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Describing cyclic extensions of Bol loops

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Abstract. It was shown in [4] that if a Moufang loop G factorizes as $G = NH$ where N is a normal subloop and $H = \langle u \rangle = \langle u^3 \rangle$ is a cyclic group then the structure of G is determined by the binary operation of N, the intersection $N \cap H$ and how u permutes the elements of N as a semi-automorphism of N. Here it is shown that if G is Moufang with $H = \langle u \rangle \neq \langle u^3 \rangle$ or if G is a Bol loop, not necessarily Moufang, then the structure of G is determined by the binary operation of N, the intersection $N \cap H$, how u permutes the elements of N and either of the two binary operations $x *_1 y = (xu)(u \backslash y)$ or $x *_{-1} y = (xu^{-1})(u^{-1} \backslash y)$ of N.

1. Introduction

A *quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $\cdot$ such that for any $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ respectively. A quasigroup is a *loop* if it contains a two-sided identity element. A *right Bol loop* is a loop Q which, for all $x, y, z \in Q$, satisfies the right Bol relation

$$((zx)y)x = z((xy)x).$$

Similarly, a loop Q is a *left Bol loop* provided it satisfies the right Bol relation

$$x(y(xz)) = (x(yx))z.$$

Recently the structure and construction of Bol loops has caught the attention of many including Chein and Goodaire [1, 2] along with Foguel, Kinyon and Phillips [3].

Here the focus will be on (right) Bol loops of the form $Q = NH$ where $N \trianglelefteq Q$ and $H = \langle u \rangle$ is cyclic. It is well known for groups that the binary operation of Q depends only on the binary operation of N, the intersection $N \cap H$ and how H acts on N. Likewise, it was shown in [4] that the same is true for Moufang loops as long as $H = \langle u \rangle = \langle u^3 \rangle$. Generalizing to Bol loops, it is shown here how such extensions depend on the maps

$$\begin{aligned} f_{m,n} &: N \longrightarrow N \\ & g \longmapsto (u^m(gu^n))u^{-m-n}. \end{aligned}$$

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Furthermore, the structure of Q depends not only on the structure of N, the intersection $N \cap H$ and the maps $f_{n,m}$ but also on the Bol loops $(N, *_i)$ where

$$x *_i y = (xu^i)(u^i \backslash y).$$

For instance, there exists a nonassociative right Bol loop $Q = NH$ of order eight (the mirror version of the left Bol loop LeftBolLoop(8,2) in GAP [5]) where $N \cong C_2 \times C_2$, $H \cong C_2$ and $f_{m,n} : N \to N$ is the identity map for all $m, n \in \mathbb{Z}$. But since Q is nonassociative, $Q \ncong C_2 \times C_2 \times C_2$. This is because the cyclic extension also depends on the Bol loop $(N, *_1)$ which is not the Klein four-group but rather the cyclic group of order four.

2. Preliminaries

For every element, say a, of a quasigroup Q one can define the *right translation* $R_a : Q \longrightarrow Q$ by $(x)R_a = xa$ and the *left translation* $L_a : Q \longrightarrow Q$ by $(x)L_a = ax$. By definition of a quasigroup, all such translations are bijections of Q. Let

$$x \backslash y = yL_x^{-1} \quad \text{and} \quad x/y = xR_y^{-1}$$

and note that

$$x \backslash y = z \Longleftrightarrow y = xz \quad \text{and} \quad x/y = w \Longleftrightarrow x = wy.$$

Such operations $\backslash$ and $/$ are called the left and right divisions respectively. The *multiplication group* of Q, denoted by $\mathrm{Mlt}(Q)$, is the permutation group generated by all left and right translations of Q. If Q is a loop with an identity element 1 then the *inner mapping group* of Q, denoted by $\mathrm{Inn}(Q)$, is the stabilizer of 1 in $\mathrm{Mlt}(Q)$.

Two quasigroups $(Q_1, *)$ and $(Q_2, \circ)$ are called *isotopic* if there exist three bijections $f, g, h : Q_1 \longrightarrow Q_2$ such that $f(x * y) = g(x) \circ h(y)$ for any $x, y \in Q_1$.

