Quasigroups and Related Systems **22** (2014), 147 − 158

Pentagonal quasigroups

Stipe Vidak

Abstract. The concept of pentagonal quasigroup is introduced as IM-quasigroup satisfying the additional property of pentagonality. Some basic identities which are valid in a general pentagonal quasigroup are proved. Four different models for pentagonal quasigroups and their mutual relations are studied. Geometric interpretations of some properties and identities are given in the model $C(q)$, where q is a solution of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$.

1. Introduction

A quasigroup $(Q, \cdot)$ is called *IM-quasigroup* if it satisfies the identities of *idempotency* and *mediality*:

$$aa = a \tag{1}$$

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

Immediate consequences of these identities are the identities known as *elasticity*, *left distributivity* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{3}$$

$$a \cdot bc = ab \cdot ac, \tag{4}$$

$$ab \cdot c = ac \cdot bc. \tag{5}$$

Adding an additional identity to identities of idempotency and mediality some interesting subclasses of IM-quasigroups can be defined. For example, adding the identity $a(ab \cdot b) = b$ *golden section quasigroup* or *GS-quasigroup* is defined (see [9], [2]). Adding the identity of *semi-symmetricity*, $ab \cdot a = b$, *hexagonal quasigroup* is defined (see [10], [1]).

In this paper we study IM-quasigroups satisfying the identity of *pentagonality*:

$$(ab \cdot a)b \cdot a = b. \tag{6}$$

Such quasigroups are called *pentagonal quasigroups*.

2010 Mathematics Subject Classification: 20N05
Keywords: IM-quasigroup, regular pentagon.

Example 1.1. Let $(F,+,\cdot)$ be a field such that the equation

$$q^4-3q^3+4q^2-2q+1=0 \tag{7}$$

has a solution in F. If q is a solution of (7), we define binary operation $*$ on F by

$$a*b=(1-q)a+qb. \tag{8}$$

Then $(F,*)$ is a pentagonal quasigroup.

Idempoteny follows trivially:

$$a*a=(1-q)a+qa=a.$$

To prove mediality, we write

$$\begin{aligned}(a*b)*(c*d)&=((1-q)a+qb)*((1-q)c+qd)\\&=(1-q)((1-q)a+qb)+q((1-q)c+qd)\\&=(1-q)^2a+q(1-q)b+q(1-q)c+q^2d.\end{aligned}$$

This expression remains unchanged applying $b\leftrightarrow c$ and we conclude that

$$(a*b)*(c*d)=(a*c)*(b*d).$$

Since

$$\begin{aligned}(((a*b)*a)*b)*a&=(1-q)((a*b)*a)b+qa\\&=(1-q)((1-q)((a*b)*a)+qb)+qa\\&=(1-q)^2((a*b)*a)+(1-q)qb+qa\\&=(1-q)^2((1-q)(a*b)+qa)+(1-q)qb+qa\\&=(1-q)^3(a*b)+(1-q)^2qa+(1-q)qb+qa\\&=(1-q)^4a+(1-q)^3qb+(1-q)^2qa+(1-q)qb+qa\\&=(q^4-3q^3+4q^2-2q+1)a+(-q^4+3q^3-4q^2+2q)b,\end{aligned}$$

using (7) we get

$$(((a*b)*a)*b)*a=b,$$

which proves pentagonality.

Example 1.2. We put $F=\mathbb{C}$ in the previous example, and q is a solution of the equation (7). Since we are in the set $\mathbb{C}$, that equation has four complex solutions. These are:

$$\begin{aligned}q_{1,2}&=\frac{1}{4}(3+\sqrt{5}\pm i\sqrt{2(5+\sqrt{5})})i,\\q_{3,4}&=\frac{1}{4}(3-\sqrt{5}\pm i\sqrt{2(5-\sqrt{5})}).\end{aligned}$$

Now $C(q) = (\mathbb{C}, *)$, where $q \in \{q_1, q_2, q_3, q_4\}$, and $*$ is defined by

$$a * b = (1 - q)a + qb$$

is also a pentagonal quasigroup.

