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On intra-regular and some left regular Γ-semigroups

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Abstract. We characterize the intra-regular Γ-semigroups and the left regular Γ-semigroups M in which $x\Gamma M \subseteq M\Gamma x$ for every $x \in M$ in terms of filters and we prove, among others, that every intra-regular Γ-semigroup is decomposable into simple components, and every Γ-semigroup M for which $x\Gamma M \subseteq M\Gamma x$ is left regular, is decomposable into left simple components.

1. Introduction and prerequisites

A structure theorem concerning the intra-regular semigroups, another one concerning some left regular semigroups have been given in [3]. These are the two theorems in [3]:

Theorem II.4.9. *The following conditions on a semigroup S are equivalent:*

(1) *Every $\mathcal{N}$-class of S is simple.*
(2) *Every ideal of S is completely semiprime.*
(3) *For every $x \in S$, $x \in Sx^2S$.*
(4) *For every $x \in S$, $N(x) = \{y \in S \mid x \in SyS\}$.*
(5) $\mathcal{N} = \mathcal{I}$.
(6) *Every ideal of S is a union of $\mathcal{N}$-classes.*

Theorem II.4.5. *The following conditions on a semigroup S are equivalent:*

(1) *Every $\mathcal{N}$-class of S is left simple.*
(2) *Every left ideal of S is completely semiprime and two-sided.*
(3) *For every $x \in S$, $x \in Sx^2$ and $xS \subseteq Sx$.*
(4) *For every $x \in S$, $N(x) = \{y \in S \mid x \in Sy\}$.*
(5) $\mathcal{N} = \mathcal{L}$.
(6) *Every left ideal of S is a union of $\mathcal{N}$-classes.*

Note that we always use the term "semiprime" instead of "completely semiprime" given by Petrich in [3]. So the condition (2) in the two theorems above should be read as "Every ideal (resp. left ideal) of S is semiprime", meaning that if A is an ideal (resp. left ideal) of S, then for every $x \in S$ such that $x^2 \in A$, we have $x \in A$. In the present paper we generalize these results in case of Γ-semigroups.

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Let M be a Γ-semigroup. An equivalence relation σ on M is called *left* (resp. *right*) *congruence* (on M) if $(a,b) \in \sigma$ implies $(c\gamma a, c\gamma b) \in \sigma$ (resp. $(a\gamma c, b\gamma c) \in \sigma$) for every $c \in M$ and every $\gamma \in \Gamma$. A relation σ which is both left and right congruence on M is called a *congruence* on M. A congruence σ on M is called *semilattice congruence* if $(a\gamma b, b\gamma a) \in \sigma$ and $(a\gamma a, a) \in \sigma$ for every $a, b \in M$ and every $\gamma \in \Gamma$. A nonempty subset A of M is called a *left* (resp. *right*) *ideal* of M if $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$). A subset A of M which is both a left and right ideal of M is called an *ideal* of M. For an element a of M, we denote by $L(a)$, $R(a)$, $I(a)$ the left ideal, right ideal and the ideal of M, respectively, generated by a, and we have $L(a) = a \cup M\Gamma a$, $R(a) = a \cup a\Gamma M$, $I(a) = a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M$. We denote by $\mathcal{L}$ the equivalence relation on M defined by $\mathcal{L} := \{(a,b) \mid L(a) = L(b)\}$, by $\mathcal{R}$ the equivalence relation on M defined by $\mathcal{R} := \{(a,b) \mid R(a) = R(b)\}$ and by $\mathcal{I}$ the equivalence relation on M defined by $\mathcal{I} := \{(a,b) \mid I(a) = I(b)\}$. A nonempty subset A of M is called a *subsemigroup* of M if $a, b \in A$ and $\gamma \in \Gamma$ implies $a\gamma b \in A$, that is, $A\Gamma A \subseteq A$. A subsemigroup F of M is called a *filter* of M if $a, b \in F$ and $\gamma \in \Gamma$ such that $a\gamma b \in F$ implies $a \in F$ and $b \in F$. We denote by $\mathcal{N}$ the relation on M defined by $\mathcal{N} := \{(a,b) \mid N(a) = N(b)\}$ where $N(x)$ is the filter of M generated by x ($x \in M$). It is well known that the relation $\mathcal{N}$ is a semilattice congruence on M. So, if $z \in M$ and $\gamma \in \Gamma$, then we have $(z\gamma z, z) \in \mathcal{N}$, $(z\gamma z\gamma z, z\gamma z) \in \mathcal{N}$, $(z\gamma z\gamma z\gamma z, z\gamma z\gamma z) \in \mathcal{N}$ and so on. A subset A of M is called *semiprime* if $a \in M$ and $\gamma \in \Gamma$ such that $a\gamma a \in A$ implies $a \in A$. A Γ-semigroup $(M, \Gamma, .)$ is called *left simple* if for every left ideal L of M, we have $L = M$, that is, M is the only left ideal of M. A subsemigroup T of M is called *left simple* if the Γ-semigroup $(T, \Gamma, .)$ (that is, the set T with the same Γ and the multiplication "." on M) is left simple. Which means that for every left ideal A of T, we have $A = T$. A subsemigroup of M which is both left simple and right simple is called *simple*. If M is a Γ-semigroup and σ a semilattice congruence on M, then the class $(a)_\sigma$ of M containing a is a subsemigroup of M for every $a \in M$. Let now M be a Γ-semigroup and σ a congruence on M. For $a, b \in M$ and $\gamma \in \Gamma$, we define $(a)_\sigma \gamma (b)_\sigma := (a\gamma b)_\sigma$. Then the set $M/\sigma := \{(a)_\sigma \mid a \in M\}$ is a Γ-semigroup as well. A Γ-semigroup M is said to be a *semilattice of simple semigroups* if there exists a semilattice congruence σ on M such that the class $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$.

