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Parastrophes of quasigroups

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Abstract. Parastrophes (conjugates) of a quasigroup can be divided into separate classes containing isotopic parastrophes. We prove that the number of such classes is always 1, 2, 3 or 6. Next we characterize quasigroups having a fixed number of such classes.

1. Introduction

Denote by S_Q the set of all permutations of the set Q. We say that a quasigroup $(Q,\cdot)$ is *isotopic* to $(Q,\circ)$ if there are $\alpha,\beta,\gamma\in S_Q$ such that $\alpha(x)\circ\beta(y)=\gamma(x\cdot y)$ for all $x,y\in Q$. The triplet (α,β,γ) is called an *isotopism*. Quasigroups $(Q,\cdot)$ and $(Q,\circ)$ for which there are $\alpha,\beta,\gamma\in S_Q$ such that $\alpha(x)\circ\beta(y)=\gamma(y\cdot x)$ for all $x,y\in Q$ are called *anti-isotopic*. This fact is denoted by $(Q,\cdot)\sim(Q,\circ)$. In the case when $(Q,\cdot)$ and $(Q,\circ)$ are isotopic we write $(Q,\cdot)\approx(Q,\circ)$. It is clear that the relation $\approx$ is an equivalence and divides all quasigroups into disjoint classes containing isotopic quasigroups.

Each quasigroup $Q=(Q,\cdot)$ determines five new quasigroups $Q_i=(Q,\circ_i)$ with the operations $\circ_i$ defined as follows:

$$
\begin{array}{l}
x\circ_1 y=z\longleftrightarrow x\cdot z=y\\
x\circ_2 y=z\longleftrightarrow z\cdot y=x\\
x\circ_3 y=z\longleftrightarrow z\cdot x=y\\
x\circ_4 y=z\longleftrightarrow y\cdot z=x\\
x\circ_5 y=z\longleftrightarrow y\cdot x=z
\end{array}
$$

Such defined (not necessarily distinct) quasigroups are called *parastrophes* or *conjugates* of Q. Traditionally they are denoted as

$$Q_1=Q^{-1}=(Q,\backslash),\quad Q_2={}^{-1}Q=(Q,/),\quad Q_3={}^{-1}(Q^{-1})=(Q_1)_2,$$

$$Q_4=({}^{-1}Q)^{-1}=(Q_2)_1\ \text{ and }\ Q_5=({}^{-1}(Q^{-1}))^{-1}=((Q_1)_2)_1=((Q_2)_1)_2.$$

Each parastrophe Q_i can be obtained from Q by the permutation σ_i, where $\sigma_1=(23)$, $\sigma_2=(13)$, $\sigma_3=(132)$, $\sigma_4=(123)$, $\sigma_5=(12)$.

Generally, parastrophes Q_i do not save properties of Q. Parastrophes of a group are not a group, but parastrophes of an idempotent quasigroup also are idempotent

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quasigroups. Moreover, in some cases (described in [7]) parastrophes of a given quasigroup Q are pairwise equal or all are pairwise distinct (see also [2] and [8]). In [7] it is proved that the number of distinct parastrophes of a quasigroup is always a divisor of 6 and does not depend on the number of elements of a quasigroup.

Parastrophes of each quasigroup can be divided into separate classes containing isotopic parastrophes. We prove that the number of such classes is always 1, 2, 3 or 6. The number of such classes depends on the existence of an anti-isotopism of a quasigroup and some parastrophe of it.

2. Classification of parastrophes

As it is known (see for example [1]) a quasigroup $(Q,\cdot)$ can be considered as an algebra $(Q,\cdot,\backslash,/)$ with three binary operations satisfying the following axioms

$$x(x\backslash z)=z,\quad (z/y)y=z,\quad x\backslash xy=y,\quad xy/y=x,$$

where

$$x\backslash z=y\longleftrightarrow xy=z\quad\text{and}\quad z/y=x\longleftrightarrow xy=z.$$

We will use these axioms to show the relationship between parastrophes. But let's start with the following simple observation.

