Green's relations and the relation $\mathcal{N}$ in Γ-semigroups

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Abstract. Let M be a Γ-semigroup. For a prime ideal I of M, let σ_I be the relation on M consisted of the pairs (x,y), where x and y are elements of M such that either both x and y are elements of I or both x and y are not elements of I. Let $\mathcal{N}$ be the semilattice congruence on M defined by $x\mathcal{N}y$ if and only if the filters of M generated by x and y coincide. Then the set $\mathcal{N}$ is the intersection of the relations σ_I, where I runs over the prime ideals of M. If $\mathcal{R}$, $\mathcal{L}$, $\mathcal{I}$, $\mathcal{H}$ are the Green's relations of M and $\mathcal{A}$ the set of right ideals, $\mathcal{B}$ the set of left ideals and $\mathcal{I}$ the set of ideals of M, then we have $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{I} \subseteq \mathcal{N}$, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{I} \subseteq \mathcal{N}$ and $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{I}$, $\mathcal{R} = \bigcap\limits_{I \in \mathcal{A}} \sigma_I$, $\mathcal{L} = \bigcap\limits_{Ix \in \mathcal{B}} \sigma_I$, $\mathcal{I} = \bigcap\limits_{I \in \mathcal{M}} \sigma_I$. The relation $\mathcal{R} \circ \mathcal{L}$ $(= \mathcal{L} \circ \mathcal{R})$ is the least -with respect to the inclusion relation- equivalence relation on M containing both $\mathcal{R}$ and $\mathcal{L}$. Finally, we characterize the Γ-semigroups which have only one $\mathcal{L}$ (or $\mathcal{R}$)-class or only one $\mathcal{I}$-class.

1. Introduction and prerequisites

An ideal I of a semigroup S is called *completely prime* if for any $a, b \in I$, $ab \in I$ implies that either $a \in I$ or $b \in I$. Every semilattice congruence on a semigroup S is the intersection of congruences σ_I where I is a completely prime ideal and for all $x, y \in S$, we have $x\sigma_I y$ if and only if $x, y \in I$ or $x, y \notin I$ [6]. For semigroups, ordered semigroups or ordered Γ-semigroups, we always use the terminology weakly prime, prime (subset) instead of the terminology prime, completely prime given by Petrich. For Green's relations in semigroups we refer to [1, 6]. For Green's relations in ordered semigroups, we refer to [2]. In the present paper we mainly present the analogous results of [2] in case of Γ-semigroups.

The concept of a Γ-semigroup has been introduced by M.K. Sen in 1981 as follows: If S and Γ are two nonempty sets, S is called a Γ-*semigroup* if the following assertions are satisfied: (1) $a\alpha b \in S$ and $\alpha a \beta \in \Gamma$ and (2) $(a\alpha b)\beta c = a(\alpha b \beta)c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$ [8]. In 1986, M.K. Sen and N.K. Saha changed that definition and gave the following definition of a Γ-semigroup: Given two nonempty sets M and Γ, M is called a Γ-semigroup if (1) $a\alpha b \in M$ and (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$ [9]. Later, in [7], Saha calls a nonempty set S a Γ-semigroup $(\Gamma \neq \emptyset)$ if there is a mapping

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$S \times \Gamma \times S \to S \mid (a, \gamma, b) \to a\gamma b$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$, and remarks that the most usual semigroup concepts, in particular regular and inverse Γ-semigroups have their analogous in Γ-semigroups. Although it was very convenient to work on the definition by Sen and Saha using binary relations [9], the uniqueness condition was missing from that definition. Which means that in an expression of the form, say $a\gamma b\mu c\xi d\rho e$ or $a\Gamma b\Gamma c\Gamma d\Gamma e$, it was not known where to put the parentheses. In that sense, the definition of a Γ-semigroup given by Saha in [7] was the right one. However, adding the uniqueness condition in the definition given by Sen and Saha in [9], we do not need to define it via mappings. The revised version of the definition by Sen and Saha in [9] has been introduced by Kehayopulu in [3] as follows:

For two nonempty sets M and Γ, define $M\Gamma M$ as the set of all elements of the form $m_1\gamma m_2$, where $m_1, m_2 \in M$, $\gamma \in \Gamma$. That is,

$$M\Gamma M := \{m_1\gamma m_2 \mid m_1, m_2 \in M, \gamma \in \Gamma\}.$$

Definition 1.1. Let M and Γ be two nonempty sets. The set M is called a Γ-*semigroup* if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$.