Lemma 1. *Every quasigroup $(Q, *)$ is isotopic to a loop. For $a, b \in Q$, $(Q, *)$ is isotopic to $(Q, \circ)$ where*

$$x \circ y = (x)R_b^{-1} * (y)L_a^{-1}$$

or equivalently

$$(x * b) \circ (a * y) = x * y.$$

Here $(a * b) \circ x = x = x \circ (a * b)$ for any $x \in Q$.

A loop Q is said to be *power-associative* if for any element $x \in Q$, the subloop generated by x is a group. A loop Q is *diassociative* if for any $x, y \in Q$, the subloop generated by x and y is a group. A loop Q is *left power-alternative,* if for any $x, y \in Q$,

$$x^m(x^n y) = x^{m+n} y$$

for all integers m and n. Similarly, Q is *right power-alternative,* if for any $x, y \in Q$,

$$(yx^m)x^n = yx^{m+n}$$

for all integers m and n. A loop Q is *power-alternative* if it is both left and right power-alternative.

Robinson [6] made the simple observation that right (left) Bol loops are right (left) power-alternative. Let Q be a right Bol loop with $u \in Q$ and define $x *_i y = (xu^i)(u^i \backslash y)$ for any $x, y \in Q$. Since right Bol loops are right power-alternative,

$$\begin{aligned} x *_i y &= (xu^i)(u^i \backslash y) \\ &= (x/u^{-i})(u^i \backslash y) \end{aligned}$$

and $(Q, *_i)$ is a loop isotopic to Q.

Lemma 2. (cf. [7]) *Let Q be a (right) Bol loop. Then all loop isotopes of Q are isomorphic to Q.*

From this it follows that $(Q, *_i)$ is a right Bol loop isomorphic to the original loop Q.

3. Diassociative Bol loops

It is well known that a Bol loop is diassociative if and only if it is a Moufang loop. As mentioned in [4], cyclic extensions $Q = NH$ resulting in such loops depend on more than just how H acts on N when $\langle u^3 \rangle \subsetneqq \langle u \rangle = H$.

Theorem 1. *Let Q be a right Bol loop. Suppose $Q = NH$ where $N \trianglelefteq Q$ and $H = \langle u \rangle$. If Q is diassociative then for any $x, y \in N$ and any $m, n \in \mathbb{Z}$*

$$(xu^m)(yu^n) = (x *_{2m+n} f^m(y))u^{m+n} \tag{1}$$

where

$$\begin{aligned} f : N &\longrightarrow N \\ g &\longmapsto ugu^{-1} \end{aligned}$$

and $(Q, *_i)$ is a right Bol loop isomorphic to Q with $x *_i y = (xu^i)(u^i \backslash y) = (xu^i)(u^{-i}y)$.

Proof. By Lemma 2, $(Q, *_i)$ is isomorphic to Q. For any $x, y \in N$ and any $m, n \in \mathbb{Z}$

$$\begin{aligned} (xu^m)(yu^n) &= (xu^m)\big(u^n \cdot u^{-n}y \cdot u^n\big) \\ &= \big[(xu^m)u^n \cdot \big(u^{-n}y\big)\big]\, u^n \\ &= \big[\big(xu^{m+n}\big)\big(u^m \cdot u^{-m-n}yu^{-m} \cdot u^m\big)\big]\, u^n \\ &= \big[\big(xu^{m+n}\big)u^m \cdot \big(u^{-m-n}yu^{-m}\big)\big]\, u^m \cdot u^n \\ &= \big[\big(xu^{2m+n}\big)\big(u^{-2m-n}f^m(y)\big)\big]\, u^m \cdot u^n \\ &= (x *_{2m+n} f^m(y))\, u^{m+n}. \end{aligned}$$

□

If $2m+n \equiv 0 \pmod 3$ then, from Equation (1) of [4], the binary operation $*_{2m+n}$ is uniquely determined by $f : g \longmapsto ugu^{-1}$. But if $2m+n \not\equiv 0 \pmod 3$ then $x *_{2m+n} y$ depends on f along with either of the two operations $*_1$ or $*_{-1}$.