Lemma 2.1. *Let Q be a quasigroup. Then*

(a) $xy=y\circ_5 x,\;\; x\circ_1 y=y\circ_4 x,\;\; x\circ_2 y=y\circ_3 x,$

(b) $Q\sim Q_5,\;\; Q_1\sim Q_3,\;\; Q_2\sim Q_4,$

(c) $xy=yx\longleftrightarrow Q=Q_5\longleftrightarrow Q_1=Q_3\longleftrightarrow Q_2=Q_4,$

(d) $Q_1=Q\longleftrightarrow Q_2=Q_3\longleftrightarrow Q_4=Q_5,$

(e) $Q_2=Q\longleftrightarrow Q_1=Q_4\longleftrightarrow Q_3=Q_5.$

To describe the relationship between the parastrophes, we will need these two simple lemmas.

Lemma 2.2. *Let A,B,C,D be quasigroups. Then*

(a) $A\sim B,\; B\sim C\longrightarrow A\approx C,$

(b) $A\sim B,\; B\approx C\longrightarrow A\sim C,$

(c) $A\approx B,\; B\sim C\longrightarrow A\sim C.$

Lemma 2.3. *Let Q_i° be the i-th parastrophe of the quasigroup $Q^\circ=(Q,\circ)$. Then*

(a) $Q\approx Q^\circ$ *implies* $Q_i\approx Q_i^\circ$ *for each* $i=1,2,3,4,5,$

(b) $Q_i\approx Q_i^\circ$ *for some* $i=1,2,3,4,5$ *implies* $Q\approx Q^\circ$.

(c) *Moreover, if* $Q\approx Q^\circ$, *then for each* $i=1,\dots,5$

$$Q\sim Q_i\longleftrightarrow Q^\circ\sim Q_i^\circ,\ \text{and}\ \ Q\approx Q_i\longleftrightarrow Q^\circ\approx Q_i^\circ.$$

Now we will present a series of lemmas about anti-isotopies of quasigroups and their parastrophes.

Lemma 2.4. $Q \sim Q \longleftrightarrow Q \approx Q_5 \longleftrightarrow Q_1 \approx Q_3 \longleftrightarrow Q_2 \approx Q_4$.

Proof. Indeed,

$$Q \sim Q \longleftrightarrow \gamma(xy) = \alpha(y)\beta(x) \longleftrightarrow \gamma(xy) = \beta(x) \circ_5 \alpha(y) \longleftrightarrow Q \approx Q_5.$$

Also

$$Q \sim Q \longleftrightarrow \gamma(xy) = \alpha(y)\beta(x) \longleftrightarrow \gamma(z) = \alpha(y)\beta(z/y) \longleftrightarrow \alpha(y)\backslash\gamma(z) = \beta(z/y).$$

Thus $Q \sim Q \longleftrightarrow Q_1 \sim Q_2$. Moreover,

$$\begin{aligned} Q_1 \sim Q_2 &\longleftrightarrow \alpha(y)\backslash\gamma(z) = \beta(z/y) \longleftrightarrow \gamma(z) = \alpha(y)\beta(z/y) \\ &\longleftrightarrow \beta(z/y) = \gamma(z) \circ_4 \alpha(y) \longleftrightarrow Q_2 \approx Q_4. \end{aligned}$$

Similarly, for some $\alpha', \beta', \gamma' \in S_Q$ we have

$$\begin{aligned} Q_1 \sim Q_2 &\longleftrightarrow \gamma'(x/y) = \alpha'(y)/\beta'(x) \longleftrightarrow \gamma'(x\backslash y)\beta'(x) = \alpha'(y) \\ &\longleftrightarrow \gamma'(x\backslash y) = \beta'(x) \circ_3 \alpha'(y) \longleftrightarrow Q_1 \approx Q_3. \end{aligned}$$

This completes the proof. □

Lemma 2.5. $Q \sim Q_1 \longleftrightarrow Q \sim Q_2 \longleftrightarrow Q_1 \approx Q_2$.