Previous example $C(q)$ motivates the introduction of many geometric concepts in pentagonal quasigroups. We can regard elements of the set $\mathbb{C}$ as points of the Euclidean plane. For any two different points $a, b \in \mathbb{C}$ the equality (8) can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0}.$$

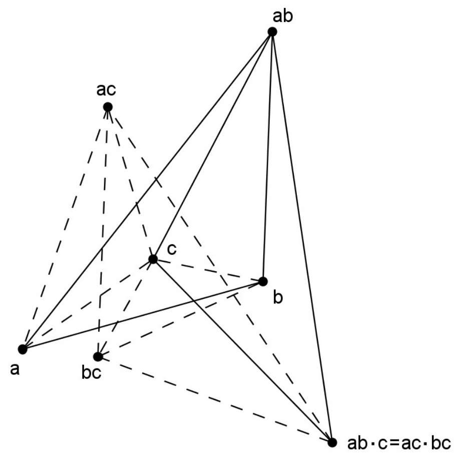

Figure 1. Right distributivity (5) in $C(q_1)$

That means that the points a, b and $a * b$ are vertices of a triangle directly similar to the triangle with vertices 0, 1 and q. In $C(q_1)$ the point $a * b$ is the third vertex of the regular pentagon determined by adjacent vertices a and b. Any identity in the pentagonal quasigroup $C(q) = (\mathbb{C}, *)$ can be interpreted as a theorem of the Euclidean geometry which can be proved directly, but the theory of pentagonal quasigroups gives a better insight into the mutual relations of such theorems. Figure 1 gives an illustration of the right distributivity (5).

In this paper we study different identities in pentagonal quasigroups and their mutual relations. We prove Toyoda-like representation theorem for pentagonal quasigroups, where they are caracterized in terms of Abelian groups with a certain type of automorphism. In the last section, motivated by quasigroups $C(q_i)$, $i = 1, 2, 3, 4$, we study four different models for pentagonal quasigroups.

2. Basic properties and identities

In pentagonal quasigroups, along with pentagonality and the identities which are valid in any IM-quasigroup, some other very useful identities hold.

Theorem 2.1. *In the IM-quasigroup* $(Q,\cdot)$ *identity* (6) *and the identities*

$$(ab\cdot a)c\cdot a = bc\cdot b, \tag{9}$$

$$(ab\cdot a)a\cdot a = ba\cdot b, \tag{10}$$

$$ab\cdot(ba\cdot a)a = b \tag{11}$$

are mutually equivalent and they imply the identity

$$a(b\cdot(ba\cdot a)a)\cdot a = b \tag{12}$$

for every a, b, $c \in Q$.

Proof. First, we prove (6) $\Leftrightarrow$ (9). We have

$$\begin{aligned} bc\cdot b &\overset{(6)}{=} bc\cdot((ab\cdot a)b\cdot a) \overset{(2)}{=} (b\cdot(ab\cdot a)b)\cdot ca \overset{(3)}{=} (b(ab\cdot a)\cdot b)\cdot ca \\ &\overset{(2)}{=} (b(ab\cdot a)\cdot c)\cdot ba \overset{(5)}{=} (bc\cdot(ab\cdot a)c)\cdot ba \overset{(2)}{=} (bc\cdot b)\cdot((ab\cdot a)c\cdot a) \end{aligned}$$

Since we have $bc\cdot b \overset{(1)}{=} (bc\cdot b)\cdot(bc\cdot b)$ and

$$(bc\cdot b)\cdot((ab\cdot a)c\cdot a) = (bc\cdot b)\cdot(bc\cdot b),$$

using cancellation in the quasigroup we get $(ab\cdot a)c\cdot a = bc\cdot b$.

Then, we prove (6) $\Leftrightarrow$ (10). We have

$$\begin{aligned} ba\cdot b &\overset{(6)}{=} ba\cdot((ab\cdot a)b\cdot a) \overset{(2)}{=} (b\cdot(ab\cdot a)b)\cdot aa \overset{(3)}{=} (b(ab\cdot a)\cdot b)\cdot aa \\ &\overset{(2)}{=} (b(ab\cdot a)\cdot a)\cdot ba \overset{(5)}{=} (ba\cdot(ab\cdot a)a)\cdot ba \overset{(2)}{=} (ba\cdot b)\cdot((ab\cdot a)a\cdot a) \end{aligned}$$

Since we have $ba\cdot b \overset{(1)}{=} (ba\cdot b)\cdot(ba\cdot b)$ and

$$(ba\cdot b)\cdot((ab\cdot a)a\cdot a) = (ba\cdot b)\cdot(ba\cdot b),$$

cancellation again gives $(ab\cdot a)a\cdot a = ba\cdot b$.