Lemma 3. *Let Q be a diassociative Bol loop with $u \in Q$. For any $x, y \in Q$ define $x *_i y = (xu^i)(u^{-i}y)$ and $f : Q \longrightarrow Q$ as $f(x) = uxu^{-1}$. If $2m+n \equiv 1 \pmod 3$, that is $2m+n = 3k+1$, then*

$$x *_{2m+n} y = f^k\big(f^{-k}(x) *_1 f^{-k}(y)\big)\,.$$

Likewise, if $2m+n \equiv 2 \pmod 3$ and $2m+n = 3k+2$ then

$$x *_{2m+n} y = f^{k+1}\big(f^{-k-1}(x) *_{-1} f^{-k-1}(y)\big)\,.$$

Proof. If $2m+n = 3k+1$ then, by using Equation (1) of [4],

$$\begin{aligned} x *_{2m+n} y &= \big(xu^{3k+1}\big)\big(u^{-3k-1}y\big) \\ &= \big(xuu^{3k}\big)\big(u^{-1}f^{-3k}(y)u^{-3k}\big) \\ &= f^k\big(f^{-k}(xu)f^{2k}(u^{-1}f^{-3k}(y))\big) \\ &= f^k\big(f^{-k}(x)u \cdot u^{-1}f^{-k}(y)\big) \\ &= f^k\big(f^{-k}(x) *_1 f^{-k}(y)\big)\,. \end{aligned}$$

Similarly, if $2m+n = 3k+2$ then $x *_{2m+n} y = f^{k+1}\big(f^{-k-1}(x) *_{-1} f^{-k-1}(y)\big)$. $\square$

Note that for any integers i and k, $x *_i y = f^k\big(f^{-k}(x) *_{i-3k} f^{-k}(y)\big)$. In other words

$$f^k(x *_i y) = f^k(x) *_{i+3k} f^k(y). \tag{2}$$

From [4] it is known that if $Q = NH$ is a Moufang loop where $N \trianglelefteq Q$ and $H = \langle u \rangle = \langle u^3 \rangle$ then the binary operation of Q depends only on the binary operation of N, the intersection $N \cap H$ and how u permutes the elements in N. Thus without loss it can be assumed that $H = \langle u \rangle \neq \langle u^3 \rangle$ in which case the binary operation of Q also depends on the loops $(N, *_1)$ and $(N, *_{-1})$.

Theorem 2. *Let Q be a right Bol loop. Suppose $Q = NH$ where $N \trianglelefteq Q$ and $H = \langle u \rangle$ with $\langle u^3 \rangle \subsetneqq H$. If Q is diassociative then for any $x, y \in N$*

$$(xu^m)(yu^n) = \begin{cases} f^k\big(f^{-k}(x) *_1 f^{m-k}(y)\big)\,u^{m+n} & \text{if } 2m+n = 3k+1; \\ f^{k+1}\big(f^{-k-1}(x) *_{-1} f^{m-k-1}(y)\big)\,u^{m+n} & \text{if } 2m+n = 3k+2; \end{cases}$$

where

$$\begin{aligned} f : N &\longrightarrow N \\ g &\longmapsto ugu^{-1} \end{aligned}$$

and $(Q, *_i)$ is a right Bol loop isomorphic to Q with $x *_i y = (xu^i)(u^{-i}y)$.

Proof. By Theorem 1, $(xu^m)\,(yu^n) = (x *_{2m+n} f^m(y))u^{m+n}$ for any $x, y \in N$. If $2m + n = 3k + 1$ then, by Lemma 3,

$$\begin{aligned} x *_{2m+n} f^m(y) &= f^k\big(f^{-k}(x) *_1 f^{-k}(f^m(y))\big) \\ &= f^k\big(f^{-k}(x) *_1 f^{m-k}(y)\big)\,. \end{aligned}$$

Hence, $(xu^m)(yu^n) = f^k\big(f^{-k}(x) *_1 f^{m-k}(y)\big)\, u^{m+n}$. Similarly, if $2m+n = 3k+2$ then $(xu^m)(yu^n) = f^{k+1}\big(f^{-k-1}(x) *_{-1} f^{m-k-1}(y)\big)\, u^{m+n}$. □

From this we see that these extensions by cyclic groups with orders divisible by three depend on the permutation $f(x) = uxu^{-1}$ along with the binary operations $*_1$ and $*_{-1}$. In Section 4 it will be shown that such extensions depend on f along with just one of the binary operations $*_1$ or $*_{-1}$.