Proof. Indeed, according to the definition of the operations $\backslash$ and $/$, we have

$$\gamma(x\backslash z) = \alpha(z)\beta(x) \longleftrightarrow \gamma(y) = \alpha(xy)\beta(x) \longleftrightarrow \alpha(xy) = \gamma(y)/\beta(x).$$

So, $Q_1 \sim Q \longleftrightarrow Q \sim Q_2$, which by Lemma 2.2 implies $Q_1 \approx Q_2$.

Conversely, if $Q_1 \approx Q_2$, then $\gamma(x\backslash y) = \alpha(x)/\beta(y)$, i.e., $\gamma(x\backslash y)\beta(y) = \alpha(x)$ for some $\alpha, \beta, \gamma \in S_Q$. From this, for $y = xz$, we obtain $\gamma(z)\beta(xz) = \alpha(x)$, i.e., $\beta(xz) = \gamma(z)\backslash\alpha(x)$. Thus, $Q \sim Q_1$, and consequently, also $Q \sim Q_2$. □

Lemma 2.6. *For any quasigroup Q*

(a) $Q_1 \sim Q \longleftrightarrow Q_1 \sim Q_3 \longleftrightarrow Q \approx Q_3 \longleftrightarrow Q_1 \approx Q_5$,

(b) $Q_2 \sim Q \longleftrightarrow Q_2 \sim Q_4 \longleftrightarrow Q \approx Q_4 \longleftrightarrow Q_2 \approx Q_5$.

Proof. Replacing in Lemma 2.5 a quasigroup Q by Q_1 we get the first two equivalences. The third equivalence is a consequence of Lemma 2.3.

Similarly, replacing Q by Q_2 we obtain (b). □

Lemma 2.7. $Q_3 \sim Q \longleftrightarrow Q \approx Q_2 \longleftrightarrow Q_1 \approx Q_4 \longleftrightarrow Q_3 \approx Q_5$.

Proof. Obviously $Q_3 \sim Q \longleftrightarrow Q_3 \approx Q_5$. Moreover,

$$Q_3 \sim Q \longleftrightarrow \gamma(xy)\alpha(y) = \beta(x) \longleftrightarrow \gamma(xy) = \beta(x)/\alpha(y) \longleftrightarrow Q \approx Q_2.$$

Analogously, $xy = z$ we obtain

$$Q_3 \sim Q \longleftrightarrow \gamma(z)\alpha(x\backslash z) = \beta(x) \longleftrightarrow Q_1 \approx Q_4.$$

This completes the proof. □

Lemma 2.8. $Q_4 \sim Q \longleftrightarrow Q \approx Q_1 \longleftrightarrow Q_2 \approx Q_3 \longleftrightarrow Q_4 \approx Q_5$.

Proof. Of course $Q_4 \sim Q \longleftrightarrow Q_4 \approx Q_5$. Since $Q_4 \sim Q \longleftrightarrow \beta(x)\gamma(xy) = \alpha(y)$, we obtain $Q_4 \sim Q \longleftrightarrow Q \approx Q_1$ and $Q_4 \sim Q \longleftrightarrow Q_2 \approx Q_3$ for $x = z/y$. $\square$

Theorem 2.9. *All parastrophes of a quasigroup Q are isotopic to Q if and only if $Q \sim Q$ and $Q \sim Q_i$ for some $i = 1, 2, 3, 4$.*

Proof. If $Q \sim Q$, then, by Lemma 2.4, we have $Q \approx Q_5$, $Q_1 \approx Q_3$ and $Q_2 \approx Q_4$. This for $Q \sim Q_i$, $i = 1, 2, 3, 4$, by Lemmas 2.6, 2.7 and 2.8, gives $Q \approx Q_1 \approx Q_2 \approx Q_3 \approx Q_4 \approx Q_5$. So, in this case all parastrophes are isotopic to Q.