Next, we prove (6) $\Leftrightarrow$ (11):

$$ab\cdot(ba\cdot a)a \overset{(2)}{=} a(ba\cdot a)\cdot ba \overset{(3)}{=} (ab\cdot a)a\cdot ba \overset{(5)}{=} (ab\cdot a)b\cdot a.$$

It remains to prove (6), (10) $\Rightarrow$ (12). We get successively:

$$\begin{aligned} a(b\cdot(ba\cdot a)a)\cdot a &\overset{(4)}{=} (ab\cdot(a\cdot(ba\cdot a)a))a \overset{(3)}{=} (ab\cdot((ab\cdot a)a\cdot a))a \\ &\overset{(10)}{=} (ab\cdot(ba\cdot b))a \overset{(5)}{=} (a\cdot ba)b\cdot a \overset{(3)}{=} (ab\cdot a)b\cdot a \overset{(6)}{=} b. \qquad \square \end{aligned}$$

The identity (9) generalises identity (6), so it is called *generalised pentagonality.*

In a pentagonal quasigroup $(Q,\cdot)$ it is often very useful to know how to "solve the equations" of the types $ax = b$ and $ya = b$ for given a, $b \in Q$. The next theorem follows immediately from the identities (12) and (6).

Theorem 2.2. *In the pentagonal quasigroup $(Q,\cdot)$ for a, $b \in Q$ the following implications hold:*

$$\begin{aligned} ax = b \quad &\Rightarrow \quad x = (b\cdot(ba\cdot a)a)a, \\ ya = b \quad &\Rightarrow \quad y = (ab\cdot a)b. \qquad \square \end{aligned}$$

3. Representation theorem

A more general example of the pentagonal quasigroup $C(q)$, where q is a solution of the equation (7) can be obtained by taking an Abelian group $(Q,+)$ with an automorphism φ which satisfies

$$\varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + \mathbf{1} = 0. \tag{13}$$

The equation $ax = b$ is equivalent with

$$\begin{aligned} a + \varphi(x-a) &= b, \\ \varphi(x) &= \varphi(a) + b - a, \\ x &= a + \varphi^{-1}(b-a), \end{aligned}$$

which means that $ax = b$ has the unique solution.
The equation $ya = b$ is equivalent with

$$\begin{aligned} y + \varphi(a-y) &= b, \\ y - \varphi(y) &= b - \varphi(a). \end{aligned}$$

Let us check that $y_0 = a + 2\varphi(b-a) - 2\varphi^2(b-a) + \varphi^3(b-a)$ satisfies the last equality:

$$\begin{aligned} &y_0 - \varphi(y_0) = \\ &\quad = a + 2\varphi(b) - 2\varphi(a) - 2\varphi^2(b) + 2\varphi^2(a) + \varphi^3(b) - \varphi^3(a) \\ &\quad -\varphi(a) - 2\varphi^2(b) + 2\varphi^2(a) + 2\varphi^3(b) - 2\varphi^3(a) - \varphi^4(b) + \varphi^4(a) \\ &\quad = (a - 3\varphi(a) + 4\varphi^2(a) - 3\varphi^3(a) + \varphi^4(a)) + (2\varphi(b) - 4\varphi^2(b) + 3\varphi^3(b) - \varphi^4(b)) \\ &\quad \overset{(1)}{=} -\varphi(a) + b = b - \varphi(a). \end{aligned}$$

Now let us assume that there exist $y_1, y_2 \in Q$ such that $y_1 a = b$ and $y_2 a = b$. That means that we have

$$\begin{aligned} y_1 a &= y_2 a \\ y_1 + \varphi(y_1) - \varphi(a) &= y_2 + \varphi(y_2) - \varphi(a) \\ (\mathbf{1} + \varphi)(y_1) &= (\mathbf{1} + \varphi)(y_2). \end{aligned}$$

