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On δ-primary co-ideals of a commutative semiring

Shahabaddin Ebrahimi Atani, Saboura Dolati Pish Hesari
and Mehdi Khoramdel

Abstract. We introduce the notion of a δ-primary co-ideal of a commutative semiring R and study some of it properties. Here δ is a mapping that assigns to each co-ideal J a co-ideal $\delta(J)$ of the same semiring. We investigate the relationship between the minimal prime co-ideals of R/I and $\delta(I)/I$, when I is a δ-primary Q-co-ideal. We also prove that every identity summand of R/I is contained in $\delta(I)/I$ and $\delta(I)$ contains all minimal prime co-ideals which contains I.

1. Introduction

The most trivial example of a semiring which is not a ring is the first algebraic structure we encounter in life: the set of nonnegative integers $\mathbb{N}$, with the usual addition and multiplication. Similarly, the set of nonnegative real numbers $\mathbb{R}^+$ with the usual addition and multiplication is a semiring which is not a ring. The nontrivial examples of semirings first appear in the work of Richard Dedekind in 1894, in connection with the algebra of ideals of a commutative ring and were later studied independently by algebraists, especially by H. S. Vandiver, who worked very hard to get them accepted as a fundamental algebraic structure, being basically the best structure which includes both rings and bounded distributive lattices. Semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas and, hence, ought to be in the literature [9, 11].

In this paper, we introduce the notion of co-ideal expansion and δ-primary co-ideals that is motivated from the notion of δ-primary ideals in semirings (resp. rings) [2] (resp. [7]). A number of results concerning of these class of co-ideals are given. For example, we investigate the relationship between the minimal prime co-ideals of R/I and $\delta(I)/I$, when I is a δ-primary Q-co-ideal. We also prove that I is δ-primary if and only if every identity-summand element of R/I is contained in $\delta(I)/I$.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. A commutative semiring R is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected

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by $a(b+c) = ab+ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r+0 = r$, $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

Definition 1.1. Let R be a semiring.

• A nonempty subset I of R is called *co-ideal*, denoted by $I \trianglelefteq^c R$, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (clearly, $0 \in I$ if and only if $I = R$) [4].

• A co-ideal I of R is called *subtractive* if $x, xy \in I$, then $y \in I$ (so every subtractive co-ideal is a strong co-ideal) [4].

• A proper co-ideal P of R is called *prime* if $x + y \in P$, then $x \in P$ or $y \in P$. A proper co-ideal I of R is called *primary* if $x+y \in I$, then $x \in I$ or $y \in$ co-rad(I) $= \{r \in R : nr \in I$ for some positive integer $n\}$ [4].

• A semiring R is called *co-semidomain*, if $a + b = 1$ $(a, b \in R)$, then either $a = 1$ or $b = 1$ [4].

• We say that a subset $T \subseteq R$ is *additively closed* if $0 \in T$ and $a + b \in T$ for all $a, b \in T$.

• If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \cdots d_n + r$ (with $d_i \in D$ for all $1 \leqslant i \leqslant n$ and $r \in R$) is a co-ideal of R containing D [4, 11].

• A semiring R is called an *I-semiring* if $r + 1 = 1$ for all $r \in R$ [6].

A strong co-ideal I of a semiring R is called a *partitioning strong co-ideal* (*Q-strong co-ideal*) if there exists a subset Q of R such that $R = \cup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$ and if $q_1, q_2 \in Q$, then $(q_1 I) \cap (q_2 I) \neq \emptyset$ if and only if $q_1 = q_2$ [4]. Let I be a Q-strong co-ideal of a semiring R and let $R/I = \{qI : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 I) \oplus (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 I + q_2 I) \subseteq q_3 I$, and $(q_1 I) \odot (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 q_2) I \subseteq q_3 I$ [4]. If q_e is the unique element in Q such that $1 \in q_e I$, then $q_e I = I$ is the identity of R/I [4]. Note that every Q-strong co-ideal is subtractive [4].

Throughout this paper we shall assume unless otherwise stated, that $q_0 I$ (resp. $q_e I$) is the zero element (resp. the identity element) of R/I.