The converse statement is obvious. $\square$

Corollary 2.10. *If $Q \sim Q$ and $Q \sim Q_i$ for some $i = 1, 2, 3, 4$, then also $Q \sim Q_i$ for other $i = 1, 2, 3, 4, 5$.*

Theorem 2.11. *A quasigroup Q has exactly two classes of isotopic parastrophes if and only if*

(1) *$Q \not\sim Q$, $Q \sim Q_1$ and $Q \not\sim Q_i$ for $i = 2, 3, 4$, or equivalently,*

(2) *$Q \not\sim Q$, $Q \sim Q_2$ and $Q \not\sim Q_i$ for $i = 1, 3, 4$.*

In this case $Q \approx Q_3 \approx Q_4$ and $Q_1 \approx Q_2 \approx Q_5$.

Proof. Let Q have exactly two classes of isotopic parastrophes. Then it must be true that $Q \approx Q_i$ for some $i = 1, 2, 3, 4, 5$ because $Q \not\approx Q_i$ for all $i = 1, 2, 3, 4, 5$ gives $Q_1 \approx Q_j$ for some j which by previous lemmas implies $Q \approx Q_k$ for some k.

Case $Q \approx Q_1$. In this case $Q_2 \approx Q_3$ and $Q_4 \approx Q_5$ (Lemma 2.8). So, the following classes of isotopic parastrophes are possible:

1) $\{Q, Q_1, Q_2, Q_3\}$, $\{Q_4, Q_5\}$,

2) $\{Q, Q_1, Q_4, Q_5\}$, $\{Q_2, Q_3\}$,

3) $\{Q, Q_1\}$, $\{Q_2, Q_3, Q_4, Q_5\}$.

In the first case from $Q_1 \approx Q_3$, by Lemma 2.4, we conclude $Q \approx Q_5$ which shows that in this case we have only one class. This contradics our assumption on the number of classes. So, this case is impossible.

In the second case, $Q \approx Q_5$, by the same lemma, implies $Q_2 \approx Q_4$ which (similarly as in previous case) is impossible. Also the third case is impossible because $Q_2 \approx Q_4$ leads to $Q_1 \approx Q_3$. Hence must be $Q \not\approx Q_1$.

Case $Q \approx Q_2$. Then, according to Lemma 2.7, $Q_1 \approx Q_4$ and $Q_3 \approx Q_5$. Thus

1) $\{Q, Q_1, Q_2, Q_4\}$, $\{Q_3, Q_5\}$, or

2) $\{Q, Q_2, Q_3, Q_5\}$, $\{Q_1, Q_4\}$, or

3) $\{Q, Q_2\}$, $\{Q_1, Q_3, Q_4, Q_5\}$.

Using the same argumentation as in the case $Q \approx Q_1$ we can see that the case $Q \approx Q_2$ is impossible.

Case $Q \approx Q_3$. By Lemmas 2.1, 2.2 and 2.5 only the following classes are possible: $\{Q, Q_3, Q_4\}$ and $\{Q_1, Q_2, Q_5\}$. In this case $Q \not\sim Q$ (Lemma 2.4) and $Q \sim Q_1$ (Lemma 2.6). Then also $Q \sim Q_2$ (Lemma 2.5).

Case $Q \approx Q_4$. Analogously as $Q \approx Q_3$.

Case $Q \approx Q_5$. Then $Q_1 \approx Q_3$ and $Q_2 \approx Q_4$. Is a similar way as for $Q \approx Q_1$ we can verify that this case is not possible.

So, if Q has exactly two classes of isotopic parastrophes, then $Q \not\sim Q$ and $Q \sim Q_1$, or $Q \not\sim Q$ and $Q \sim Q_2$.