Applying automorphism φ and multiplying by constants we get

$$\begin{aligned} (\mathbf{1} + \varphi)(y_1) &= (\mathbf{1} + \varphi)(y_2) \\ -3(\varphi + \varphi^2)(y_1) &= -3(\varphi + \varphi^2)(y_2) \\ 7(\varphi^2 + \varphi^3)(y_1) &= 7(\varphi^2 + \varphi^3)(y_2) \\ -10(\varphi^3 + \varphi^4)(y_1) &= -10(\varphi^3 + \varphi^4)(y_2) \\ 11(\varphi^4 + \varphi^5)(y_1) &= 11(\varphi^4 + \varphi^5)(y_2). \end{aligned}$$

Adding up all these equalities we get

$$(\mathbf{1} - 2\varphi + 4\varphi^2 - 3\varphi^3 + \varphi^4 + 11\varphi^5)(y_1) = (\mathbf{1} - 2\varphi + 4\varphi^2 - 3\varphi^3 + \varphi^4 + 11\varphi^5)(y_2),$$

from which using (13) and dividing by 11 we get $\varphi^5(y_1) = \varphi^5(y_2)$. Since φ is an automorphism, so is φ^5, and we can conclude $y_1 = y_2$. That shows that the equation $ya = b$ has the unique solution. Hence, $(Q, \cdot)$ is a quasigroup.
Since $a \cdot a = a + \varphi(0) = a$, idempotency is valid. Moreover,

$$\begin{aligned} ab \cdot cd &= (a + \varphi(b - a)) \cdot (c + \varphi(d - c)) \\ &= a + \varphi(b - a) + \varphi(c + \varphi(d - c) - (a + \varphi(b - a))) \\ &= a + \varphi(b - a + c - a) + \varphi(\varphi(d - c - b + a)) \end{aligned}$$

Interchanging b and c that expression remains unchanged, which gives mediality.
If we put $a \cdot b = a + \varphi(b - a)$, we get successively:

$$\begin{aligned} ab \cdot a &= a + \varphi(b - a) + \varphi(a - a - \varphi(b - a)) \\ &= a + \varphi(b - a) - \varphi^2(b - a), \end{aligned}$$

$$\begin{aligned} (ab \cdot a)b &= a + \varphi(b - a) - \varphi^2(b - a) + \varphi(b - a - \varphi(b - a) + \varphi^2(b - a)) \\ &= a + \varphi(b - a) - \varphi^2(b - a) + \varphi(b - a) - \varphi^2(b - a) + \varphi^3(b - a) \\ &= a + 2\varphi(b - a) - 2\varphi^2(b - a) + \varphi^3(b - a), \end{aligned}$$

$$\begin{aligned} (ab \cdot a)b \cdot a &= a + 2\varphi(b - a) - 2\varphi^2(b - a) + \varphi^3(b - a) \\ &\quad + \varphi(a - a - 2\varphi(b - a) + 2\varphi^2(b - a) - \varphi^3(b - a)) \\ &= a + 2\varphi(b - a) - 2\varphi^2(b - a) + \varphi^3(b - a) - 2\varphi^2(b - a) \\ &\quad + 2\varphi^3(b - a) - \varphi^4(b - a) \\ &= a + 2\varphi(b - a) - 4\varphi^2(b - a) + 3\varphi^3(b - a) - \varphi^4(b - a) \overset{(13)}{=} b. \end{aligned}$$

That proves pentagonality in $(Q,\cdot)$.

Based on Toyoda's representation theorem [4], next theorem shows that this is in fact the most general example of pentagonal quasigroups.