Proposition 1. *Suppose $Q = NH$ is a loop where $N \trianglelefteq Q$ and $H = \langle u\rangle$. For any $x, y \in N$ let*

$$\begin{aligned} f : N &\longrightarrow N \\ g &\longmapsto (ug)u^{-1} \end{aligned}$$

and $x *_i y = (xu^i)(u^{-i}y)$. If Q is a diassociative right Bol loop then

$$(x *_{t-s} y) *_t (z *_{t+s} f^s(x)) = x *_{2r-s} ((y *_t z) *_{2r} f^s(x)) \tag{3}$$

for any $r, s, t \in \mathbb{Z}$ and any $x, y, z \in N$. Furthermore, Q is a diassociative right Bol loop if and only if equations (1), (2) and (3) hold.

Proof. Let $x, y, z \in Q$ and $k, m, n \in \mathbb{Z}$. Using Equations (1) and (2) it follows that

$$\begin{aligned} &\Big((xu^k)\,(f^{-k}(y)u^m)\Big)\Big((f^{-k-m}(z)u^n)(xu^k)\Big) = \\ &\quad= (x *_{2k+m} y)\, u^{k+m} \cdot \Big(f^{-k-m}(z) *_{2n+k} f^n(x)\Big)u^{n+k} \\ &\quad= \Big((x *_{2k+m} y) *_{3k+2m+n} f^{k+m}(f^{-k-m}(z) *_{2n+k} f^n(x))\Big)u^{2k+m+n} \\ &\quad= \Big((x *_{2k+m} y) *_{3k+2m+n} (z *_{4k+3m+2n} f^{k+m+n}(x))\Big)u^{2k+m+n} \end{aligned}$$

and

$$\begin{aligned} &(xu^k)\Big[(f^{-k}(y)u^m)(f^{-k-m}(z)u^n) \cdot (xu^k)\Big] = \\ &\quad= (xu^k)\,\Big[\Big(f^{-k}(y) *_{2m+n} f^{-k}(z)\Big)u^{m+n} \cdot (xu^k)\Big] \\ &\quad= (xu^k)\,\Big[\Big(f^{-k}(y) *_{2m+n} f^{-k}(z)\Big) *_{k+2m+2n} f^{m+n}(x) \cdot u^{k+m+n}\Big] \\ &\quad= \Big[x *_{3k+m+n} f^k\Big(\Big(f^{-k}(y) *_{2m+n} f^{-k}(z)\Big) *_{k+2m+2n} f^{m+n}(x)\Big)\Big]u^{2k+m+n} \\ &\quad= \Big[x *_{3k+m+n} \Big(f^k\Big(f^{-k}(y) *_{2m+n} f^{-k}(z)\Big) *_{4k+2m+2n} f^{k+m+n}(x)\Big)\Big]u^{2k+m+n} \\ &\quad= \Big[x *_{3k+m+n} \Big((y *_{3k+2m+n} z) *_{4k+2m+2n} f^{k+m+n}(x)\Big)\Big]u^{2k+m+n}. \end{aligned}$$

By letting $s = k + m + n$, $t = 3k + 2m + n$ and $r = 2k + m + n$ (i.e., $k = r - s$, $m = t + s - 2r$ and $n = s - t + r$)

$$\big((xu^k)(f^{-k}(y)u^m)\big)\big((f^{-k-m}(z)u^n)(xu^k)\big) = (x *_{t-s} y) *_t (z *_{t+s} f^s(x))$$

and

$$(xu^k)\big[(f^{-k}(y)u^m)(f^{-k-m}(z)u^n) \cdot (xu^k)\big] = x *_{2r-s} ((y *_t z) *_{2r} f^s(x)).$$