Conversely, if $Q \not\sim Q$ and $Q \sim Q_1$, or equivalently, $Q \not\sim Q$ and $Q \sim Q_2$, then by Lemmas 2.5 and 2.6 we have two classes: $\{Q, Q_3, Q_4\}$ and $\{Q_1, Q_2, Q_5\}$. Since $Q_1 \not\approx Q_3$ (Lemma 2.4), these classes are disjoint. □

Theorem 2.12. *A quasigroup Q has exactly three classes of isotopic parastrophes if and only if*

(1) *$Q \not\sim Q$, $Q \sim Q_3$ and $Q \not\sim Q_i$ for $i = 1, 2, 4$, or*

(2) *$Q \not\sim Q$, $Q \sim Q_4$ and $Q \not\sim Q_i$ for $i = 1, 2, 3$, or*

(3) *$Q \sim Q$, $Q \sim Q_5$ and $Q \not\sim Q_i$ for $i = 1, 2, 3, 4$.*

In the first case we have $\{Q, Q_2\}$, $\{Q_1, Q_4\}$ and $\{Q_3, Q_5\}$; in the second $\{Q, Q_1\}$, $\{Q_2, Q_3\}$ and $\{Q_4, Q_5\}$; in the third $\{Q, Q_5\}$, $\{Q_1, Q_3\}$ and $\{Q_2, Q_4\}$.

Proof. Suppose that a quasigroup Q has exactly three classes of isotopic parastrophes. From the above lemmas it follows that in this case $Q \approx Q_i$ for some i.

Case $Q \approx Q_1$. Then, by Lemma 2.8, we have three classes $\{Q, Q_1\}$, $\{Q_2, Q_3\}$, $\{Q_4, Q_5\}$ and $Q \sim Q_4$. Since $Q_1 \not\approx Q_3$ we also have $Q \not\sim Q$ (Lemma 2.4).

Case $Q \approx Q_2$. In this case $\{Q, Q_2\}$, $\{Q_1, Q_4\}$, $\{Q_3, Q_5\}$ and $Q \sim Q_3$ (Lemma 2.7). Analogously as in the previous case $Q_1 \not\approx Q_3$ gives $Q \not\sim Q$.

Case $Q \approx Q_3$. This case is impossible because by Lemmas 2.5 and 2.6 it leads to two classes.

Case $Q \approx Q_4$. Analogously as $Q \approx Q_3$.

Case $Q \approx Q_5$. Then $Q_1 \approx Q_3$, $Q_2 \approx Q_4$ and $Q \sim Q$. Since classes $\{Q, Q_5\}$, $\{Q_1, Q_3\}$, $\{Q_2, Q_5\}$ are disjoint $Q \not\sim Q_i$ for each $i = 1, 2, 3, 4$.

The converse statement is obvious. □

As a consequence of the above results we obtain

Corollary 2.13. *Parastrophes of a quasigroup Q are non-isotopic if and only if $Q \not\sim Q$ and $Q \not\sim Q_i$ for all $i = 1, 2, 3, 4$.*

Corollary 2.14. *The number of non-isotopic parastrophes of a quasigroup Q is always 1, 2, 3, or 6.*

Depending on the relationship between parastrophes quasigroups can be divided into six types presented below.

type	classes of isotopic parastrophes
A	$\{Q, Q_1, Q_2, Q_3, Q_4, Q_5\}$
B	$\{Q, Q_3, Q_4\}$, $\{Q_1, Q_2, Q_5\}$
C	$\{Q, Q_2\}$, $\{Q_1, Q_4\}$, $\{Q_3, Q_5\}$
D	$\{Q, Q_1\}$, $\{Q_2, Q_3\}$, $\{Q_4, Q_5\}$
E	$\{Q, Q_5\}$, $\{Q_1, Q_3\}$, $\{Q_2, Q_4\}$
F	$\{Q\}$, $\{Q_1\}$, $\{Q_2\}$, $\{Q_3\}$, $\{Q_4\}$, $\{Q_5\}$

Our results are presented in the following table where "+" means that the corresponding relation holds. The symbol "$-$" means that this relation has no place.