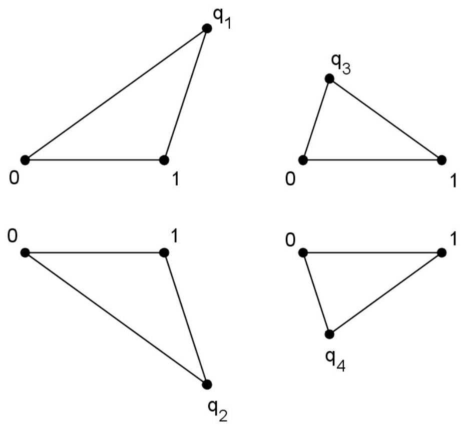

Figure 2. Four characteristic triangles for pentagonal quasigroups

Theorem 3.1. *For every pentagonal quasigroup* $(Q,\cdot)$ *there is an Abelian group* $(Q,+)$ *with an automorphism* φ *such that* (13) *and* $a \cdot b = a + \varphi(b-a)$ *for all* a, $b \in Q$.

Proof. Since $(Q,\cdot)$ is a pentagonal quasigroup, it is also an IM-quasigroup. According to the version of Toyoda's theorem for IM-quasigroups, there is an Abelian group $(Q,+)$ with an automorphism φ such that $a\cdot b = a+\varphi(b-a)$ for all a, $b \in Q$. The identity of pentagonality (6) is equivalent to (13), which is proved by computation done prior to this theorem. $\square$

4. Four models for pentagonal quasigroups

Depending on the choice of $q \in \{q_1, q_2, q_3, q_4\}$ and if we regard complex numbers as points of the Euclidean plane, we can get four different *characteristic triangles* in $C(q)$ with vertices 0, 1 and q_i, $i = 1,2,3,4$, see Figure 2. Each of the $C(q_i)$, $i = 1,2,3,4$, gives one model for pentagonal quasigroups.

Points q_1 and q_2 are the third vertices of two regular pentagons determined by its two adjacent vertices 0 and 1, while q_3 and q_4 are intersection points of two diagonals of the same two pentagons.

Let us observe a pentagonal quasigroup $(Q,\cdot)$ in the model $C(q_1)$. In the Figure 3 we can spot characteristic triangles from the models $C(q_2)$, $C(q_3)$ and $C(q_4)$.

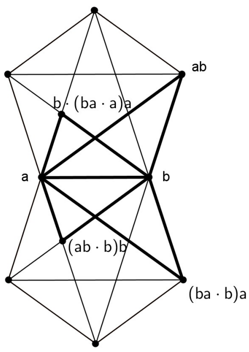

Figure 3. Models for pentagonal quasigroups

Characteristic triangle of the model $C(q_2)$ has vertices a, b and $(ba \cdot b)a$. We will denote

$$a \circ b = (ba \cdot b)a.$$

Characteristic triangle of the model $C(q_3)$ has vertices a, b and $b \cdot (ba \cdot a)a$. We will denote

$$a * b = b \cdot (ba \cdot a)a.$$

Characteristic triangle of the model $C(q_4)$ has vertices a, b and $(ab \cdot b)b$. We will denote

$$a \diamond b = (ab \cdot b)b.$$

The main goal of this section is to prove that $(Q, \circ)$, $(Q, *)$ i $(Q, \diamond)$ are also pentagonal quasigroups. It will be enough to prove that if $(Q, \cdot)$ is a pentagonal quasigroup, then so is $(Q, *)$, because we will show

$$b * (((b * a) * a) * a) = (ba \cdot b)a = a \circ b,$$

$$b \circ (((b \circ a) \circ a) \circ a) = (ab \cdot b)b = a \diamond b.$$

$$b \diamond (((b \diamond a) \diamond a) \diamond a) = ab.$$

In a quasigroup $(Q, \cdot)$ operations of left and right division are defined by

$$a \backslash c = b \Leftrightarrow ab = c \Leftrightarrow c/b = a.$$

Formula is an expression built up from variables using the operations $\cdot$, $\backslash$ and $/$. More precisely:

(1) elements of the set Q (variables) are formulae;

(2) if φ and ψ are formulae, then so are $\varphi \cdot \psi$, $\varphi \backslash \psi$ and φ / ψ.

A formula φ containing at most two variables gives rise to a new binary operation $Q \times Q \to Q$, which will also be denoted by φ.

In [3] the next corollary was proved. We will use it in the proof of the next theorem.

