Quasigroups and Related Systems **23** (2015), 211 − 216

A note on (m,n)-ideals in regular duo ordered semigroups

Limpapat Bussaban and Thawhat Changphas

Abstract The purpose of this note is to prove that, for a regular duo ordered semigroup, every (m, n)-ideal is a two-sided ideal. The result obtained is more general than that of the result for regular duo semigroups (without order) proved by Lajos.

1. Introduction

Let S be a semigroup (without order). Then S is said to be *regular* if $a \in aSa$ for any $a \in S$, i.e., if for any $a \in S$ there exists $x \in S$ such that $a = axa$. The semigroup S is called a *duo semigroup* if every one-sided (left or right) ideal of S is a two-sided ideal of S. In [9], S. Lajos introduced the concept of (m, n)-ideal of S as follows: let m, n be non-negative integers. A subsemigroup A of S is called an (m, n)-*ideal* of S if

$$A^m S A^n \subseteq A.$$

Here, $A^0 S = SA^0 = S$. The author proved in [10] that every (m, n)-ideal of a regular duo semigroup is two-sided ideal.

In this paper, using the concept of (m, n)-ideals for ordered semigroups introduced and studied by J. Sanborisoot and the second author in [11], we extend the results obtained by S. Lajos in [10] to ordered semigroups.

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup opration, that is, for any a, b, c in S,

$$a \leqslant b \Rightarrow ac \leqslant bc, \quad ca \leqslant cb,$$

is called an *ordered semigroup*.

The subset $(A]$ of S is defined to be the set of all elements $x \in S$ such that $x \leqslant a$ for some $a \in A$, that is,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

2010 Mathematics Subject Classification: 20M12, 20M17, 06F05
Keywords: semigroup, ordered semigroup, regular duo ordered semigroup, (m, n)-ideal, bi-ideal, π-ideal
The research was supported by the National Research Council of Thailand (NRCT), Project No. 580009

Note that the following conditions hold: (1) $A \subseteq (A]$; (2) $(A](B] \subseteq (AB]$; (3) If $A \subseteq B$, then $(A] \subseteq (B]$ (cf. [6]).

A non-empty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *left* (resp. *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);
(ii) $(A] = A$.

And, A is called a *two-sided ideal*, or simply an *ideal* of S if it is both a left and a righ ideal of S [6, 8].

A subsemigroup B of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *bi-ideal* [7] of S if it satisfies the following conditions:

(i) $BSB \subseteq B$;
(ii) $(B] = B$.

Let m, n be non-negative integers. A subsemigroup A of an ordered semigroup $(S, \cdot, \leqslant)$ is called an (m, n)-*ideal* of S if it satisfies the following conditions:

(i) $A^m SA^n \subseteq A$;
(ii) $(A] = A$.

Here, let $A^0 S = SA^0 = S$ [11].

An ordered semigroup $(S, \cdot, \leqslant)$ is *regular* if, for every $a \in S$, $a \in (aSa]$, i.e., if for any $a \in S$, $a \leq axa$ for some $x \in S$ [7]. It was proved in [3] that the following holds for a regular ordered semigroup.

Theorem 1.1. *Let $(S, \cdot, \leqslant)$ be a regular ordered semigroup. Then a non-empty subset A of S is a bi-ideal of S if and only if there exists a left ideal L of S and a right ideal R of S such that $A = (RL]$.*

As in [9], the concept of π-ideal of an ordered semigroup $(S, \cdot, \leqslant)$ are defined by: a subsemigroup S_n of S will be called *attainable* if there are subsemigroups S_i $(i = 1, 2, \ldots, n-1)$ of S such that

$$S_n \subseteq S_{n-1} \subseteq \ldots \subseteq S_2 \subseteq S_1 \subseteq S_0 = S$$

holds, where S_i $(i = 1, 2, \ldots, n)$ is an one-sided (left or right) ideal of S_{i-1}. With every such chain above, we use the letters l (resp. r) in which the i-th for a subsemigroup S_i of S which is contained in S_{i-1} is a left (resp. a right) ideal of S_{i-1}. If S_i is a two-sided ideal of S_{i-1}, then either of l and of r can be choosen. And, a product of the letters l and r will be denoted by π. Now, a subsemigroup A of S will be called a π-*ideal* of S if A is attainable.