$Q \sim Q$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$Q \approx Q_5$
$Q \sim Q_1$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$Q \approx Q_3$
$Q \sim Q_2$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$Q \approx Q_4$
$Q \sim Q_3$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$Q \approx Q_2$
$Q \sim Q_4$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$Q \approx Q_1$
type	A	B	B	C	D	E	F	

The parastrophe Q_5 plays no role in our research since always is $Q \sim Q_5$.

3. Parastrophes of selected quasigroups

In this section we present characterizations of parastrophes of several classical types of quasigroups. We start with parastrophes of IP-quasigroups.

As a consequence of our results, we get the following well-known fact (see for example [1])

Proposition 3.1. *All parastrophes of an IP-quasigroup are isotopic.*

Proof. Indeed, in any IP-quasigroup Q there are permutations $\alpha, \beta \in S_Q$ such that $\alpha(x) \cdot xy = y = yx \cdot \beta(x)$ for all $x, y \in Q$. So, $Q \approx Q_1 \approx Q_2$, i.e., Q is a quasigroup of type A. □

Corollary 3.2. *Parastrophes of a group are isotopic.*

The same is true for the parastrophes of Moufang quasigroups since groups and Moufang quasigroups are IP-quasigroups.

Also parastrophes of T-quasigroups, linear and alinear quasigroups (studied in [3]) are isotopic. This fact follows from more general result proved below.

Theorem 3.3. *All parastrophes of a quasigroup isotopic to a group are isotopic.*

Proof. Let $G = (G, \circ)$ be a group. Then $\varphi(x \circ y) = \varphi(y) \circ \varphi(x)$ for $\varphi(x) = x^{-1}$. Since $(Q, \cdot) \approx (G, \circ)$, for some α, β, γ we have

$$\gamma(xy) = \alpha(x) \circ \beta(y) = \varphi^{-1}(\varphi\beta(y) \circ \varphi\alpha(x)) = \varphi^{-1}\gamma\left(\alpha^{-1}\varphi\beta(y) \cdot \beta^{-1}\varphi\alpha(x)\right).$$

Thus $\gamma^{-1}\varphi\gamma(xy) = \alpha^{-1}\varphi\beta(y) \cdot \beta^{-1}\varphi\alpha(x)$. So, $Q \sim Q$.

Moreover, from $\gamma(xy) = \alpha(x) \circ \beta(y)$ for $xy = z$ we obtain

$$\alpha(x) \backslash\!\backslash \gamma(z) = \beta(x \backslash z) \quad \text{and} \quad \gamma(z) /\!/ \beta(y) = \alpha(z/y),$$

where $\backslash\!\backslash$ and $/\!/$ are inverse operations in a group G. Thus $Q_1 \approx G_1$ and $Q_2 \approx G_2$. Since $G \approx G_1 \approx G_2$, also $Q \approx Q_1 \approx Q_2$. This shows that a quasigroup isotopic to a group is a quasigroup of type A. Hence (Lemma 2.3) all its parastrophes are isotopic. □

D-loops (called also *loops with anti-automorphic property*) are defined as loops with the property $(xy)^{-1} = y^{-1}x^{-1}$, where x^{-1} denotes the inverse element [5].

Theorem 3.4. *Let Q be a D-loop. Then*

(1) *all parastrophes of Q coincide with Q, or*

(2) *Q has three classes of isotopic parastrophes: $\{Q, Q_5\}$, $\{Q_1, Q_3\}$, $\{Q_2, Q_4\}$.*

The second case holds if and only if $Q \not\sim Q_1$ or $Q \not\approx Q_1$.