(2) If $m_1, m_2, m_3, m_4 \in M$, $\gamma_1, \gamma_2 \in \Gamma$ such that $m_1 = m_3$, $\gamma_1 = \gamma_2$ and $m_2 = m_4$, then $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.

(3) $(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$ for all $m_1, m_2, m_3 \in M$ and $\gamma_1, \gamma_2 \in \Gamma$.

In other words, Γ is a set of binary operations on M such that:

$(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$ for all $m_1, m_2, m_3 \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$.

According to that "associativity" relation, each of the elements $(m_1\gamma_1 m_2)\gamma_2 m_3$, and $m_1\gamma_1(m_2\gamma_2 m_3)$ is denoted by $m_1\gamma_1 m_2\gamma_2 m_3$.

Using conditions (1) – (3) one can prove that for an element of M of the form

$$m_1\gamma_1 m_2\gamma_2 m_3\gamma_3 m_4 \dots \gamma_{n-1} m_n \gamma_n m_{n+1},$$

or a subset of M of the form

$$m_1\Gamma_1 m_2\Gamma_2 m_3\Gamma_3 m_4 \dots \Gamma_{n-1} m_n \Gamma_n m_{n+1},$$

one can put a parenthesis in any expression beginning with some m_i and ending in some m_j [3, 4, 5].

The example below based on Definition 1.1 shows what a Γ-semigroup is.

Example 1.2. [4] Consider the two-elements set $M := \{a, b\}$, and let $\Gamma = \{\gamma, \mu\}$ be the set of two binary operations on M defined in the tables below:

γ	a	b
a	a	b
b	b	a

μ	a	b
a	b	a
b	a	b

One can check that $(x\rho y)\omega z = x\rho(y\omega z)$ for all $x, y, z \in M$ and all $\rho, \omega \in \Gamma$. So, M is a Γ-semigroup.

Example 1.3. [5] Consider the set $M := \{a, b, c, d, e\}$, and let $\Gamma = \{\gamma, \mu\}$ be the set of two binary operations on M defined in the tables below:

γ	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

μ	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

Since $(x\rho y)\omega z = x\rho(y\omega z)$ for all $x, y, z \in M$ and all $\rho, \omega \in \Gamma$, M is a Γ-semigroup.

Let now M be a Γ-semigroup. A nonempty subset A of M is called a *subsemigroup* of M if $A\Gamma A \subseteq A$, that is, if $a\gamma b \in A$ for every $a, b \in A$ and every $\gamma \in \Gamma$. A nonempty subset A of M is called a *left ideal* of M if $M\Gamma A \subseteq A$, that is, if $m \in M$, $\gamma \in \Gamma$ and $a \in A$, implies $m\gamma a \in A$. It is called a *right ideal* of M if $A\Gamma M \subseteq A$, that is, if $a \in A$, $\gamma \in \Gamma$ and $m \in M$, implies $a\gamma m \in A$. A is called an *ideal* of M if it is both a left and a right ideal of M. For an element a of M, we denote by $R(a)$, $L(a)$, $I(a)$, the right ideal, left ideal and the ideal of M, respectively, generated by a, and we have $R(a) = a \cup a\Gamma M$, $L(a) = a \cup M\Gamma a$, $I(a) = a \cup a\Gamma M \cup M\Gamma a \cup M\Gamma a\Gamma M$. An ideal A of M is called a *prime ideal* of M if $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in A$, then $a \in A$ or $b \in A$. Equivalently, if B and C are subsets of M such that $B \neq \emptyset$ (or $C \neq \emptyset$), $\gamma \in \Gamma$ and $B\gamma C \subseteq A$, then $B \subseteq A$ or $C \subseteq A$. A subsemigroup F of M is called a *filter* of M if $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in F$, implies $a \in F$ and $b \in F$. For an element a of M, we denote by $N(a)$ the filter of M generated by a and by $\mathcal{N}$ the equivalence relation on M defined by $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$. An ideal A of M is a prime ideal of M if and only if $M \backslash A = \emptyset$ or $M \backslash A$ is a subsemigroup of M. A nonempty subset F of M is a filter of M if and only if $M \backslash F = \emptyset$ or $M \backslash F$ is a prime ideal of M. An equivalence relation σ on M is called a *left congruence* on M if $(a, b) \in \sigma$ implies $(c\gamma a, c\gamma b) \in \sigma$ for every $c \in M$ and every $\gamma \in \Gamma$. It is called a *right congruence* on M if $(a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ for every $c \in M$ and every $\gamma \in \Gamma$. It is called a *congruence* on M if it is both a left and a right congruence on M. A *semilattice congruence* σ is a congruence on M such that