Hence, Q satisfies the Moufang identities if and only if Equations (1), (2) and (3) hold. □

Note that by letting $y = 1$, Equation (3) simplifies to

$$x *_t (z *_{t+s} f^s(x)) = x *_{2r-s} (z *_{2r} f^s(x)).$$

Since the right hand side of the equality is independent of t,

$$x *_{t_1} (z *_{t_1+s} f^s(x)) = x *_{t_2} (z *_{t_2+s} f^s(x))$$

for any $t_1, t_2 \in \mathbb{Z}$. Similarly, by letting $z = 1$ in Equation (3) and using a similar argument it follows that

$$(x *_{t_1} y) *_{t_1+s} f^s(x) = x *_{t_2} (y *_{t_2+s} f^s(x))$$

for any $t_1, t_2 \in \mathbb{Z}$. Therefore, from Equation (3) it follows that if Q is a diassociative right Bol loop then

$$\begin{aligned}(x *_{n-k} y) *_n (z *_{n+k} f^k(x)) &= x *_{m-k} ((y *_n z) *_m f^k(x))\\ &= (x *_{\ell-k} (y *_n z)) *_\ell f^k(x)\end{aligned}$$

for any $x, y, z \in Q$ and $k, m, n, \ell \in \mathbb{Z}$.

4. The general case

Here Theorem 1 will be generalized for cyclic extensions resulting in arbitrary right Bol loops. The following is a useful lemma that will be used to prove the main result.

Lemma 4. *If Q is a right Bol loop with $u \in Q$ then $u\backslash x = \big(u^{-1}(xu)\big)\,u^{-1}$ for any $x \in Q$.*

Proof. For any $x \in Q$,

$$\begin{aligned}u\left[\big(u^{-1}(xu)\big)\,u^{-1}\right] &= \big((uu^{-1})(xu)\big)\,u^{-1}\\ &= (xu)u^{-1}\\ &= x.\end{aligned}$$

□

Theorem 3. *If $Q = NH$ is a right Bol loop with $N \trianglelefteq Q$ and $H = \langle u \rangle \leq Q$ then for any $x, y \in N$ and any $m, n \in \mathbb{Z}$*

$$(xu^m)(yu^n) = (x *_{2m+n} f_{m,n}(y))\, u^{m+n} \tag{4}$$

where

$$\begin{aligned} f_{m,n} : N &\longrightarrow N \\ g &\longmapsto (u^m (gu^n))\, u^{-m-n} \end{aligned}$$

and $(Q, *_i)$ is a right Bol loop isomorphic to Q with $x *_i y = (xu^i)(u^i \backslash y)$.

Proof. By Lemma 2, $(Q, *_i)$ is isomorphic to Q. Furthermore, for any $x, y \in N$ and any $m, n \in \mathbb{Z}$

$$\begin{aligned} (xu^m)(yu^n) &= \big((xu^{2m+n}u^{-m-n})(yu^n)\big)u^{-m-n}u^{m+n} \\ &= \big[(xu^{2m+n})\big[(u^{-m-n}(yu^n))u^{-m-n}\big]\big]u^{m+n} \\ &= \big[(xu^{2m+n})\big[(u^{-2m-n}u^m(yu^n))u^m \cdot u^{-2m-n}\big]\big]u^{m+n} \\ &= \big[(xu^{2m+n})\big[\big[u^{-2m-n}((u^m(yu^n))u^m)\big]u^{-2m-n}\big]\big]u^{m+n} \\ &= \big[(xu^{2m+n})\big[\big[u^{-2m-n}\big((u^m(yu^n))u^{-m-n} \cdot u^{2m+n}\big)\big]u^{-2m-n}\big]\big]u^{m+n} \\ &= \big[(xu^{2m+n})\big[\big[u^{-2m-n}\big(f_{m,n}(y)u^{2m+n}\big)\big]u^{-2m-n}\big]\big]u^{m+n} \\ &= \big[(xu^{2m+n})\big(u^{2m+n}\backslash f_{m,n}(y)\big)\big]u^{m+n} \\ &= (x *_{2m+n} f_{m,n}(y))u^{m+n}. \end{aligned}$$

□

Since right Bol loops are right power-alternative, it should be noted that $f_{m,n} : g \mapsto (u^m (gu^n))\, u^{-m-n}$ is the identity map whenever $m = 0$. Therefore, from Theorem 3, it follows that in any right Bol loop $x(yu^n) = (x *_n y)u^n$.

