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Free covering semigroups of topological n-ary semigroups

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Abstract. Connections between the topology of an n-ary semigroup and the topology of its free covering semigroup are described.

Investigations of topological n-ary groups and n-ary semigroups were initiated by Čupona in 1970 when he studied the problem of embedding of some topological universal algebras into topological semigroups (see [4, 5] and [7]). By the way it turned out that topological n-ary groups may be defined in various ways. For $n > 2$ these definitiona are equivalent, but for $n = 2$ some of these are not valid (for details see [14]). Properties of topological n-ary groups are strongly connected with the properties of their retracts [15]. On the other hand, each toplogical n-ary group can be embedded into some topological group (see [3] and [6]). The topology of this covering group is strongly connected with the topology of an initial n-ary group. Namely, Endres proved in [10] that topological properties of this topological group may be moved to its initial topological n-ary groups and conversely. For n-ary semigroups the situation is more complicated.

The topology on the n-ary semigroups can be defined by the systems of some maps [1]. Conditions under which a topology determined by the families of left invariant derivations is compatible with the n-ary operation are given in [9]. The problem of the embedding of some locally compact n-ary semigroups into locally compact n-ary groups is studied in [12]. Conditions under which an n-ary semigroup with a locally compact topology can be topologically embedded into a locally compact binary group as an open set were found in [11]. In [13] for topological n-ary semigrous in which all translations (shifts) are continous and open the covering semigroup is constructed in this way that the topology of an initial topological n-ary semigroup can be extended to a topology compatible with the semigroup operation on the covering semigroup.

Below we describe connections between the the topology of an n-ary semigroup and the topology of its free covering semigroup. For this we use the construction of free covering semigroup proposed in [8] and the following proposition from [2] (Chapter 1, §6, Proposition 6).

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Proposition 1. *Let ρ be an equivalence relation on a topological space X. Then a map f of X/ρ into a topological space Y is continuous if and only if $f \circ \varphi$, where φ is a cannonical map of X onto X/ρ, is continuous on X.* $\square$

Let $(G, [\,])$ be an n-ary semigroup. Further, for simplicity we will omit the operator symbol $[\,]$ and instead of $[...[[x_1 \ldots x_n]x_{n+1} \ldots x_{2n-1}]x_{2n} \ldots x_p]$, where $p = k(n-1)+1$, we will write $[x_1, \ldots, x_p]$. Additionaly we put $[x] = x$ for $k = 0$.

Let F be the set of non-empty words over G, i.e.,

$$F = \bigcup_{k\in\mathbb{N}} G^k = \{(x_1, \ldots, x_k) \,|\, k \in \mathbb{N}, x_j \in G\}.$$

On F we introduce the product $(x_1, \ldots, x_k) \diamond (y_1, \ldots, y_t) = (x_1, \ldots, x_k, y_1, \ldots, y_t)$. Then F with this product is a free semigroup.

Two elements $\alpha = (a_1, \ldots, a_p)$, $\beta = (b_1, \ldots, b_q)$ of F are *strongly linked* if and only if there exists an element $(d_1, \ldots, d_t) \in F$ and two sequences of non-negative integers $k_1 < k_2 < \ldots, k_p = t$, $m_1 < m_2 < \ldots < m_q = t$ such that

$$a_1 = [d_1 \ldots d_{k_1}], \quad a_2 = [d_{k_1+1} \ldots d_{k_2}], \; \ldots \; a_p = [d_{k_{p-1}+1} \ldots d_{k_p}],$$

$$b_1 = [d_1 \ldots d_{m_1}], \quad b_2 = [d_{m_1+1} \ldots d_{m_2}], \; \ldots \; b_q = [d_{m_{p-1}+1} \ldots d_{m_p}].$$

Two strongly linked elements $\alpha, \beta \in F$ are denoted by $\alpha \sim \beta$. The relation $\sim$ is reflexive and symmetric. Its transitive closure $\approx$ is a congruence on F (for details see [8]). The quotient semigroup $(F/\!\approx, *)$ is called the *free covering semigroup* of an n-ary semigroup $(G, [\,])$ and is denoted by F^*. The equivalence class of α, i.e., an element of F^* induced by α, is denoted by α^*.

The set $G^* = \{a^* \,|\, a \in G\}$ is an n-ary subsemigroup of $F/\!\approx$ with the operation

$$[a_1^* a_2^* \ldots a_n^*] = a_1^* * a_2^* * \cdots * a_n^*.$$

The canonical mapping $\varphi(a) = a^*$ is an isomorphism from G onto G^*. So, we can identify the element $a \in G$ with the class a^* and G with G^*. Moreover, since $a^* = \beta^*$ if and only if $a = [b_1 \ldots b_q]$, we can write F^* in the form

$$F^* = G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_{n-1},$$

where $G_j = \{a_1 * a_2 * \cdots * a_j \,|\, a_1, \ldots, a_j \in G\}$ and $G_i \cap G_j = \emptyset$ for $i \neq j$.