(1) $(a\gamma a, a) \in \sigma$ for every $a \in M$ and every $\gamma \in \Gamma$ and
(2) $(a\gamma b, b\gamma a) \in \sigma$ for every $a, b \in M$ and every $\gamma \in \Gamma$.

The relation $\mathcal{N}$ defined above is a semilattice congruence on M.

2. Main results

For a Γ-semigroup M, the Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{I}$, $\mathcal{H}$ are the equivalence relations on M defined by:

$\mathcal{R} = \{(x, y) \mid R(x) = R(y)\}$, $\qquad \mathcal{L} = \{(x, y) \mid L(x) = L(y)\}$,

$$\mathcal{I} = \{(x,y) \mid I(x) = I(y)\}, \qquad \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

The relation $\mathcal{R}$ is a left congruence and the relation $\mathcal{L}$ is a right congruence on M. Let now M be a Γ-semigroup. For a subset I of M we denote by σ_I the relation on M defined by

$$\sigma_I = \{(x,y) \mid x,y \in I \text{ or } x,y \notin I\}.$$

Exactly as in case of semigroups, for a Γ- semigroup the following holds:

Lemma 2.1. *Let M be a Γ-semigroup. If F is a filter of M, then*

($\star$) *$M \backslash F = \emptyset$ or $M \backslash F$ is a prime ideal of M.*

In particular, any nonempty subset F of M satisfying ($\star$) is a filter of M. □

Proposition 2.2. *Let M be a Γ-semigroup and I a prime ideal of M. Then the set σ_I is a semilattice congruence on M.*

Proof. Clearly σ_I is a relation on M which is reflexive and symmetric. Let $(a,b) \in \sigma_I$ and $(b,c) \in \sigma_I$. Then $a,b \in I$ or $a,b \notin I$ and $b,c \in I$ or $b,c \notin I$. If $a,b \in I$ and $b,c \in I$, then $a,c \in I$, so $(a,c) \in \sigma_I$. The case $a,b \in I$ and $b,c \notin I$ is impossible and so is the case $a,b \notin I$ and $b,c \in I$. If $a,b \notin I$ and $b,c \notin I$, then $a,c \notin I$, then $(a,c) \in \sigma_I$, and σ_I is transitive. Let $(a,b) \in \sigma_I$, $c \in M$ and $\gamma \in \Gamma$. Then $(a\gamma c, b\gamma c) \in \sigma_I$. Indeed: If $a,b \in I$ then, since I is an ideal of M, we have $a\gamma c$, $b\gamma c \in I$, so $(a\gamma c, b\gamma c) \in \sigma_I$. Let $a,b \notin I$. If $c \in I$ then, since I is an ideal of M, we have $a\gamma c$, $b\gamma c \in I$, so $(a\gamma c, b\gamma c) \in \sigma_I$. If $c \notin I$, then $a\gamma c, b\gamma c \notin I$. This is because if $a\gamma b \in I$ then, since I is a prime ideal of M, we have $a \in I$ or $c \in I$ which is impossible. For $b\gamma c \in I$, we also get a contradiction. Thus we obtain $(a\gamma c, b\gamma c) \in \sigma_I$, and σ_I is a right congruence on M. Similarly σ_I is a left congruence on M, so σ_I is a congruence on M.