2. Definition and basic structure

We begin with the key definition of this paper.

Definition 2.1. Let R be a semiring with co-Id(R) its set of co-ideals.

(i) A *co-ideal expansion* is a function δ : co-Id$(R) \longrightarrow$ co-Id(R), which satisfies the following conditions:

(1) $I \subseteq \delta(I)$ for each co-ideal I of R;

(2) $I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$ for all co-ideals I, J of R.

(*ii*) A *Q-co-ideal* (resp. *subtractive co-ideal*) *expansion* is a co-ideal expansion which assigns to each Q-co-ideal (resp. subtractive co-ideal) I of a semiring R to another Q-co-ideal (resp. subtractive co-ideal) $\delta(I)$ of the same semiring.

Remark 2.2. Since the intersection of any collection of co-ideals is a co-ideal of R, the intersection of any collection of co-ideal expansions is a co-ideal expansion.

The proof of the following lemma is well-known, but we give the details for convenience.

Lemma 2.3. *If I is a Q-strong co-ideal of R and q_eI is the identity element in R/I, then $q_eI \oplus qI = q_eI$ and $qI \oplus qI = qI$ for each $qI \in R/I$.*

Proof. Let $q_eI \oplus qI = q'I$, where q' is the unique element in Q such that $q_eI + qI \subseteq q'I$. Since I is co-ideal, $q_eI + qI \subseteq I = q_eI$ which gives $q'I = q_eI = I$. Finally, $qI \oplus qI = qI \odot (q_eI \oplus q_eI) = qI \odot q_eI = qI$. □

Proposition 2.4. *Let I be a co-ideal of a semiring R.*

(1) *The set* $\mathrm{cl}(I) = \{a \in R : \ ac = d \ for\ some\ c, d \in I\}$ *is a co-ideal of R (we call* $\mathrm{cl}(I)$ *the co-closure of I).*

(2) *I is subtractive if and only if* $\mathrm{cl}(I) = I$.

Proof. (1). Let $a_1, a_2 \in \mathrm{cl}(I)$; we show that $a_1a_2 \in \mathrm{cl}(I)$. By assumption, there exist $c_1, c_2, d_1, d_2 \in I$ such that $a_1c_1 = d_1$, $a_2c_2 = d_2$, hence $(a_1a_2)(c_1c_2) = d_1d_2$. Since I is a co-ideal of R, we have $a_1a_2 \in \mathrm{cl}(I)$. Now, let $a \in \mathrm{cl}(I)$ and $r \in R$; we show that $a+r \in \mathrm{cl}(I)$. Since $a \in \mathrm{cl}(I)$, there exist $c, d \in I$ such that $ac = d$. As I is a co-ideal of R, $(a + r)c = ac + cr \in I$; so $a + r \in \mathrm{cl}(I)$. Thus $\mathrm{cl}(I)$ is a co-ideal of a semiring R.

(2). Assume that I is a subtractive co-ideal of R (so it is a strong co-ideal) and let $x \in I$. Then $x = x1 \in I$ gives $I \subseteq \mathrm{cl}(I)$. For the reverse incusion, let $y \in \mathrm{cl}(I)$. Then $yc \in I$ for some $c \in I$; hence $y \in I$ since I is subtractive, and so we have equality. The other implication is clear. □

Example 2.5. (1) For each $I \in \text{co-Id}(R)$, define $\delta_1(I) = I$, $\delta_2(I) = \text{co-rad}(I)$ and $\delta_3(I) = \mathrm{cl}(I)$. Then δ_1, δ_2 and δ_3 are expansions of co-ideals.

(2) By [4, Proposition 2.1], if I is a proper co-ideal of R, then there exists a maximal co-ideal M of R such that $I \subseteq M$. Now for each proper co-ideal I, let $\delta_4(I)$ be the intersection of all maximal co-ideals containing I, and $\delta_4(R) = R$. Then δ_4 is an expansion of co-ideals. □

Theorem 2.6. Let R be a semiring.

(1) $\delta_1(I) \subseteq \delta_2(I) \subseteq \delta_3(I) \subseteq \delta_4(I)$ for each strong co-ideal I of R.

