f(x)/s)\backslash f(x) = s$, we have $s \in V$. Further, if $t \in \varphi_f(V)$, then there is $u \in U_0$ such that $t = \varphi_f(g(u)\backslash f(u)) = f(u)/(g(u)\backslash f(u)) = g(u) \in W$. Thus $\varphi_f(V) \subseteq W$, and we can conclude that φ_f is continuous. The proof of the continuity of φ_f^{-1} is analogous. $\square$

Let (E, X, σ) be a sheaf space of po-quasigroups. Consider the following condition:

(C) *if f, g are continuous local sections over the same open set $U \subseteq X$ such that* $\sup\{f(u), g(u)\}$ *exists for each $u \in U$, then the set* $\{\sup\{f(u), g(u)\} | \, u \in U\}$ *is open in E.*

Lemma 3.8. *Let (E, X, σ) be a sheaf space of po-quasigroups where fibres E_x are lattice ordered quasigroups. Then (E, X, σ) is a sheaf space of l-quasigroups if and only if (E, X, σ) satisfies the condition* (C).

Proof. Suppose that (E, X, σ) satisfies the condition (C). Firstly we will show that $\vee$ is continuous. Let (s, t) be an arbitrary point of $E \Delta E$, i.e., $s, t \in E_x$ for some $x \in X$. Let $W_{s \vee t}$ be an open set in E, $s \vee t \in W_{s \vee t}$. By Proposition 3.2($i$) there exist an open set $U \subseteq X$, $x \in U$, and continuous local sections f, g over U with $f(x) = s$, $g(x) = t$. By (C), the set $W_{\sup} = \{f(u) \vee g(u) | \, u \in U\}$ is open in E. Denote $W_0 = W_{\sup} \cap W_{s \vee t}$. By Proposition 3.1($i$), the set $U_0 = \sigma(W_0)$ is open in X which implies that $f(U_0)$ and $g(U_0)$ are open in E, and $\{(f(u), \, g(u)) | \, u \in U_0\} = (f(U_0) \times g(U_0)) \cap (E \Delta E)$ is open in $E \Delta E$ containing the point $(s, t) \in E \Delta E$. Since $f(U_0) \vee g(U_0) \equiv \{(f(u) \vee g(u)) | \, u \in U_0\} \subseteq W_0 \subseteq W_{s \vee t}$, we can conclude that $\vee$ is continuous.

We are going to show that $\wedge$ is continuous. Let $s, t \in E_x$. Let $W_{s \wedge t}$ be an open set in E, $s \wedge t \in W_{s \wedge t}$. In view of Proposition 3.3(i) for the proof of the continuity of $\wedge$ we may suppose that $\sigma|_{W_{s \wedge t}}$ is a homeomorphism. Denote $f = (\sigma|_{W_{s \wedge t}})^{-1}$. Clearly, f is a continuous local section over $U = \sigma(W_{s \wedge t})$. By Lemma 3.7, the mapping $\varphi_f : E_U \to E_U$; $E_z \ni r \mapsto f(z)/r$ is a homeomorphism. Thus $W = \varphi_f(f(U))$ is open in E and $f(x)/(s \wedge t) \in W$. By (L3), $f(x)/(s \wedge t) = (f(x)/s) \vee (f(x)/t)$ and since $\vee$ is continuous, there exist neighborhoods V_s of $f(x)/s$ and V_t of $f(x)/t$, $\sigma(V_s) = \sigma(V_t) \subseteq U$, such that $V_s \vee V_t \subseteq W$. Denote $W_s = \varphi_f^{-1}(V_s)$ and $W_t = \varphi_f^{-1}(V_t)$. Since $\varphi_f^{-1}(f(x)/s) = (f(x)/s) \backslash f(x) = s$, we have $s \in W_s$. Analogously, $t \in W_t$. Further, if $p \in W_s$, $r \in W_t$, $\sigma(p) = \sigma(r) = z$, then $\varphi_f(p) \vee \varphi_f(r) = (f(z)/p) \vee (f(z)/r) \in V_s \vee V_t \subseteq W$, which yields $f(z)/(p \wedge r) \in W$. Hence $\varphi_f^{-1}(f(z)/(p \wedge r)) = p \wedge r \in f(U) \subseteq W_{s \wedge t}$. Thus $W_s \wedge W_t \subseteq W_{s \wedge t}$, and we can conclude that $\wedge$ is continuous.