σ_I is a semilattice congruence on M. In fact: Let $a \in M$ and $\gamma \in \Gamma$. Then $(a\gamma a, a) \in \sigma_I$. Indeed: If $a \notin I$, then $a\gamma a \notin I$. This is because if $a\gamma a \in I$ then, since I is a prime ideal of M, we have $a \in I$ which is impossible. Since $a, a\gamma a \notin I$, we have $(a, a\gamma a) \in \sigma_I$. If $a \in I$ then, since I is an ideal of M, we have $a\gamma a \in I$, so $(a, a\gamma a) \in \sigma_I$. Let now $a,b \in M$ and $\gamma \in \Gamma$. Then $(a\gamma b, b\gamma a) \in \sigma_I$. In fact: If $a\gamma b \in I$ then, since I is a prime ideal of M, we have $a \in I$ or $b \in I$. Then, since I is an ideal of M, we have $b\gamma a \in I$. Since $a\gamma b, b\gamma a \in I$, we have $(a\gamma b, b\gamma a) \in \sigma_I$. If $a\gamma b \notin I$, then $b\gamma a \notin I$. This is because if $b\gamma a \in I$ then, since I is a prime ideal of M, we have $b \in I$ or $a \in I$. Since I is an ideal of M, we have $a\gamma b \in I$ which is impossible. Since $a\gamma b, b\gamma a \notin I$, we have $(a\gamma b, b\gamma a) \in \sigma_I$. □

Theorem 2.3. *Let M be a Γ-semigroup and $\mathcal{P}(M)$ the set of prime ideals of M. Then*

$$\mathcal{N} = \bigcap_{I \in \mathcal{P}(M)} \sigma_I.$$

Proof. $\mathcal{N} \subseteq \sigma_I$ for every $I \in \mathcal{P}(M)$. In fact: Let $(a,b) \in \mathcal{N}$ and $I \in \mathcal{P}(M)$. Then $(a,b) \in \sigma_I$. Indeed: Let $(a,b) \notin \sigma_I$. Then $a \in I$ and $b \notin I$ or $a \notin I$ and $b \in I$. Let $a \in I$ and $b \notin I$. Since $b \in M \backslash I$, we have $\emptyset \neq M \backslash I \subseteq M$. Since

$M\backslash(M\backslash I) = I$ and I is a prime ideal of M, the set $M\backslash(M\backslash I)$ is a prime ideal of M. By Lemma 2.1, $M\backslash I$ is a filter of M. Since $b \in M\backslash I$, we have $N(b) \subseteq M\backslash I$. Since $N(a) = N(b)$, we have $a \in M\backslash I$ which is impossible. If $a \notin I$ and $b \in I$, we also get a contradiction.

Let now $(a,b) \in \sigma_I$ for every $I \in \mathcal{P}(M)$. Then $(a,b) \in \mathcal{N}$. In fact: Let $(a,b) \notin \mathcal{N}$. Then $N(a) \neq N(b)$, from which $a \notin N(b)$ or $b \notin N(a)$ (This is because if $a \in N(b)$ and $b \in N(a)$, then $N(a) \subseteq N(b) \subseteq N(a)$, so $N(a) = N(b)$). Let $a \notin N(b)$. Then $a \in M\backslash N(b)$. Since $b \in N(b)$, $b \notin M\backslash N(b)$. Since $a \in M\backslash N(b)$ and $b \notin M\backslash N(b)$, we have $(a,b) \notin \sigma_{M\backslash N(b)}$. Since $N(b)$ is a filter of M and $M\backslash N(b) \neq \emptyset$, by Lemma 2.1, $M\backslash N(b) \in \mathcal{P}(M)$. We have $M\backslash N(b) \in \mathcal{P}(M)$ and $(a,b) \notin \sigma_{M\backslash N(b)}$ which is impossible. If $b \notin N(a)$, by symmetry we get a contradiction. $\square$

For two relations ρ and σ on a set X, their composition $\rho \circ \sigma$ is defined by

$$\rho \circ \sigma = \{(a,b) \mid \exists\, x \in X : (a,x) \in \rho \text{ and } (x,b) \in \sigma\}.$$

If $\mathcal{B}_X$ is the set of relations on X, then the composition " $\circ$ " is an associative operation on $\mathcal{B}_X$, and so $(\mathcal{B}_X, \circ)$ is a semigroup.