(2) If I is a subtractive co-ideal of R, then $\delta_1(I) = \delta_2(I) = \delta_3(I)$.

(3) δ_1, δ_2 and δ_3 are Q-co-ideal expansions.

Proof. (1). It is clear that $\delta_1(I) \subseteq \delta_2(I)$. Let $x \in \delta_2(I) = \text{co-rad}(I)$. So there exists $n \in \mathbb{N}$ such that $nx \in I$; hence $x \in cl(I)$, and so $\delta_2(I) \subseteq \delta_3(I)$. Now, let $x \in \delta_3(I) = cl(I)$. So there exists $a \in I$ such that $ax \in I$. It suffices to show that $x \in \cap M$, where M is a maximal co-ideal of R containing I. Let $x \notin M$ for some maximal co-ideal M of R containing I. So $F(M \cup \{x\}) = R$, which implies $0 = ax^n + r$ for some $a \in M$. Since $x \in \text{cl}(I)$, there exists $b \in I$ such that $bx \in I \subseteq M$. As M is a co-ideal of R, $0 = ab^nx^n + rb^n \in M$, a contradiction. Thus $\delta_3(I) \subseteq \delta_4(I)$.

(2). Suppose that I is a subtractive co-ideal of R and let $x \in \delta_2(I)$. So there exists $n \in \mathbb{N}$ such that $nx \in I$; hence $x \in I$, and so $\delta_1(I) = \delta_2(I)$ by (1). Now, let $x \in \text{cl}(I)$. Then $ax \in I$ for some $a \in I$, so $x \in I$ since I is subtractive. Thus $\delta_2(I) = \delta_3(I)$.

(3). It is clear that δ_1 is a Q-co-ideal expansion. We show that δ_2 is Q-co-ideal expansion. For this lLet I be a Q-co-ideal. Since we have $I \subseteq \text{co-rad}(I)$, $R = \cup\{qI : q \in Q\} \subseteq \cup\{q(\text{co-rad}(I)) : q \in Q\}$, so $R = \cup\{q(\text{co-rad}(I)) : q \in Q\}$. Let $x \in q_1(\text{co-rad}(I)) \cap q_2(\text{co-rad}(I))$, so $x = q_1a_1 = q_2a_2$, where $a_1, a_2 \in$co-rad(I). Thus there exist positive integer elements n, m such that $na_1, ma_2 \in I$. Suppose, without loss of generality, $n \geqslant m$. Hence $nx = q_1(na_1) = q_2(na_2) \in q_1I \cap q_2I$. So $q_1 = q_2$ which gives co-rad(I) is a Q-strong co-ideal of R.

Now, we show that δ_3 is a Q-co-ideal expansion. It is clear that we have $R = \cup\{q(\text{cl}(I)) : q \in Q\}$. Let $x \in q_1(\text{cl}(I)) \cap q_2(\text{cl}(I))$. So $x = q_1a_1 = q_2a_2$ for some $a_1, a_2 \in \text{cl}(I)$. Since I is a Q-co-ideal of R, there exists $q \in Q$ such that $xI \subseteq qI$. Since $a_1, a_2 \in \text{cl}(I)$, there exist $b_1, b_2 \in I$ such that $a_1b_1, a_2b_2 \in I$. Hence $xb_1 = q_1a_1b_1 \in q_1I \cap qI$ and $xb_2 = q_2a_2b_2 \in q_2I \cap qI$ for some $b_1, b_2 \in I$. Thus $q_1 = q = q_2$. □

3. δ-primary co-ideals

In this section, we investigate δ-primary co-ideals of a commutative semiring R which unify prime co-ideals and primary co-ideals of R.

Definition 3.1. Let R be a semiring and δ be a co-ideal expansion. A proper co-ideal I of a semiring R is called *δ-primary* if $a+b \in I$ and $a \notin I$, then $b \in \delta(I)$.

One can easily show that I is δ_1-primary if and only if it is a prime co-ideal of R and I is δ_2-primary if and only if I is a primary co-ideal of R.