Conversely, let (E, X, σ) be a sheaf space of l-quasigroups. Suppose that f, g are continuous local sections over the same open set $U \subseteq X$. We are going to show that $W_{\sup} = \{f(u) \vee g(u) \mid u \in U\}$ is open in E. Let $x \in U$. By Proposition 3.3(i) there exists an open set W in E, $f(x) \vee g(x) \in W$, such that $\sigma|_W : W \to \sigma(W)$ is a homeomorphism. Since $\vee$ is continuous, there exist an open set $U_0 \subseteq U \subseteq X$, $x \in U_0$, such that $W_0 = f(U_0) \vee g(U_0) \subseteq W$. Clearly, $W_0 \subseteq W_{\sup}$ and, since $W_0 = E_{U_0} \cap W$, by Lemma 3.4, W_0 is open. Thus we can conclude that $W_{\sup}$ can be covered by open sets, which means that $W_{\sup}$ is open in the topology of E. $\square$

The sheaf space of l-groups is defined as a triplet (E, X, σ) such that each fibre E_x is an l-group, the mappings $\cdot$, $\vee$, $\wedge$ are continuous from $E \Delta E$ to E and $^{-1}$ is continuous from E to E (see [2]). In view of Lemma 3.7 and Lemma 3.8 we have

Corollary 3.9. *Let (E, X, σ) be a sheaf space of po-quasigroups satisfying* (C)*. If E_x is an l-group for each $x \in X$, then (E, X, σ) is a sheaf space of l-groups.*

Proof. Clearly, $\cdot$ is continuous from $E \Delta E$ to E and, by Lemma 3.8, the lattice operations $\vee$ and $\wedge$ are also continuous. Consider the global section $e : X \to E$; $e(x) = e_x$, where e_x is the identity element of E_x; e is a continuous global section (see [4]). Since for $s \in E_x$ we have $s^{-1} = e_x/s = e(x)/s$ and, by Lemma 3.7, $s \mapsto e(x)/s$ is a homeomorphism, we can conclude that $^{-1}$ is a continuous mapping from E to E. □

Let (E, X, σ) be a sheaf space of po-quasigroups. Clearly, the direct product $\prod_{x \in X} E_x$ of po-quasigroups E_x is a po-quasigroup. Denote by $\mathcal{R}$ the set of all continuous global sections of σ and define the relation $\leqslant$ on $\mathcal{R}$ by

$$g \leqslant h \Longleftrightarrow g(x) \leqslant h(x) \text{ for all } x \in X. \tag{4}$$

Let $\mathcal{R} \neq \emptyset$. Then $\mathcal{R}$ with the operations $\cdot$, $/$, $\backslash$ defined componentwise and the relation $\leqslant$ defined by (4) is a po-quasigroup. Moreover, it is easy to see that

Lemma 3.10. *If $\mathcal{R} \neq \emptyset$, then $\mathcal{R}$ is a po-subquasigroup of the direct product $\prod_{x \in X} E_x$.*

The following theorem generalizes the analogous result valid for lattice ordered groups (see [2, Theorem 49.4]).

Theorem 3.11. *Let $\mathcal{Q}$ be a po-quasigroup and let X be a topological space. Suppose that for each $x \in X$ there exists a convex congruence relation θ_x on $\mathcal{Q}$ such that the following conditions are satisfied*

(i) *for all $g, h \in Q$, the set $U_{gh} = \{x \in X \mid [g]\theta_x = [h]\theta_x\}$ is open in X,*

(ii) *if $[g]\theta_x \leqslant [h]\theta_x$ for each $x \in X$, then $g \leqslant h$.*

Then $\mathcal{Q}$ can be o-embedded into a po-quasigroup of the continuous global sections of some sheaf space of po-quasigroups over X. Especially, if $\mathcal{Q}$ is an l-quasigroup and θ_x are directed convex congruence relations on $\mathcal{Q}$ satisfying (i) *and* (ii)*, then $\mathcal{Q}$ can be o-embedded into an l-quasigroup of the continuous global sections of some sheaf space of l-quasigroups over X.*

Proof. Let $\mathcal{Q}$ be a po-quasigroup such that (i) and (ii) are valid. We follow the idea of the construction of a sheaf space which was used for l-groups in the proof of Theorem 49.4 in [2]. Denote

$$E = \bigcup_{x \in X} E_x,$$

where $E_x = \mathcal{Q}/\theta_x \times \{x\}$ and define

$$\sigma : E \to X; \quad ([g]\theta_x, x) \mapsto x.$$

Clearly, σ is a surjection. Further, for each $g \in Q$ we define

$$\widehat{g} : X \to E; \quad x \mapsto ([g]\theta_x, x)$$

and consider the finest topology τ on E such that each $\widehat{g}$ is continuous. Denote