Proposition 2. *Suppose Q is a right Bol loop with $u \in Q$. Let*

$$\begin{aligned} f_{i,j} : Q &\longrightarrow Q \\ g &\longmapsto \big(u^i\,(gu^j)\big)\, u^{-i-j} \end{aligned}$$

and $(Q, *_i)$ be the right Bol loop isomorphic to Q with $x *_i y = (xu^i)(u^i \backslash y)$. Then by knowing the maps $f_{i,j}$ along with $(Q, *_n)$ and $(Q, *_{n+1})$ for some fixed integer n, the Bol loop $(Q, *_k)$ is uniquely determined for any integer k.

Proof. Since Q is a right Bol loop,

$$z\big((xu^{n-2m})u^{3m-n} \cdot (xu^{n-2m})\big) = \big((z(xu^{n-2m}))u^{3m-n}\big)(xu^{n-2m})$$

for any $z, x \in N$ and any $n, m \in \mathbb{Z}$. Therefore, by Theorem 3,

$$\begin{aligned}
& z\big((xu^{n-2m})u^{3m-n} \cdot (xu^{n-2m})\big) = \big((z(xu^{n-2m}))u^{3m-n}\big)(xu^{n-2m}) \\
\Longrightarrow \quad & z\big((xu^{m})(xu^{n-2m})\big) = \big(((z *_{n-2m} x)u^{n-2m})u^{3m-n}\big)(xu^{n-2m}) \\
\Longrightarrow \quad & z\big((x *_n f_{m,n-2m}(x))u^{n-m}\big) = ((z *_{n-2m} x)u^{m})(xu^{n-2m}) \\
\Longrightarrow \quad & (z *_{n-m} (x *_n f_{m,n-2m}(x)))u^{n-m} = ((z *_{n-2m} x) *_n f_{m,n-2m}(x))u^{n-m} \\
\Longrightarrow \quad & z *_{n-m} (x *_n f_{m,n-2m}(x)) = (z *_{n-2m} x) *_n f_{m,n-2m}(x).
\end{aligned} \tag{5}$$

By letting $m = -1$, Equation (5) becomes

$$z *_{n+1} (x *_n f_{-1,n+2}(x)) = (z *_{n+2} x) *_n f_{-1,n+2}(x).$$

Whereas, by replacing n with $n+1$ and m with 1, Equation (5) becomes

$$z *_n (x *_{n+1} f_{1,n-1}(x)) = (z *_{n-1} x) *_{n+1} f_{1,n-1}(x).$$

Hence, with the operations $*_n$ and $*_{n+1}$, both $z *_{n+2} x$ and $z *_{n-1} x$ can be determined for any elements z and x. By induction, $*_k$ can then be obtained for any $k \in \mathbb{Z}$. $\square$

Since $*_0$ is just the original binary operation of N, by letting n be either 0 or -1, it immediately follows that for any integer k the Bol loop $(Q, *_k)$ is uniquely determined by the maps $f_{i,j}$, the subloop N and either $(N, *_1)$ or $(N, *_{-1})$.

Corollary 1. *If $Q = NH$ is a right Bol loop with $N \trianglelefteq Q$ and $H = \langle u \rangle \leq Q$ then for any $x, y \in N$ and any $m, n \in \mathbb{Z}$ the product $(xu^m)(yu^n)$ is uniquely determined by the subloops N and $N \cap H$ along with the maps*

$$\begin{aligned}
f_{m,n} : N &\longrightarrow N \\
g &\longmapsto (u^m\,(gu^n))\,u^{-m-n}
\end{aligned}$$

and one of the two Bol loops $(N, *_1)$ or $(N, *_{-1})$ where $x *_i y = (xu^i)(u^i \backslash y)$.