Corollary 4.1. *If $(Q,\cdot)$ is a medial quasigroup, then binary operation defined by the formula φ is also medial.* $\square$

Theorem 4.2. *Let $(Q,\cdot)$ be a pentagonal quasigroup and let $*\colon Q\times Q\to Q$ be a binary operation defined by*

$$a*b=b\cdot(ba\cdot a)a.$$

Then $(Q,*)$ is a pentagonal quasigroup.

Proof. First we prove that $(Q,*)$ is a quasigroup, i.e., that for given a, $b\in Q$ there exist unique x, $y\in Q$ such that $a*x=b$ and $y*a=b$. If we put $x=ab\cdot a$, we get

$$\begin{aligned} a*x &= x\cdot(xa\cdot a)a=(ab\cdot a)\cdot((ab\cdot a)a\cdot a)a \overset{(10)}{=} (ab\cdot a)\cdot(ba\cdot b)a\\ &\overset{(5)}{=} (ab\cdot(ba\cdot b))a \overset{(5)}{=} (a\cdot ba)b\cdot a \overset{(3)}{=} (ab\cdot a)b\cdot a \overset{(6)}{=} b. \end{aligned}$$

Let us now assume that there exist x_1, $x_2\in Q$ such that $a*x_1=a*x_2$. That means that we have

$$x_1\cdot(x_1a\cdot a)a=x_2\cdot(x_2a\cdot a)a.$$

Multiplying by a from the left and applying (4) we get

$$ax_1\cdot(a\cdot(x_1a\cdot a)a)=ax_2\cdot(a\cdot(x_2a\cdot a)a).$$

Now using (3) and (10) we get

$$ax_1\cdot(x_1a\cdot x_1)=ax_2\cdot(x_2a\cdot x_2).$$

After applying (5) and (3) the equality becomes

$$(ax_1\cdot a)x_1=(ax_2\cdot a)x_2,$$

so multiplying from the right by a and using (6), we finally get $x_1=x_2$.
If we now put $y=(ba\cdot a)a$, we get

$$\begin{aligned} y*a &= a\cdot(ay\cdot y)y=a(((a\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\ &\overset{(3),(10)}{=} a(((ba\cdot b)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\ &\overset{(2)}{=} a(((ba\cdot(ba\cdot a))\cdot ba)\cdot(ba\cdot a)a) \overset{(5)}{=} a(((b\cdot ba)a\cdot ba)\cdot(ba\cdot a)a)\\ &\overset{(5)}{=} a(((b\cdot ba)b\cdot a)\cdot(ba\cdot a)a) \overset{(5)}{=} a\cdot((b\cdot ba)b\cdot(ba\cdot a))a\\ &\overset{(2)}{=} a\cdot(((b\cdot ba)\cdot ba)\cdot ba)a \overset{(4)}{=} a\cdot(b(ba\cdot a)\cdot ba)a \overset{(4),(3)}{=} a(b\cdot(ba\cdot a)a)\cdot a\\ &\overset{(4),(3),(10)}{=} (ab\cdot(ba\cdot b))a \overset{(5)}{=} (a\cdot ba)b\cdot a \overset{(3)}{=} (ab\cdot a)b\cdot a \overset{(6)}{=} b. \end{aligned}$$

Let us now assume that there exist y_1, $y_2\in Q$ such that $y_1*a=y_2*a$. We get

$$a\cdot(ay_1\cdot y_1)y_1=a\cdot(ay_2\cdot y_2)y_2.$$

Using cancellation we get

$$(ay_1 \cdot y_1)y_1 = (ay_2 \cdot y_2)y_2.$$

Multiplying by $y_1 a$ from the left and using (11) we get

$$a = y_1 a \cdot (ay_2 \cdot y_2)y_2.$$

Applying (11) once again gives

$$y_2 a \cdot (ay_2 \cdot y_2)y_2 = y_1 a \cdot (ay_2 \cdot y_2)y_2,$$

wherefrom cancelling first with $(ay_2 \cdot y_2)y_2$ and then with a, we finally get $y_2 = y_1$.
Idempotency of $*$ follows immediately from idempotency of $\cdot$.
Mediality of $*$ follows from Corollary 4.1 by putting $\varphi = *$.
Let us now prove $(a * b) * a = (a \cdot (ab \cdot b)b)b$.