Remark 3.2. Let I, J be co-ideals of the semiring R. The *co-ideal quotient* of I, J, denoted by (I : J), is the set $\{r \in R : r+J \subseteq I\} = \{r \in R : r+x \in I \text{ for all } x \in J\}$ such that (I : J) is closed under multiplication. For each $a \in R$, (I : a) denotes the set $\{r \in R : r + a \in I\}$ such that (I : a) is closed under multiplication. By [4, Lemma 2.4], $(I : J)$ is a co-ideal of R with $I \subseteq (I : J)$, and $(I : a)$ is a co-ideal of R for each $a \in R$. Also, by [4, Example 2.2], the condition "$(I : J)$ is closed under multiplication" is not superficial in the above definition. □

Theorem 3.3. *Let R be a semiring and δ be a co-ideal expansion.*

(1) *If P is a δ-primary co-ideal of R and $I \nsubseteq \delta(P)$, then $(P : I) = P$.*

(2) *For any δ-primary co-ideal P and any subset T of R, $(P : T)$ is δ-primary co-ideal of R.*

(3) *The union of any directed collection of δ-primary co-ideals is δ-primary.*

(4) *If $\delta(I) \subseteq \text{co-rad}(I)$ for every δ-primary co-ideal I, then $\delta(I) = \text{co-rad}(I)$.*

Proof. (1). It is clear that $P \subseteq (P : I)$. Let $x \in (P : I)$ and $a \in I \setminus \delta(P)$. So $x + a \in P$. Since P is δ-primary and $a \notin \delta(P)$, $x \in P$. So $(P : I) \subseteq P$, which gives $(P : I) = P$.

(2). Let $a + b \in (P : T)$ and $a \notin (P : T)$ for some $a, b \in R$. So $a + t \notin P$ and $a + b + t \in P$ for some $t \in T$. This implies $b \in \delta(P) \subseteq \delta(P : T)$ since P is δ-primary. Thus $(P : T)$ is a δ-primary co-ideal of R.

(3). Let $\sum = \{I_i : i \in D\}$ be a directed collection of primary co-ideals and $I = \cup_{i \in D} I_i$. Let $a + b \in I$ and $a \notin I$. So there is $i \in D$ such that $a + b \in I_i$ and $a \notin I_i$. So $b \in \delta(I_i) \subseteq \delta(I)$. Hence I is δ-primary.

(4). If $I = \text{co-rad}(I)$, then $\delta(I) = I = \text{co-rad}(I)$. Suppose $I \neq \text{co-rad}(I)$. Let $x \in \text{co-rad}(I)$. Then $nx \in I$ for some the least positive integer $n > 1$. Now $nx \in I$ and $(n-1)x \notin I$ gives $x \in \delta(I)$, and so we have equality. □

Definition 3.4. Let R be a semiring. A co-ideal expansion δ is said to be *intersection preserving* if $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for all co-ideals I, J of R.

Theorem 3.5. *Let R be a semiring.*

(1) *$\delta_1, \delta_2, \delta_3$ and δ_4 are intersection preserving co-ideal expansions.*

(2) *Assume that δ is an intersection preserving co-ideal expansion and let $Q_1, \ldots, Q_n$ be δ-primary co-ideals of R with $P = \delta(Q_i)$ for all $1 \leqslant i \leqslant n$. Then $Q = \bigcap_{i=1}^{n} Q_i$ is δ-primary.*

Proof. (1). It is clear that δ_1 is intersection preserving co-ideal expansion. By [4, Lemma 2.2], δ_2 is intersection preserving co-ideal expansion. We show that $\text{cl}(I \cap J) = \text{cl}(I) \cap \text{cl}(J)$. It is clear that $\text{cl}(I \cap J) \subseteq \text{cl}(I) \cap \text{cl}(J)$. Let $x \in \text{cl}(I) \cap \text{cl}(J)$. So there exist $a \in I$ and $b \in J$ such that $ax \in I$ and $bx \in J$. Since I, J are co-ideals of R, $a + b \in I \cap J$. Hence $x(a + b) = xa + xb \in I \cap J$, so $x \in \text{cl}(I \cap J)$. Thus $\text{cl}(I \cap J) = \text{cl}(I) \cap \text{cl}(J)$. By an argument like that in [2, Lemma 2.2], $\delta_4(I \cap J) = \delta_4(I) \cap \delta_4(J)$.