Theorem 2.4. *Let M be a Γ-semigroup, $\mathcal{A}$ the set of right ideals, $\mathcal{B}$ the set of left ideals and $\mathcal{M}$ the set of ideals of M. Then we have*

(1) $\mathcal{R} = \bigcap\limits_{I \in \mathcal{A}} \sigma_I, \quad \mathcal{L} = \bigcap\limits_{I \in \mathcal{B}} \sigma_I, \quad \mathcal{I} = \bigcap\limits_{I \in \mathcal{M}} \sigma_I.$

(2) $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{I} \subseteq \mathcal{N}, \quad \mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{I}\ (\subseteq \mathcal{N})$ *and* $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{I}$.

(3) *In particular, if M is commutative, then* $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{I} = \mathcal{L} \circ \mathcal{R}$.

Proof. (1). Let $(x,y) \in \mathcal{R}$ and $I \in \mathcal{A}$. If $x \in I$, then

$$y \in R(y) = R(x) = x \cup x\Gamma M \subseteq I \cup I\Gamma M = I,$$

so $y \in I$. Then $x, y \in I$, and $(x,y) \in \sigma_I$. If $x \notin I$, then $y \notin I$. This is because $y \in I$ implies $x \in I$ which is impossible. Since $x, y \notin I$, we have $(x,y) \in \sigma_I$. Let now $(x,y) \in \sigma_I$ for every $I \in \mathcal{A}$. Since $x \in R(x)$ and $(x,y) \in \sigma_{R(x)}$, we have $y \in R(x)$, then $R(y) \subseteq R(x)$. Since $y \in R(y)$ and $(x,y) \in \sigma_{R(y)}$, we have $x \in R(y)$, so $R(x) \subseteq R(y)$. Then $R(x) = R(y)$, and $(x,y) \in \mathcal{R}$. The rest of the proof is similar.

(2). Let $(x,y) \in \mathcal{R}$. Then $R(x) = R(y)$, so $x \cup x\Gamma M = y \cup y\Gamma M$. Then

$$M\Gamma(x \cup x\Gamma M) = M\Gamma(y \cup y\Gamma M),$$

and $M\Gamma x \cup M\Gamma x\Gamma M = M\Gamma y \cup M\Gamma y\Gamma M$. Then we have

$$I(x) = x \cup x\Gamma M \cup M\Gamma x \cup M\Gamma x\Gamma M = y \cup y\Gamma M \cup M\Gamma y \cup M\Gamma y\Gamma M = I(y),$$

and $(x,y) \in \mathcal{I}$. Moreover, $\mathcal{I} \subseteq \mathcal{N}$. Indeed: By Theorem 2.3, $\mathcal{N} = \bigcap\limits_{I \in \mathcal{P}(S)} \sigma_I$, where $\mathcal{P}(S)$ is the set of prime ideals of M. Since $\mathcal{P}(M) \subseteq \mathcal{M}$, by (1), we have

$$\mathcal{I} = \bigcap_{I \in \mathcal{M}} \sigma_I \subseteq \bigcap_{I \in \mathcal{P}(S)} \sigma_I = \mathcal{N}.$$