$$\begin{aligned}
(a * b) * a &= a \cdot (a(a * b) \cdot (a * b))(a * b) \\
&= a \cdot (a(b \cdot (ba \cdot a)a) \cdot (b \cdot (ba \cdot a)a))(b \cdot (ba \cdot a)a) \\
&\overset{(4),(3),(10)}{=} a \cdot ((ab \cdot (ba \cdot b)) \cdot (b \cdot (ba \cdot a)a))(b \cdot (ba \cdot a)a) \\
&\overset{(4),(3)}{=} a \cdot ((ab \cdot a)b \cdot (b \cdot (ba \cdot a)a))(b \cdot (ba \cdot a)a) \\
&\overset{(4),(3),(10)}{=} (a \cdot (ab \cdot a)b)(ab \cdot (ba \cdot b)) \cdot (ab \cdot (ba \cdot b)) \\
&\overset{(4),(3)}{=} ((a \cdot (ab \cdot a)b) \cdot (ab \cdot a)b) \cdot (ab \cdot a)b \\
&\overset{(2)}{=} ((a \cdot (ab \cdot a)b) \cdot (ab \cdot a))((ab \cdot a)b \cdot b) \\
&\overset{(2),(6)}{=} (a \cdot ab)b \cdot ((ab \cdot a)b \cdot b) \\
&\overset{(5)}{=} ((a \cdot ab) \cdot (ab \cdot a)b)b \overset{(2)}{=} (a(ab \cdot a) \cdot (ab \cdot b))b \\
&\overset{(3)}{=} ((a \cdot ab)a \cdot (ab \cdot b))b \overset{(2)}{=} (((a \cdot ab) \cdot ab) \cdot ab)b \\
&\overset{(4)}{=} (a(ab \cdot b) \cdot ab)b \overset{(4)}{=} (a \cdot (ab \cdot b)b)b
\end{aligned}$$

Now we prove $((a * b) * a) * b = (ba \cdot a)a$. Let us denote $c = (a * b) * a$. We have

$$\begin{aligned}
((a * b) * a) * b &= b \cdot (bc \cdot c)c \\
&= b(((b \cdot (a \cdot (ab \cdot b)b)b) \cdot (a \cdot (ab \cdot b)b)b) \cdot (a \cdot (ab \cdot b)b)b) \\
&\overset{(3)}{=} b(((b(a \cdot (ab \cdot b)b) \cdot b) \cdot (a \cdot (ab \cdot b)b)b) \cdot (a \cdot (ab \cdot b)b)b) \\
&\overset{(4)}{=} b((((ba \cdot (b \cdot (ab \cdot b)b)) \cdot b) \cdot (a \cdot (ab \cdot b)b)b) \cdot (a \cdot (ab \cdot b)b)b) \\
&\overset{(4),(11)}{=} b((((ba \cdot b)a \cdot b) \cdot (a \cdot (ab \cdot b)b)b) \cdot (a \cdot (ab \cdot b)b)b)
\end{aligned}$$

$$\begin{aligned}
&\overset{(6)}{=} b((a\cdot(a\cdot(ab\cdot b)b)b)\cdot(a\cdot(ab\cdot b)b)b)\\
&\overset{(4),(1)}{=} (ba\cdot(ba\cdot(b\cdot(ab\cdot b)b))b)\cdot(ba\cdot(b\cdot(ab\cdot b)b))b\\
&\overset{(4),(11)}{=} (ba\cdot((ba\cdot b)a\cdot b))((ba\cdot b)a\cdot b)\overset{(6)}{=}(ba\cdot a)a.
\end{aligned}$$