But if $[Q, N] = 2$ then much of the Cayley table of $(N, *_1)$ may be determined by N and the maps $f_{i,j}$.

Proposition 3. *Suppose $Q = NH$ is a right Bol loop with $N \trianglelefteq Q$ and $H = \langle u \rangle \leq Q$ where $[Q : N] = 2$. Then for any $z \in N$ and*

$$y \in \{x *_2 f_{1,0}(x) \mid x \in N\} = \{(x(f_{1,0}(x)u^2))u^{-2} \mid x \in N\} \subseteq N,$$

$z *_1 y = (zu)(u \backslash y)$ is uniquely determined by the binary operation of N and

$$\begin{aligned}
f_{1,0} : N &\longrightarrow N \\
g &\longmapsto (ug)u^{-1}.
\end{aligned}$$

Proof. By Lemma 4, for any $a, x \in Q$,

$$\begin{aligned} a *_2 f_{1,0}(x) &= \big(au^2\big)\big(u^2 \backslash f_{1,0}(x)\big) \\ &= \big(au^2\big)\big(u^{-2}\big(f_{1,0}(x)u^2\big) \cdot u^{-2}\big) \\ &= \big(au^2 \cdot u^{-2}\big)\big(f_{1,0}(x)u^2\big) \cdot u^{-2} \\ &= a\big(f_{1,0}(x)u^2\big) \cdot u^{-2}. \end{aligned}$$

Since Q is a right Bol loop, for any $x, z \in N$, $z((xu)x) = ((zx)u)x$. Thus, by Theorem 3,

$$\begin{aligned} & z((xu)x) = ((zx)u)x \\ \Longrightarrow\quad & z((x *_2 f_{1,0}(x))u) = ((zx) *_2 f_{1,0}(x))\, u \\ \Longrightarrow\quad & (z *_1 (x *_2 f_{1,0}(x)))\, u = ((zx) *_2 f_{1,0}(x))\, u \\ \Longrightarrow\quad & z *_1 (x *_2 f_{1,0}(x)) = (zx) *_2 f_{1,0}(x) \\ \Longrightarrow\quad & z *_1 \big(x\big(f_{1,0}(x)u^2\big) \cdot u^{-2}\big) = (zx)\big(f_{1,0}(x)u^2\big) \cdot u^{-2}. \end{aligned}$$

Since $[Q : N] = 2$, $u^2 \in N$. Hence, for any elements $z \in N$ and

$$y \in \{x *_2 f_{1,0}(x) \mid x \in N\} = \{(x(f_{1,0}(x)u^2))u^{-2} \mid x \in N\} \subseteq N,$$

$z *_1 y$ is uniquely determined by $f_{1,0}$ and the binary operation of N. □

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References

[1] **O. Chein and E.G. Goodaire**, *A new construction of Bol loops of order 8k*, J. Algebra **287** (2005), 103–122.

[2] **O. Chein and E.G. Goodaire**, *A new construction of Bol loops: the "odd" case*, Quasigroups and Related Systems **13** (2005), 87–98.

[3] **T. Foguel, M.K. Kinyon and J.D. Phillips**, *On twisted subgroups and Bol loops of odd order*, Rocky Mountain J. Math. **36** (2006), 183–212.

[4] **S.M. Gagola III**, *Cyclic extensions of Moufang loops induced by semi-automorphisms*, J. Algebra Appl. **13** (2014), no. 4, 1350128, 7 p.

[5] **The GAP Group**, *GAP-Groups, Algorithms, and Programming*, Version 4.4.12, 2008, http://www.gap-system.org.

[6] **D.A. Robinson**, *Bol loops*, Trans. Amer. Math. Soc. **123** (1966), 341–354.

[7] **D.A. Robinson**, *A Bol loop isomorphic to all loop isotopes*, Proc. Amer. Math. Soc. **19** (1968), 671–672.

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