(2). Let $x + y \in Q$ and $x \notin Q$. So $x \notin Q_i$ for some $1 \leqslant i \leqslant n$. Since $x + y \in Q_i$ and Q_i is δ-primary, $y \in \delta(Q_i)$. As δ is intersection preserving, $\delta(Q) = \delta(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n} \delta(Q_i) = P$, we have $y \in \delta(Q)$. Thus Q is δ-primary. □

Definition 3.6. Let R be a semiring with co-ideal expansion δ. An element x of R is called *δ-co-nilpotent* if $x \in \delta(F(\{1\}))$.

Remark 3.7. Let I be a Q-strong co-ideal of a semiring R with a subtractive co-ideal expansion δ. Then an inspection will show that $\overline{\delta} : \mathrm{Id}(R/I) \to \mathrm{Id}(R/I)$ is a subtractive co-ideal expansion of R/I, where $\overline{\delta}(J/I) = \delta(J)/I$ for each co-ideal J/I of R/I (see [4, Theorem 3.4 and Theorem 3.5]). So $\overline{\delta}(\{q_eI\}) = \overline{\delta}(\{I\}) = \delta(I)/I$.

Theorem 3.8. *Let I be a Q-strong co-ideal of a semiring R with a subtractive co-ideal expansion δ. If J is a subtractive co-ideal of R with $J \supseteq I$, then J/I is a $\overline{\delta}$-primary co-ideal of R/I if and only if J is a δ-primary co-ideal of R.*

Proof. Suppose that J/I is a $\overline{\delta}$-primary co-ideal of R/I; we show J is a δ-primary co-ideal of R. Let $a + b \in J$ and $a \notin J$. Since I is a Q-strong co-ideal of R, there exist $q_1, q_2 \in Q$ such that $a \in q_1I$ and $b \in q_2I$. Let $q_1I \oplus q_2I = q_3I$, where q_3 is the unique element of Q such that $q_1I + q_2I \subseteq q_3I$. It follows that $a + b = q_3d \in J$ for some $d \in I$; so $q_3 \in J$ since J is subtractive; hence $q_1I \oplus q_2I = q_3I \in J/I$. Clearly, $q_1I \notin J/I$. Now J/I $\overline{\delta}$-primary gives, $q_2I \in \overline{\delta}(J/I) = \delta(J)/I$; so $q_2 \in \delta(J)$. Hence $b \in \delta(J)$.

Conversely, assume that J is a δ-primary co-ideal of R. We show J/I is $\overline{\delta}$-primary. Let $q_1I \oplus q_2I \in J/I$ and $q_1I \notin J/I$ (so $q_1 \notin J$). Let q_3 be the unique element of Q such that $q_1I \oplus q_2I = q_3I$, where $q_1I + q_2I \subseteq q_3I$. Since $q_3I \in J/I$, $q_3 \in J$. Therefore $q_1 + q_2 = q_3j \in J$ for some $j \in I$. As J is δ-primary and $q_1 \notin J$, $q_2 \in \delta(J)$. Therefore $q_2I \in \overline{\delta}(J/I) = \delta(J)/I$. Thus J/I is a $\overline{\delta}$-primary co-ideal of a semiring R. □

An element r of a commutative semiring R with identity is said to be *identity-summand* if there exists $1 \neq a \in R$ such that $r + a = 1$. The set of all identity-summand elements of R is denoted by $S(R)$.

Theorem 3.9. *Let I be a Q-strong co-ideal of a semiring R with a subtractive co-ideal expansion δ. Then the following statements are equivalent:*

(1) *I is δ-primary,*
(2) $S(R/I) \subseteq \{qI : q \in Q \cap \delta(I)\} = \delta(I)/I$,
(3) *every identity-summand of R/I is $\overline{\delta}$-co-nilpotent,*
(4) *$P/I \subseteq \delta(I)/I$ for every $P/I \in \min(R/I)$, where $\min(R/I)$ is the set of all minimal prime ideals of R/I.*

Proof. (1) $\Rightarrow$ (2). Let I be a δ-primary and $qI \in S(R/I)$. Hence there exists $I \neq q'I \in R/I$ such that $qI \oplus q'I = q_eI = I$; so $q + q' \in I$. Since I is δ-primary and $q' \notin I$, $q \in \delta(I)$. Thus $qI \in \delta(I)/I$.