Finally, we prove $(((a*b)*a)*b)*a=b$. If we put $d=((a*b)*a)*b$, we have

$$\begin{aligned}
(((a*b)*a)*b)*a &= a\cdot(ad\cdot d)d\\
&= a(((a\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\
&\overset{(3)}{=} a((((ab\cdot a)a\cdot a)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\
&\overset{(10)}{=} a(((ba\cdot b)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\
&\overset{(3)}{=} a(((b\cdot ab)\cdot(ba\cdot a)a)\cdot(ba\cdot a)a)\\
&\overset{(5),(11)}{=} a((b\cdot(ba\cdot a)a)b\cdot(ba\cdot a)a)\\
&\overset{(4)}{=} ((ab\cdot(a\cdot(ba\cdot a)a))\cdot ab)(a\cdot(ba\cdot a)a)\\
&\overset{(4),(11),(3)}{=} ((ab\cdot a)b\cdot ab)((ab\cdot a)a\cdot a)\\
&\overset{(5)}{=} ((ab\cdot a)a\cdot b)((ab\cdot a)a\cdot a)\\
&\overset{(4)}{=} (ab\cdot a)a\cdot ba\overset{(5)}{=}(ab\cdot a)b\cdot a\overset{(6)}{=} b. \qquad\square
\end{aligned}$$

In the end we state three more theorems which express multiplications in quasigroups $(Q,*)$, $(Q,\circ)$ and $(Q,\diamond)$ in terms of multiplication in quasigroup $(Q,\cdot)$. First statements in these theorems follow immediately from Theorem 4.2. Second statements can be proved by rather tedious calculations similar to those in the proof of Theorem 4.2 or using some automated theorem prover. We omit these proofs in this paper.

Theorem 4.3. *Let $(Q,*)$ be a pentagonal quasigroup and let $\circ\colon Q\times Q\to Q$ be a binary operation defined by*

$$a\circ b=b*(((b*a)*a)*a).$$

Then $(Q,\circ)$ is a pentagonal quasigroup. Furthermore

$$a\circ b=(ba\cdot b)a. \qquad\square$$

Theorem 4.4. *Let $(Q,\circ)$ be a pentagonal quasigroup and let $\diamond\colon Q\times Q\to Q$ be a binary operation defined by*

$$a\diamond b=b\circ(((b\circ a)\circ a)\circ a).$$

Then $(Q, \diamond)$ *is a pentagonal quasigroup. Furthermore*

$$a \diamond b = (ab \cdot b)b. \qquad \square$$

Theorem 4.5. *Let* $(Q, \diamond)$ *be a pentagonal quasigroup and let* $\odot\colon Q \times Q \to Q$ *be a binary operation defined by*

$$a \odot b = b \diamond (((b \diamond a) \diamond a) \diamond a).$$

Then $(Q, \odot)$ *is a pentagonal quasigroup. Furthermore*

$$a \odot b = ab. \qquad \square$$

References

[1] **M. Bombardelli and V. Volenec**, *Vectors and transfers in hexagonal quasigroup*, Glasnik Matematički **42 (62)** (2007), 363 – 373.

[2] **Z. Kolar-Begović and V. Volenec**, *Affine regular pentagons in GS-quasigroups*, Quasigroups Related Systems **12** (2004), 103 – 112.

[3] **V. Krčadinac and V. Volenec**, *A note on medial quasigroups*, Math. Commun. **11** (2006), 83 – 85.

[4] **K. Toyoda**, *On axioms of linear functions*, Proc. Imp. Acad. Tokyo **17** (1941), 221 – 227.

[5] **V. Volenec**, *Admissible identities in complex IM-quasigroups I*, Rad JAZU [413] **4** (1985), 61 – 85.

[6] **V. Volenec**, *Admissible identities in complex IM-quasigroups II*, Rad JAZU [421] **5** (1986), 45 – 78.

[7] **V. Volenec**, *Geometry of medial quasigroups*, Rad JAZU [421] **5** (1986), 79 – 91.

[8] **V. Volenec**, *Geometry of IM-quasigroups*, Rad HAZU [456] **10** (1991), 139 – 146.

[9] **V. Volenec**, *GS-quasigroups*, Čas. Pěst. Mat. **115** (1990), 307 – 318.

[10] **V. Volenec**, *Hexagonal quasigroups*, Archivum Mathematicum (Brno) **27a** (1991), 113 – 122.

Received August 20, 2013

Department of Mathematics
University of Zagreb
Bijenička 30
HR-10000 Zagreb
Croatia
Email: svidak@math.hr