In what follows, for the product π, we let m and n be the numbers of the factors l and r, respectively. The following two theorems can be found in [2].

Theorem 1.2. *Let A be a subset of an ordered semigroup $(S, \cdot, \leqslant)$. Then the following three statements are equivalent:*

(1) *A is an lr-ideal of S;*

(2) *A is an rl-ideal of S;*

(3) *A is an* $(1,1)$*-ideal of* S.

Consequently,

Corollary 1.3. *Let A be a subset of an ordered semigroup* $(S,\cdot,\leqslant)$. *Then A is a* π*-ideal of S if and only if A is an* $r^m l^n$*-ideal of* S.

Theorem 1.4. *Let* $(S,\cdot,\leqslant)$ *be an ordered semigroup. Then a subset A of S is a* π*-ideal of S if and only if A is an* (m,n)*-ideal of* S.

2. Main results

An ordered semigroup $(S,\cdot,\leqslant)$ will be called a *duo ordered semigroup* if every one-sided (left or right) ideal of S is a two-sided ideal of S. An ordered semigroup S will be called a *regular duo ordered semigroup* if it is both regular and duo [3].

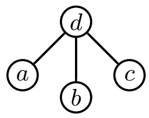
Example 2.1. Let $S=\{a,b,c,d\}$ be an ordered semigroup such that the multiplication and the partial order are defined by:

$\cdot$	a	b	c	d
a	c	d	d	d
b	c	c	d	d
c	d	d	d	d
d	d	d	d	d

$$\leq=\{(a,a),(b,b),(c,c),(d,d),(a,d),(b,d),(c,d)\}$$

We give a covering relation and the figure of S by:

$$\prec=\{(a,d),(b,d),(c,d)\}$$



Then we obtain $(S,\cdot,\leqslant)$ is a regular duo ordered semigroup.

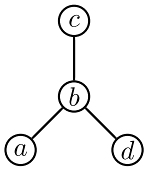
Example 2.2. Let $S=\{a,b,c,d\}$ be an ordered semigroup such that the multiplication and the partial order are defined by:

$\cdot$	a	b	c	d
a	d	b	b	d
b	b	b	b	b
c	b	b	c	b
d	d	b	b	d

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,c),(d,b),(d,c)\}$$

We give a covering relation and the figure of S by:

$$\prec = \{(a,b),(d,b),(b,c)\}$$



Then we obtain $(S,\cdot,\leqslant)$ is a regular duo ordered semigroup.

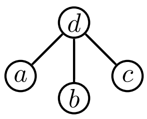
Example 2.3. Let $S=\{a,b,c,d\}$ be an ordered semigroup such that the multiplication and the partial order are defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	d
c	a	b	b	d
d	a	d	d	d

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,d),(b,d),(c,d)\}$$

We give a covering relation and the figure of S as follows:

$$\prec = \{(a,d),(b,d),(c,d)\}$$



Then we obtain that $(S,\cdot,\leqslant)$ is a regular duo ordered semigroup.

The following theorem was shown in [3].

Theorem 2.4. *Let $(S,\cdot,\leqslant)$ be a regular duo ordered semigroup. Then every bi-ideal of S is a two-sided ideal of S.*

We now present the main result of this paper.

Theorem 2.5. *Let $(S,\cdot,\leqslant)$ be a regular duo ordered semigroup, and let m,n be non-negative integers such that $m+n>0$. Then every (m,n)-ideal of S is a two-sided ideal of S.*

Proof. Let A be an (m,n)-ideal of S; thus $A^m S A^n \subseteq A$ and $(A] = A$. There are four cases to consider:
Case 1: $m = 0, n \neq 0$. If $n = 1$, then A is a left ideal of S; hence A is a two-sided ideal of S, since S is a regular duo ordered semigroup. Suppose that every $(0,k)$-ideal of S is a two-sided ideal of S for integers $k \geq 1$. Assume now that A is an $(0, k+1)$-ideal of S. By Theorem 1.4, there are subsemigroups $L_i (i = 1, 2, \ldots, k)$ of S such that

$$A = L_{k+1} \subseteq L_k \subseteq L_{k-1} \subseteq \ldots \subseteq L_2 \subseteq L_1 \subseteq L_0 = S$$

where $L_{i-1}L_i \subseteq L_i (i = 1, 2, 3, \ldots, k+1)$.