$\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{I}$. In fact: If $(a, b) \in \mathcal{L} \circ \mathcal{R}$, then there exists $c \in M$ such that $(a, c) \in \mathcal{L}$ and $(c, b) \in \mathcal{R}$. Then $L(a) = L(c)$ and $R(c) = R(b)$, $a \in c \cup M\Gamma c$ and $c \in b \cup b\Gamma M$. Then we get $a \in b \cup b\Gamma M \cup M\Gamma(b \cup b\Gamma M) = b \cup b\Gamma M \cup M\Gamma b \cup M\Gamma b\Gamma M = I(b)$, and so $I(a) \subseteq I(b)$. Since $(b, c) \in \mathcal{R}$ and $(c, a) \in \mathcal{L}$, we have

$$b \in c \cup c\Gamma M \subseteq a \cup M\Gamma a \cup (a \cup M\Gamma a)\Gamma M = a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M = I(a),$$

and $I(b) \subseteq I(a)$. Then $I(a) = I(b)$, and $(a, b) \in \mathcal{I}$.

(3). Let now M be commutative. Then we have

$$\begin{aligned}(a, b) \in \mathcal{L} &\Longleftrightarrow L(a) = L(b) \Longleftrightarrow a \cup M\Gamma a = b \cup M\Gamma b\\ &\Longleftrightarrow a \cup a\Gamma M = b \cup b\Gamma M \Longleftrightarrow (a, b) \in \mathcal{R}.\end{aligned}$$

$\mathcal{I} \subseteq \mathcal{H}$. Indeed:

$$\begin{aligned}(a, b) \in I &\Longrightarrow I(a) = I(b)\\ &\Longrightarrow a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M = b \cup M\Gamma b \cup b\Gamma M \cup M\Gamma b\Gamma M\\ &\Longrightarrow a \cup M\Gamma a \cup M\Gamma M\Gamma a = b \cup M\Gamma b \cup M\Gamma M\Gamma b\\ &\Longrightarrow a \cup M\Gamma a = b \cup M\Gamma b \Longrightarrow L(a) = L(b) \Longrightarrow (a, b) \in \mathcal{L} = \mathcal{R} = \mathcal{H}.\end{aligned}$$

Since $\mathcal{I} \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{I}$ (by (2)), we have $\mathcal{H} = \mathcal{I}$.

$\mathcal{I} \subseteq \mathcal{L} \circ \mathcal{R}$. Indeed: If $(a, b) \in \mathcal{I}$, then $\mathcal{I} = \mathcal{L} = \mathcal{R}$. Since $(a, b) \in \mathcal{L}$ and $(a, b) \in \mathcal{R}$, we have $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Besides, by (2), $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{I}$. Thus we get $\mathcal{I} = \mathcal{L} \circ \mathcal{R}$. □

Corollary 2.5. *Let M be a Γ-semigroup, A a right ideal, B a left ideal and I an ideal of M. Then*

$$A = \bigcup_{x \in A} (x)_{\mathcal{R}},\ B = \bigcup_{x \in B} (x)_{\mathcal{L}},\ I = \bigcup_{x \in I} (x)_{\mathcal{I}}.$$

Proof. Let A be a right ideal of M. If $t \in A$, then $t \in (t)_{\mathcal{R}} \subseteq \bigcup_{x \in A} (x)_{\mathcal{R}}$. Let $t \in (x)_{\mathcal{R}}$ for every $x \in A$. Then, by Theorem 2.4, we have $(t, x) \in \mathcal{R} = \bigcap_{I \in \mathcal{A}} \sigma_I$. Since $(t, x) \in \sigma_A$ and $x \in A$, we have $t \in A$. The proof of the rest is similar. □

Finally, we prove that the relation $\mathcal{R} \circ \mathcal{L}$, which is equal to $\mathcal{L} \circ \mathcal{R}$, is the least – with respect to the inclusion relation – equivalence relation on M containing both $\mathcal{R}$ and $\mathcal{L}$.

For a set X, denote by $E(X)$ the set of equivalence relations on X and by $\sup_{E(X)}\{\rho, \sigma\}$ the supremum of ρ and σ in $E(X)$.