Proof. Let Q be a D-loop. Then $Q \sim Q$. Thus all its parastrophes are isotopic to Q or they are divided into three classes $\{Q, Q_5\}$, $\{Q_1, Q_3\}, \{Q_2, Q_4\}$ (see Table). By Lemmas 2.6 and 2.8 they are disjoint if and only if $Q \not\sim Q_1$ or $Q \not\approx Q_1$. □

Corollary 3.5. *A D-loop Q has three classes of isotopic parastrophes if and only if $Q \not\sim Q_2$ or $Q \not\approx Q_2$.*

In [5] is proved that parastrophes of a D-loop Q are isomorphic to one of the quasigroups Q, Q_1, Q_2. Comparing this fact with our results we obtain

Theorem 3.6. *For a D-loop Q the following conditions are equivalent:*

(1) *all parastrophes of Q are isomorphic,*

(2) *Q and Q_1 are isomorphic,*

(3) *Q and Q_2 are isomorphic,*

(4) *Q_1 and Q_2 are isomorphic.*

Example 3.7. Consider the following three loops.

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	3	4
3	3	6	1	2	4	5
4	4	5	2	1	6	3
5	5	3	4	6	1	2
6	6	4	5	3	2	1

$\circ_1$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	5	6	4	3
3	3	4	1	5	6	2
4	4	3	6	1	2	5
5	5	6	2	3	1	4
6	6	5	4	2	3	1

$\circ_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	2	4
4	4	6	5	1	3	2
5	5	4	6	2	1	3
6	6	3	2	5	4	1

The first loop is a D-loop, the second and the third are parastrophes of the first. They are not D-loops and are not isotopic to the first. So this D-loop has three classes of isotopic parastrophes. In this case $Q = Q_5$, $Q_1 = Q_3$ and $Q_2 = Q_4$. □

4. Some consequences

Note first of all that the proofs of our results remain true also for the case when $\alpha = \beta = \gamma$. In this case an anti-isotopism is an anti-isomorphism and an isotopism is an isomorphism. So, the above results will be true if we replace an anti-isotopism by an anti-isomorphism, and an isotopism by an isomorphism. Moreover, an isotopism of parastrophes can be characterized by the identities:

$$\alpha_1(x)\cdot\beta_1(yx) = \gamma_1(y), \tag{1}$$
$$\beta_2(xy)\cdot\alpha_2(x) = \gamma_2(y), \tag{2}$$
$$\beta_3(yx)\cdot\alpha_3(x) = \gamma_3(y), \tag{3}$$
$$\alpha_4(x)\cdot\beta_4(xy) = \gamma_4(y), \tag{4}$$
$$\beta_5(xy) = \gamma_5(y)\cdot\alpha_5(x), \tag{5}$$

where $\alpha_i, \beta_i, \gamma_i$ are fixed permutations of the set Q.

Namely, from our results it follows that

$$\begin{aligned}
&Q \ \text{ satisfies (1)} \longleftrightarrow Q_1 \sim Q \longleftrightarrow Q_3 \approx Q,\\
&Q \ \text{ satisfies (2)} \longleftrightarrow Q_2 \sim Q \longleftrightarrow Q_4 \approx Q,\\
&Q \ \text{ satisfies (3)} \longleftrightarrow Q_3 \sim Q \longleftrightarrow Q_2 \approx Q,\\
&Q \ \text{ satisfies (4)} \longleftrightarrow Q_4 \sim Q \longleftrightarrow Q_1 \approx Q,\\
&Q \ \text{ satisfies (5)} \longleftrightarrow Q \sim Q \longleftrightarrow Q_5 \approx Q.
\end{aligned}$$

Lemma 2.3 shows that these identities are universal in some sense, i.e., if one of these identities is satisfied in a quasigroup Q, then in a quasigroup isotopic to Q is satisfied the identity of the same type, i.e., it is satisfied with other permutations.

Since $Q \sim Q_1 \longleftrightarrow Q \sim Q_2$ we have

Proposition 4.1. *A quasigroup Q satisfies for some $\alpha_1, \beta_1, \gamma_1 \in S_Q$ the identity* (1) *if and only if for some $\alpha_2, \beta_2, \gamma_2 \in Q_S$ it satisfies the identity* (2).