Let τ be a topology on G and let τ_k be a topology on the Cartesian product G^k obtained as a product of k topologies τ defined on k factors G. The sum of all topologies τ_k, $(k = 1, 2, 3, \ldots)$, where $\tau_1 = \tau$, is denoted by $\tau_{_F}$. By $\tau_{_{F^*}}$ we denote the factor topology on a free covering semigroup F^*.

Theorem 1. *Let $(G, [\,])$ be an n-ary semigroup. Then:*

1) *The semigroup $(F, \diamond)$ endowed with a topology $\tau_{_F}$ is a topological semigroup.*

2) *Each subset G^k is an open-closed subset of a topological space $(F^*, \tau_{_{F^*}})$.*

3) *The free covering semigroup* $(F^*, *)$ *endowed with the topology* τ_{F^*} *has continuous left and right shifts.*

Proof. The first statement is obvious. The second follows from the fact that

$$\varphi^{-1}(G_i) = \bigcup_{k=0}^{\infty} G^{k(n-1)+i} \in \tau_F$$

and it are saturated by the relation $\approx$.

To prove the third statement observe that each right shift $R_a(x) = x \diamond a$ by an element $a \in F$ is continuous on a semigroup $(F, \diamond)$. Therefore, the composition $\varphi \circ R_a$ is continuous. Since $\varphi \circ R_a = r_a \circ \varphi$, where $r_a(x^*) = x^* * a^*$, Proposition 1 implies that $r_a : F^* \to F^*$ also is continuous.

Similarly we can prove the continuity of each left shift of F^*. □

An n-ary semigroup $(G, [\ ])$ with a topology τ is called a *topological* n*-ary semigroup* if (G, τ) is a topological space such that the n-ary operation $[\ ]$ is continuous (in all variables together).

Theorem 2. *Let* $(G, [\ ])$ *be a topological* n*-ary semigroup with the topology* τ*. Then the restriction of the topology* τ_{F^*} *to* G *coincides with the topology* τ*.*

Proof. Let $U \in \tau_{F^*}$ be an arbitrary non-empty subset of G. Then $\varphi^{-1}(U) \in \tau_F$. Hence, $U = \varphi^{-1}(U) \cap G \in \tau$.

Conversely, let $U \in \tau$ and $\alpha = (a_1, \ldots, a_p)$, where $p = t(n-1)+1$, be an arbitrary element of $\varphi^{-1}(U)$. Then $a_1^* * a_2^* * \cdots * a_p^* \in U$. Thus $a_1^* * a_2^* * \cdots * a_p^* = [a_1 a_2 \ldots a_p]$. Since the operation $[\ ]$ is continuous, there are some open (in the topology τ) neighborhoods $V_1, \ldots, V_p$ of points $a_1^*, \ldots, a_p^*$ such that for all $x_i \in V_i$, $i = 1, \ldots, p$, we have $[x_1 \ldots x_p] \in U$. Consequently, $\varphi(x_1, \ldots, x_p) = x_1^* * \cdots * x_p^* = [x_1 \ldots x_p] \in U$, i.e., $\varphi^{-1}(U) \supset V_1 \times \cdots \times V_p \in \tau_F$. Hence, $\varphi^{-1}(U) \in \tau_F$ and $\varphi^{-1}(U)$ is saturated by the relation $\approx$. This implies that $U \in \tau_{F^*}$. □

It is clear that the Cartesian product $F \times F$ with the operation $(x, y) \otimes (s, t) = (x \diamond s, y \diamond t)$ and the topology $\tau_{F\times F} = \tau_F \times \tau_F$ is a topological semigroup.

Consider on $F \times F$ the relation $\approx_{F\times F}$ defined by

$$(x, y) \approx_{F\times F} (s, t) \iff x \approx s \text{ and } y \approx t.$$

This relation is a congruence on $F \times F$ and $(F \times F)^* = F \times F/\approx_{F\times F}$ with the standard factor-operation $*$ is a semigroup. Then obviously

$$(x, y)^* * (s, f)^* = ((x, y) \otimes (s, t))^* = (x \diamond s, y \diamond t)^* = (x^* * s^*, y^* * t^*)$$

and $(x, y)^* = (x^*, y^*)$. Therefore $(F \times F)^* = F^* \times F^*$. Hence, the canonical map ω of $F \times F$ onto $(F \times F)^*$ has the form $\omega = (\varphi, \varphi)$.

By $\tau_{(F\times F)^*}$ we denote the factor topology on the semigroup $(F \times F)^*$.

Theorem 3. *The operation* $*$ *from* $F^* \times F^*$ *with the topology* $\tau_{(F\times F)^*}$ *to* F^* *is a continuous.*

Proof. Since $\diamond$, φ, ω are continuous and $(\varphi\circ\diamond)(\alpha,\beta)=(*\circ\omega)(\alpha,\beta)$ for all $\alpha,\beta\in F$, the proof follows from Proposition 1. $\square$

Corollary 1. *If the topologies $\tau_{(F\times F)^*}$ and $\tau_{F^*}\times\tau_{F^*}$ coincides, then the semigroup $(F^*,*)$ endowed with the topology τ_{F^*} is a topological semigroup.* $\square$

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