Lemma 2.6. *If ρ and σ are equivalence relations on a set X such that $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \circ \sigma$ is also an equivalence relation on X and $\rho \circ \sigma = \sup_{E(X)}\{\rho, \sigma\}$.* □

Lemma 2.7. *If ρ and σ are symmetric relations on a set X such that $\rho \circ \sigma \subseteq \sigma \circ \rho$, then $\rho \circ \sigma = \sigma \circ \rho$.* □

Theorem 2.8. *If M is a Γ-semigroup, then $\mathcal{R} \circ \mathcal{L} = sup_{E(M)}\{\mathcal{R}, \mathcal{L}\}$.*

Proof. We prove that $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, then the rest of the proof is a consequence of Lemma 2.6. According to Lemma 2.7, it is enough to prove that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. Let $(a, b) \in \mathcal{R} \circ \mathcal{L}$. Then there exists $c \in M$ such that $(a, c) \in \mathcal{R}$ and $(c, b) \in \mathcal{L}$. Since $R(a) = R(c)$ and $L(c) = L(b)$, we have $a \in c \cup c\Gamma M$ and $b \in c \cup M\Gamma c$. Then $a = c$ or $a = c\gamma x$ and $b = c$ or $b = y\mu c$ for some $x, y \in M$, $\gamma, \mu \in \Gamma$.

We consider the cases:

(A) Let $a = c$ and $b = c$. Then $(a, b) = (c, c)$. Since $c \in M$, $(c, c) \in \mathcal{L}$ and $(c, c) \in \mathcal{R}$, we have $(c, c) \in \mathcal{L} \circ \mathcal{R}$. So $(a, b) \in \mathcal{L} \circ \mathcal{R}$.

(B) Let $a = c$ and $b = y\mu c$ for some $y \in M$, $\mu \in \Gamma$. Then $(a, b) = (c, y\mu c)$. Since $(b, b) \in \mathcal{R}$, we have $(b, y\mu c) \in \mathcal{R}$. Since $c \in M$, $(c, b) \in \mathcal{L}$ and $(b, y\mu c) \in \mathcal{R}$, we have $(c, y\mu c) \in \mathcal{L} \circ \mathcal{R}$, so $(a, b) \in \mathcal{L} \circ \mathcal{R}$.

(C) Let $a = c\gamma x$ for some $\gamma \in \Gamma$, $x \in M$ and $b = c$. Then $(a, b) = (c\gamma x, c)$. Since $(a, a) \in \mathcal{L}$, we have $(c\gamma x, a) \in \mathcal{L}$. Since $a \in M$, $(c\gamma x, a) \in \mathcal{L}$ and $(a, c) \in \mathcal{R}$, we have $(c\gamma x, c) \in \mathcal{L} \circ \mathcal{R}$, so $(a, b) \in \mathcal{L} \circ \mathcal{R}$.

(D) Let $a = c\gamma x$ and $b = y\mu c$ for some $x, y \in M$, $\gamma, \mu \in \Gamma$. Then $(a, b) = (c\gamma x, y\mu c) \in \mathcal{L} \circ \mathcal{R}$. Indeed: We have $b\gamma x = y\mu c\gamma x = y\mu a$. Since $(c, b) \in \mathcal{L}$ and $\mathcal{L}$ is a right congruence on M, we have $(c\gamma x, b\gamma x) \in \mathcal{L}$. Since $(a, c) \in \mathcal{R}$ and $\mathcal{R}$ is a left congruence on M, we have $(y\mu a, y\mu c) \in \mathcal{R}$, so $(b\gamma x, y\mu c) \in \mathcal{R}$. Since $b\gamma x \in M$, $(c\gamma x, b\gamma x) \in \mathcal{L}$ and $(b\gamma x, y\mu c) \in \mathcal{R}$, we have $(c\gamma x, y\mu c) \in \mathcal{L} \circ \mathcal{R}$. □

Each Γ-semigroup M has an $\mathcal{L}$-class, an $\mathcal{R}$-class, and an $\mathcal{I}$-class. The set M is nonempty and, for each $x \in M$, $(x)_{\mathcal{L}}$ is a nonempty $\mathcal{L}$-class of M, $(x)_{\mathcal{R}}$ is a nonempty $\mathcal{R}$-class of M and $(x)_{\mathcal{I}}$ is a nonempty $\mathcal{I}$-class of M.