We have

$$\begin{aligned} SA^k &= SAA^{k-1} \subseteq S(ASA]A^{k-1} \subseteq (SASA^k] \subseteq (L_1 A^k] \subseteq (L_1 L_2 A^{k-1}] \\ &\subseteq (L_2 A^{k-1}] \subseteq \ldots \subseteq (L_k A] \subseteq (A] = A. \end{aligned}$$

Hence A is an $(0,k)$-ideal of S, and so A is a two-sided ideal of S.

Case 2: $m \neq 0, n = 0$. This can be proceed as the case before.

Case 3: $m \neq 0, n \neq 0$. Let A be an $(1,n)$-ideal of S. If $n = 1$, then A is a bi-ideal of S. By Theorem 2.4, A is a two-sided ideal of S.

Let $n > 1$. By Theorem 1.4, there are subsemigroups $R, L_i (i = 1, 2, \ldots, n-1)$ of S such that

$$A = L_n \subseteq L_{n-1} \subseteq L_{n-2} \subseteq \ldots \subseteq L_2 \subseteq L_1 \subseteq R \subseteq S$$

where $L_{i-1}L_i \subseteq L_i \ (i = 2, 3, \ldots, n)$, $RL_1 \subseteq L_1$, $RS \subseteq S$.

We consider

$$\begin{aligned} SA^n &\subseteq S(ASA]A^{n-1} \subseteq (SASA^n] \subseteq (RA^n] \subseteq (RL_1 A^{n-1}] \\ &\subseteq (L_1 A^{n-1}] \subseteq \ldots \subseteq (L_{n-1} A] \subseteq (A] = A. \end{aligned}$$

Then A is an $(0,n)$-ideal of S, and so A is a two-sided ideal of S.

Suppose that every (k,n)-ideal of S is a two-sided ideal of S for integer $k \geqslant 1$. Assume that A is an $(k+1, n)$-ideal of S. By Theorem 1.4, there are subsemigroups $R_j \ (j = 1, 2, \ldots, k+1)$, $L_i \ (i = 1, 2, \ldots, n-1)$ of S such that

$$\begin{aligned} A = L_n &\subseteq L_{n-1} \subseteq L_{n-2} \subseteq \ldots \subseteq L_2 \subseteq L_1 \\ &\subseteq R_{k+1} \subseteq R_k \subseteq \ldots \subseteq R_2 \subseteq R_1 \subseteq R_0 = S, \end{aligned}$$

where

$$L_{i-1}L_i \subseteq L_i \quad (i = 2, 3, \ldots, n),$$

$$R_{k+1}L_1 \subseteq L_1,$$

$$R_j R_{j-1} \subseteq R_j \quad (j = 1, 2, \ldots, k+1).$$

Consider:

$$\begin{aligned} A^kSA^n &\subseteq A^{k-1}(ASA]SA^n \subseteq (A^kSASA^n] \subseteq (A^kR_1A^n] \subseteq (A^{k-1}R_2R_1A^n] \\ &\subseteq (A^{k-1}R_2A^n] \subseteq \ldots \subseteq (R_{k+1}A^n] \subseteq (R_{k+1}L_1A^{n-1}] \subseteq (L_1A^{n-1}] \\ &\subseteq (L_1L_2A^{n-2}] \subseteq (L_2A^{n-2}] \subseteq \ldots \subseteq (L_{n-1}A] \subseteq (A] = A. \end{aligned}$$

Hence A is an (k,n)-ideal of S. Therefore, A is a two-sided ideal of S.

This completes the proof of the theorem. □

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Received June 16, 2015

Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

E-mail: lim.bussaban@gmail.com, thacha@kku.ac.th