$$\mathbb{B} = \{\widehat{g}(U) \,|\, U \text{ is open in } X, g \in Q\}.$$

Let $\widehat{g}(U), \widehat{h}(V) \in \mathbb{B}$. By (i), $T = \{x \in X \,|\, \widehat{g}(x) = \widehat{h}(x)\}$ is an open set in X. Let $W = T \cap U \cap V$. Clearly, W is open in X, and $\widehat{g}(W) = \widehat{h}(W) \subseteq \widehat{g}(U) \cap \widehat{h}(V)$. Conversely, if $t \in \widehat{g}(U) \cap \widehat{h}(V)$, then $t = ([g]\theta_u, u) = ([h]\theta_u, u)$, $u \in W$, which yields $t \in \widehat{g}(W)$. Therefore $\widehat{g}(U) \cap \widehat{h}(V) = \widehat{g}(W)$ and, since W is open in X, we can conclude that $\widehat{g}(U) \cap \widehat{h}(V) \in \mathbb{B}$. Thus $\mathbb{B}$ is a basis for some topology τ_B on E. By (i), for any $\widehat{h}(U) \in \mathbb{B}$ and $g \in Q$ the set $(\widehat{g})^{-1}(\widehat{h}(U)) = U \cap \{x \in X \mid [g]\theta_x = [h]\theta_x\}$ is open in X, which yields $\tau_B \subseteq \tau$. On the other hand, let V be a τ-open set in E. For every $v = ([g]\theta_x, x) \in V$ the set $U = (\widehat{g})^{-1}(V)$ is open in X, $\widehat{g}(U) \subseteq V$ and $v \in \widehat{g}(U)$. Thus V is covered by τ_B-open sets. Therefore $\tau \subseteq \tau_B$ and so $\tau = \tau_B$.

Let $s \in E$, $s = ([g]\theta_x, x)$ and let U be a neighborhood of $x = \sigma(s)$ in X. Then $V = \widehat{g}(U)$ is open in E, $s \in V$ and

$$\sigma\,|_V \circ \widehat{g}\,|_U = \mathrm{id}_U, \quad \widehat{g}\,|_U \circ \sigma\,|_V = \mathrm{id}_V.$$

Thus $\sigma : E \to X : ([g]\theta_x, x) \mapsto x$ is a continuous mapping and $\sigma\,|_V : V \to U$ is a homeomorphism. We have that $\sigma : E \to X$ is a local homeomorphism with the fibres $E_x = \{\widehat{g}(x) \mid g \in Q\}$. Each fibre E_x is a po-quasigroup under the operations

$$\widehat{g}(x) \cdot \widehat{h}(x) = (\widehat{gh})(x); \quad (\widehat{g}(x)/\widehat{h}(x) = (\widehat{g/h})(x); \quad \widehat{g}(x)\backslash\widehat{h}(x) = (\widehat{g\backslash h})(x)$$

and the partial order

$$\widehat{g}(x) \leqslant \widehat{h}(x) \text{ iff there exist } g' \in [g]\theta_x, h' \in [h]\theta_x \text{ such that } g' \leqslant h'.$$

For every open set W in E such that $\widehat{gh}(x) \in W$ there exists an open set U in X, $x \in U$, such that $\widehat{gh}(U) \subseteq W$. Since $V = \{(\widehat{g}(u), \widehat{h}(u)) \,|\, u \in U\}$ is open in $E\Delta E$ and $\widehat{g}(u) \cdot \widehat{h}(u) = \widehat{gh}(u)$ for each $u \in U$, we can conclude that the operation $\cdot$ is continuous. Analogously, the operations $\backslash, /$ are continuous. Thus (E, X, σ) is a sheaf space of po-quasigroups.

Let $\mathcal{R}$ be a po-quasigroup of all continuous global sections of (E, X, σ). Define

$$\Phi : \mathcal{Q} \to \mathcal{R}; \quad g \mapsto \widehat{g}.$$

Clearly, Φ preserves the quasigroup operations. Further, by (ii), we have

$$g \leqslant h \Leftrightarrow [g]\theta_x \leqslant [h]\theta_x \text{ for all } x \in X \Leftrightarrow \widehat{g}(x) \leqslant \widehat{h}(x) \text{ for all } x \in X \Leftrightarrow \widehat{g} \leqslant \widehat{h}.$$

Thus Φ is an o-embedding of $\mathcal{Q}$ into $\mathcal{R}$.

If $\mathcal{Q}$ is an l-quasigroup and θ_x are directed convex congruence relations on $\mathcal{Q}$, then $\mathcal{Q}/\theta_x$ are l-quasigroups, which yields that the fibres E_x are lattice ordered quasigroups under the lattice operations

$$\widehat{g}(x) \vee \widehat{h}(x) = (\widehat{g \vee h})(x); \quad \widehat{g}(x) \wedge \widehat{h}(x) = (\widehat{g \wedge h})(x).$$

By the same way as in the case of the quasigroup operations we can see that the mappings $\vee$ and $\wedge$ are continuous. Thus (E, X, σ) is a sheaf space of l-quasigroups. Clearly, $\mathcal{R}$ is an l-quasigroup and $\Phi : g \mapsto \widehat{g}$ is an o-embedding of $\mathcal{Q}$ into $\mathcal{R}$. $\square$

Remark. Let (E, X, σ) be the sheaf space constructed in the proof of Theorem 3.11. Let X be a Hausdorff space. Then E is a Hausdorff space if for all $g, h \in Q$, the set $U_{gh} = \{x \in X \mid [g]\theta_x = [h]\theta_x\}$ is open and also close in X. To prove this statement it suffices to use the same topological arguments as in the proof of Theorem 49.4 in [2].

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Department of Physics, Mathematics and Techniques, University of Prešov, Ul. 17. novembra 1, 081 16 Prešov, Slovakia
E-mails: jan.brajercik@unipo.sk, milan.demko@unipo.sk