Definition 2.9. A Γ-semigroup M is called *left* (resp. *right*) *simple* if M has only one $\mathcal{L}$ (resp. $\mathcal{R}$)-class. M called *simple* if M has only one $\mathcal{I}$-class.

A right ideal, left ideal or ideal A of a Γ-semigroup M is called *proper* if $A \neq M$.

By Theorem 2.4, we have the following:

Corollary 2.10. *A Γ-semigroup M is left* (*resp. right*) *simple if and only if M does not contain proper left* (*resp. right*) *ideals. M is simple if and only if does not contain proper ideals.*

Proof. (⇒) Let M be left simple, A a left ideal of M and $x \in M$. Then $x \in A$. Indeed: Suppose $x \notin A$. Take an element $a \in A\,(A \neq \emptyset)$. Since $(x, a) \notin \sigma_A$, by Theorem 2.4(1), we have $(x, a) \notin \mathcal{L}$. Then $x \neq a$ and $(x)_{\mathcal{L}} \neq (a)_{\mathcal{L}}$ which is impossible.
(⇐) Suppose M does not contain proper left ideals. Let $x \in M$ ($M \neq \emptyset$). Then, for each $t \in M$ such that $t \neq x$, we have $(t)_{\mathcal{L}} = (x)_{\mathcal{L}}$. In fact: Let $t \in M$, $t \neq x$. By the assumption, we have $L(x) = M$ and $L(t) = M$, then $(x, t) \in \mathcal{L}$, so $(t)_{\mathcal{L}} = (x)_{\mathcal{L}}$. Then $(x)_{\mathcal{L}}$ is the only $\mathcal{L}$-class of M, and M is left simple. The other cases are proved in a similar way. □

Corollary 2.11. *Let M be a Γ-semigroup. Then M is left* (*resp. right*) *simple if and only if $M\Gamma a = M$* (*resp. $a\Gamma M = M$*) *for every $a \in M$. M is simple if and only if $M\Gamma a\Gamma M = M$ for every $A \subseteq M$.*

Proof. Let M be left simple and $a \in M$. Since $M\Gamma a$ is a left ideal of M, by Corollary 2.10, we have $M\Gamma a = M$. Conversely, let $M\Gamma a = M$ for every $a \in M$ and A a left ideal of M. Take an element $x \in A$ $(A \neq \emptyset)$. Then $M = M\Gamma x \subseteq M\Gamma A \subseteq A$, so $A = M$. By Corollary 2.10, M is left simple. □

Remark 2.12. If M is a Γ-semigroup, then we have $M\Gamma a = M$ for every $a \in M$ if and only if $M\Gamma A = M$ for every nonempty subset A of M. We have $a\Gamma M = M$ for every $a \in M$ if and only if $A\Gamma M = M$ for every nonempty subset A of M. Also $M\Gamma a\Gamma M = M$ for every $a \in M$ if and only if $M\Gamma A\Gamma M = M$ for every nonempty subset A of M. Let us prove the third one: $\Rightarrow$. Let $a \in M$. Since $\{a\} \subseteq M$, by hypothesis, we have $M\Gamma\{a\}\Gamma M = M$, so $M\Gamma a\Gamma M = M$. $\Leftarrow$. Let $\emptyset \neq A \subseteq M$. Take an element $a \in A$. By hypothesis, we have $M = M\Gamma a\Gamma M \subseteq M\Gamma A\Gamma M \subseteq (M\Gamma M)\Gamma M \subseteq M\Gamma M \subseteq M$, so $M\Gamma A\Gamma M = M$.

Conclusion. In this paper we mainly gave the analogous results of [3] in case of Γ-semigroups. Analogous results of [3] for ordered Γ-semigroups can be also obtained. If we want to get a result on a Γ-semigroup or an ordered Γ semigroup, then we have to prove it first on a semigroup or on an ordered semigroup, respectively. We never work directly in Γ-semigroups or in ordered Γ-semigroups. The paper serves as an example to show the way we pass from semigroups to Γ-semigroups (also from ordered semigroups to ordered Γ-semigroups).

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