(2) $\Rightarrow$ (3). If $qI \in S(R/I)$, then $qI \in \delta(I)/I$ by (2). By Remark 3.7, $\delta(I)/I = \overline{\delta}(\{q_eI\})$, which gives qI is $\overline{\delta}$-co-nilpotent.

(3) $\Rightarrow$ (1). Let $a + b \in I, a \notin I$. Since I is a Q-co-ideal of R, there exist $q_1, q_2 \in Q$ such that $a \in q_1I, b \in q_2I$. Let $q_1I \oplus q_2I = q_3I$, where q_3 is the unique element of Q such that $q_1I + q_2I \subseteq q_3I$. So $a + b \in q_3I \cap I$, which gives $q_3 = q_e$. Hence $q_1I \oplus q_2I = q_eI = I$. So q_2I is an identity summand element of R/I. Thus

q_2I is δ-co-nilpotent, hence $q_2I \in \bar{\delta}(\{q_eI\}) = \delta(I)/I$, which implies $q_2 \in \delta(I)$. Hence $b \in q_2I \subseteq \delta(I)$.

(1) $\Rightarrow$ (4). Let $P/I \in \min(R/I)$. At first, we show that $R/I \setminus P/I = (P/I)^c$ is a maximal additively closed subset of R/I with $q_eI \notin (P/I)^c$. Set

$$\sum = \{S : \ (P/I)^c \subseteq S, \ S \textit{ is an additively closed subset of } R/I \textit{ and } q_eI \notin S\}$$

by Zorn's Lemma, has a maximal element M. Obviously, $(P/I)^c \subseteq M$.

Consider the set

$$\Delta = \{L : \ L/I \textit{ is a co-ideal of } R/I \textit{ and } L/I \cap M = \emptyset\}.$$

Since $\{I\}$ is a co-ideal of R/I and $\{I\} \cap M = \emptyset$, $\Delta \neq \emptyset$. By Zorn's Lemma, Δ has maximal element T/I.

We show that T/I is prime. Let $q_1I \oplus q_2I \in T/I$ and $q_1I, q_2I \notin T/I$. Then $T/I \subsetneqq J_i = F(T/I \cup \{q_iI\})$. Thus $J_i \cap M \neq \emptyset$ for each $i = 1, 2$. Let $X_i \in J_i \cap M$ for each $i = 1, 2$. We show $J_1 \cap J_2 = T/I$. It is clear that $T/I \subseteq J_1 \cap J_2$. For $qI \in J_1 \cap J_2$ we have

$$qI = r_1I \oplus c_1I \odot (q_1I)^n = r_2I \oplus c_2I \odot (q_2I)^m$$

for some $r_1I, r_2I \in R/I$, $c_1I, c_2I \in T/I$ and $n, m \in \mathbb{N}$. Since

$$c_1I \odot (q_1I \oplus q_2I)^n = c_1I \odot (q_1I)^n \oplus (q_2I) \odot (tI) \in T/I$$

for some $tI \in R/I$, we have

$$qI \oplus q_2I \odot tI = r_1I \oplus c_1I \odot (q_1I)^n \oplus (q_2I) \odot (tI) \in T/I.$$

Hence $(q_2I) \odot (tI) \in (T/I : qI)$.

It can be easily checked that $(T/I : qI)$ is a co-ideal of R. So $q_2I \odot tI \oplus q_2I \in (T/I : qI)$. By Lemma 2.3, $q_eI \oplus tI = tI$, hence