As a consequence we obtain the following classification of quasigroups>

Theorem 4.2. *Let Q be a quasigroup. Then*

- *Q is type A if and only if it satisfies all of the identities* $(1)-(5)$,
- *Q is type B if and only if it satisfies only* (1) *and* (2),
- *Q is type C if and only if it satisfies only* (3),
- *Q is type D if and only if it satisfies only* (4),
- *Q is type E if and only if it satisfies only* (5),
- *Q is type F if and only if it satisfies none of the identities* $(1)-(5)$.

If all permutations used in $(1)-(5)$ are the identity permutations, then these equations have of the form:

$$x \cdot yx = y, \tag{6}$$
$$xy \cdot x = y, \tag{7}$$
$$yx \cdot x = y, \tag{8}$$
$$x \cdot xy = y, \tag{9}$$
$$xy = yx. \tag{10}$$

Basing on our results we conclude that

$$\begin{aligned} Q \text{ satisfies } (6) &\longleftrightarrow Q = Q_4,\\ Q \text{ satisfies } (7) &\longleftrightarrow Q = Q_3,\\ Q \text{ satisfies } (8) &\longleftrightarrow Q = Q_2,\\ Q \text{ satisfies } (9) &\longleftrightarrow Q = Q_1,\\ Q \text{ satisfies } (10) &\longleftrightarrow Q = Q_5. \end{aligned}$$

Since Q satisfies (7) $\longleftrightarrow Q_5 = Q_2 \longleftrightarrow ((Q_1)_2)_1 = Q_2 \longleftrightarrow Q_1 = ((Q_2)_1)_2 \longleftrightarrow Q_1 = Q_5 \longleftrightarrow$ Q satisfies (6), we see that identities (7) and (6) are equivalent, i.e., Q satisfies (7) if and only if it satisfies (6).

As a consequence we obtain the stronger version of Theorem 4 in [7].

Theorem 4.3. *Parastrophes of a quasigroup Q can be characterized by the identities* $(6)-(10)$ *in the following way:*

- *$Q = Q_i$ for $1 \leqslant i \leqslant 5$ if and only if it satisfies all of the identities* $(6)-(10)$,
- *$Q = Q_3 = Q_4,\ Q_1 = Q_2 = Q_5$ if and only if Q satisfies only* (7) *and* (6),
- *$Q = Q_2,\ Q_1 = Q_4,\ Q_3 = Q_5$ if and only if Q satisfies only* (8),
- *$Q = Q_1,\ Q_2 = Q_3,\ Q_4 = Q_5$ if and only if Q satisfies only* (9),
- *$Q = Q_5,\ Q_1 = Q_3,\ Q_2 = Q_4$ if and only if Q satisfies only* (10),
- *$Q \neq Q_i \neq Q_j$ for all $1 \leqslant i < j \leqslant 5$ if and only if Q satisfies none of the identities* $(6)-(10)$.

Corollary 4.4. *Parastrophes of a commutative quasigroup Q coincide with Q or are divided into three classes:* $\{Q = Q_5\}$, $\{Q_1 = Q_3\}$, $\{Q_2 = Q_4\}$.

Corollary 4.5. *For a commutative quasigroup Q the following conditions are equivalent:*

(1) *all parastrophes of Q coincide with Q,*

(2) $Q = Q_1$,

(3) $Q = Q_2$,

(4) $Q_1 = Q_2$,

(5) *Q satisfies at least one of the identities* $(6) - (9)$.

Proof. We prove only the equivalence (1) $\longleftrightarrow$ (2). Other equivalences can be proved in a similar way.

For a commutative Q we have $Q = Q_5$, $Q_1 = Q_3$, $Q_2 = Q_4$. If $Q = Q_1$, then $Q = Q_1 = Q_3 = Q_5$. Hence $Q_1 = Q_5 = ((Q_2)_1)_2$ which gives $(Q_1)_2 = (Q_2)_1$. So, $Q_3 = Q_4$, i.e., (2) implies (1). The converse implication is obvious. □

Corollary 4.6. *Parastrophes of a boolean group coincide with this group.*

Note finally that identities $(6)-(10)$ can be used to determine some autotopisms of quasigroups [4].

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