$$q_2I = q_2I(q_eI \oplus tI) = q_2I \odot tI \oplus q_2I \in T/I.$$

Therefore $(c_2I) \odot (q_2I)^m \in (T/I : qI)$. So $qI = r_2I \oplus c_2I \odot (q_2I)^m \in (T/I : qI)$, because $(T/I : qI)$ is a co-ideal of R/I. Thus $qI \oplus qI = qI \in T/I$. Therefore $J_1 \cap J_2 = T/I$. Hence $X_1 + X_2 \in T/I \cap M$, a contradiction. Thus $q_1I \in T/I$ or $q_2I \in T/I$, which gives T/I is a prime co-ideal of R/I. Since $T/I \cap M = \emptyset$, $M \subseteq (T/I)^c$. So $(P/I)^c \subseteq M \subseteq (T/I)^c$, which implies $T/I \subseteq P/I$. Since $P/I \in \min(R/I)$, $T/I = P/I$. Thus $(P/I)^c$ is a maximal additively closed subset of R/I which $I \notin (P/I)^c$.

Now, let $qI \in P/I$. Then

$$T = \{q'I \oplus n(qI) : I \neq q'I \in (P/I)^c, \ n \in \mathbb{N} \cup \{0\}\}$$

is an additively closed subset of R/I which properly contains $(P/I)^c$. But we showed that $(P/I)^c$ is a maximal additively closed subset of R/I which $I \notin (P/I)^c$. So $I \in T$. Hence $q'I \oplus n(qI) = I$ for some $q'I \in (P/I)^c$, $n \in \mathbb{N}$. Thus $q'I \oplus qI = I$ by Lemma 2.3. So $q+q' \in I$. Since $q' \notin I$ and I is δ-primary, $q \in \delta(I)$ we conclude that $qI \in \delta(I)/I$.

(4) $\Rightarrow$ (1). Let $a+b \in I$ and $a \notin I$. Since I is a Q-strong co-ideal of R, there exist $q_1, q_2 \in Q$ such that $a \in q_1 I, b \in q_2 I$. Let $q_1 I \oplus q_2 I = q_3 I$. Since $a+b \in q_e I \cap q_3 I$, $q_3 = q_e$. So $q_1 I \oplus q_2 I = I$. It is clear that $I \in P/I$. So $q_1 I \oplus q_2 I \in P/I$. Hence $q_1 I \in P/I$ or $q_2 I \in P/I$. Since $P/I \subseteq \delta(I)/I$, $q_1 \in \delta(I)$ or $q_2 \in \delta(I)$. □

Theorem 3.10. *Let I be a Q-strong co-ideal of a semiring R with a subtractive co-ideal expansion δ. If I is δ-primary, then $P \subseteq \delta(I)$ for every subtractive co-ideal $P \in \min(I)$. The converse holds if $\min(R/I)$ is finite.*

Proof. At first we show that if P is subtractive and $P \in \min(I)$, then $P/I \in \min(R/I)$. Let T/I be a prime co-ideal of R/I and $T/I \subseteq P/I$. Since R/I is I-semiring, T/I is a subtractive co-ideal of R/I by [6, Proposition 2.5]. So $T/I = L/I$ where L is a subtractive prime co-ideal of R and $I \subseteq L$ by [4, Theorem 3.6, Theorem 3.7]. We show $L \subseteq P$. Let $x \in L$. Since I is a Q-co-ideal of R, $x = qa$ for some $q \in Q$ and $a \in I$. Because L is subtractive, $q \in L$. Thus $qI \in L/I \subseteq P/I$. Hence $q \in P$. So $x \in P$. Thus $L \subseteq P$ which implies $L = P$ because $P \in \min(I)$. Therefore $P/I = L/I = T/I$. Now, let $x \in P$. Since I is a Q-co-ideal of R, $x = qa$ for some $q \in Q$ and $a \in I$. Since P is subtractive $q \in P$. Hence $qI \in P/I$, where $P/I \in \min(R/I)$ by the above argument. Hence $qI \in \delta(I)/I$ by Theorem 3.9. Thus $q \in \delta(I)$, which gives $x \in \delta(I)$.

Conversely, by [6,Theorem 2.8], $I = \cap_\Lambda P_\alpha/I$, where $P_\alpha/I \in \min(R/I)$. By [6, Proposition 2.5], P_α/I is a subtractive co-ideal of R/I for each $\alpha \in \Lambda$. So $P_\alpha/I = Q\alpha/I$, where Q_α is a subtractive co-ideal of R and $I \subseteq Q_\alpha$. We show that $I = \cap_\Lambda Q_\alpha$. It is clear that $I \subseteq \cap_\Lambda Q_\alpha$. Let $x \in \cap_\Lambda Q_\alpha$. Since I is a Q-co-ideal of R, $x = qa$ for some $q \in Q$ and $a \in I$. So $q \in \cap_\Lambda Q_\alpha$, because $Q_\alpha^{,}$ s are subtractive co-ideals of R. Thus $qI \in \cap Q_\alpha/I = \cap P_\alpha/I = \{q_e I\}$, hence $q = q_e$ and $x \in I$. Therefore $I = \cap_\Lambda Q_\alpha$. Let $L \in \min(I)$. Hence $I = \cap_\Lambda Q_\alpha \subseteq L$. Since $\min(R/I)$ is finite, Λ is finite, which gives $Q_\alpha \subseteq L$, because Q_α is prime by [4, Theorem 3.7]. Thus $Q_\alpha = L$. Now, we show that I is δ-primary. Let $a+b \in I$ for some $a, b \in R$. Hence $a+b \in Q_\alpha$, where Q_α is a subtractive co-ideal and $Q_\alpha \in \min(I)$. By assumption, $Q_\alpha \subseteq \delta(I)$. Because Q_α is prime $a \in Q_\alpha \subseteq \delta(I)$ or $b \in Q_\alpha \subseteq \delta(I)$, which gives I is δ-primary. □

Definition 3.11. A co-ideal I of a semiring R with a co-ideal expansion δ is called a *δ-weakly primary* if $1 \neq a+b \in I$, then $a \in I$ or $b \in \delta(I)$ for each $a, b \in R$.

Theorem 3.12. *Let J be a subtractive co-ideal of an I-semiring R with a subtractive co-ideal expansion δ. Then the following are equivalent.*

(1) *J is δ-weakly primary.*

(2) *For each* $a \in R \setminus \delta(J)$, $(J : a) = J \cup (1 : a)$.
(3) $(J : a) = J$ *or* $(J : a) = (1 : a)$.

Proof. (1) $\Rightarrow$ (2). Let $a \in R \setminus \delta(J)$ and $b \in (J : a)$. Then $a + b \in J$. If $a + b = 1$, then $b \in (1 : a)$. If $a + b \neq 1$, then J δ-weakly primary gives $b \in J$. So $(J : a) \subseteq J \cup (1 : a)$. The converse inclusion is clear.

(2) $\Rightarrow$ (3). Let $(J : a) \neq J$ and $(J : a) \neq (1 : a)$. Then there exists $d \in (J : a)$ and $c \in (J : a)$ such that $d \notin J$ and $c \notin J$. Since J is subtractive, $(J : a)$ is a subtractive co-ideal of R, $cd \in (J : a)$. Therefore $cd \in J$ or $cd \in (1 : a)$. This implies that $c = cd + c \in J$ or $d = cd + d \in (1 : a)$, a contradiction.

(3) $\Rightarrow$ (1). Let $1 \neq a + b \in J$ and $a \notin \delta(J)$. Then $b \in (J : a) = J$. $\square$

Theorem 3.13. *Let R be an I-semiring with a subtractive co-ideal expansion δ. If J is a subtractive δ-weakly primary co-ideal of R which is not δ-primary, then $J = \{1\}$.*

Proof. Let $\{1\} \neq J$. We show that J is δ-primary co-ideal of R. Let $a + b \in J$. If $a + b \neq 1$, then J δ-weakly primary gives $a \in J$ or $b \in \delta(J)$. So we may assume that $a + b = 1$. As $J \neq \{1\}$, there exists $1 \neq c \in J$. So $1 \neq c = ac + bc \in J$ implies that $ac \in J$ or $bc \in \delta(J)$. As J and $\delta(J)$ are subtractive, $a \in J$ or $b \in \delta(J)$. Hence J is δ-primary, a contradiction. $\square$

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Faculty of Mathematical Sciences
University of Guilan
P.O.Box 1914, Rasht, Iran
E-mails: ebrahimi@guilan.ac.ir, saboura_dolati@yahoo.com, mehdikhoramdel@gmail.com