2. Intra-regular Γ-semigroups

We characterize here the intra-regular Γ-semigroups in terms of filtres and we prove that every intra-regular Γ-semigroup is decomposable into simple subsemigroups.

Definition 1. (cf. [2]) A Γ-semigroup M is called *intra-regular* if

$$x \in M\Gamma x\gamma x\Gamma M$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Lemma 2. (cf. [1]) *If M is a Γ-semigroup, then $\mathcal{I} \subseteq \mathcal{N}$.*

Theorem 3. *Let M be a Γ-semigroup. The following are equivalent:*

(1) *M is intra-regular.*
(2) *$N(x) = \{y \in M \mid x \in M\Gamma y\Gamma M\}$ for every $x \in M$.*
(3) *$\mathcal{N} = \mathcal{I}$.*
(4) *For every ideal I of M, we have $I = \bigcup_{x \in I} (x)_{\mathcal{N}}$.*
(5) *$(x)_{\mathcal{N}}$ is a simple subsemigroup of M for every $x \in M$.*
(6) *M is a semilattice of simple semigroups.*
(7) *Every ideal of M is semiprime.*

Proof. (1) $\Longrightarrow$ (2). Let $x \in M$ and $T := \{y \in M \mid x \in M\Gamma y\Gamma M\}$. T is a filter of M. In fact: Take an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since M is intra-regular, we have

$$x \in M\Gamma x\gamma x\Gamma M = (M\Gamma x)\gamma x\Gamma M \subseteq (M\Gamma M)\gamma x\Gamma M \subseteq M\Gamma x\Gamma M,$$

then $x \in T$, and T is a nonempty subset of M. Let $a, b \in T$ and $\gamma \in \Gamma$. Then $a\gamma b \in T$. Indeed: Since $b \in T$, we have $x \in M\Gamma b\Gamma M$. Since $a \in T$, $x \in M\Gamma a\Gamma M$. Since M is intra-regular, we have

$$\begin{aligned} x \in M\Gamma x\gamma x\Gamma M &\subseteq M\Gamma(M\Gamma b\Gamma M)\gamma(M\Gamma a\Gamma M)\Gamma M \\ &= (M\Gamma M)\Gamma(b\Gamma M\gamma M\Gamma a)\Gamma(M\Gamma M) \\ &\subseteq M\Gamma(b\Gamma M\gamma M\Gamma a)\Gamma M. \end{aligned}$$

We prove that $b\Gamma M\gamma M\Gamma a \subseteq M\Gamma(a\gamma b)\Gamma M$. Then we have

$$x \in M\Gamma\Big(M\Gamma(a\gamma b)\Gamma M\Big)\Gamma M \subseteq M\Gamma(a\gamma b)\Gamma M,$$

and $a\gamma b \in T$. For this purpose, let $b\delta u\gamma v\rho a \in b\Gamma M\gamma M\Gamma a$, where $u, v \in M$ and $\delta, \rho \in \Gamma$. Since M is intra-regular, $b\delta u\gamma v\rho a \in M$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} b\delta u\gamma v\rho a &\in M\Gamma(b\delta u\gamma v\rho a)\gamma(b\delta u\gamma v\rho a)\Gamma M \\ &= (M\Gamma b\delta u\gamma v)\rho(a\gamma b)\delta(u\gamma v\rho a\Gamma M) \\ &\subseteq M\Gamma(a\gamma b)\Gamma M, \end{aligned}$$

so $b\delta u\gamma v\rho a \in M\Gamma(a\gamma b)\Gamma M$. Let $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. Then $a, b \in T$. Indeed: Since $a\gamma b \in T$, we have

$$x \in M\Gamma(a\gamma b)\Gamma M = M\Gamma a\gamma(b\Gamma M) \subseteq M\Gamma a\Gamma M \text{ and}$$

$$x \in (M\Gamma a)\gamma b\Gamma M \subseteq M\Gamma b\Gamma M,$$

so $a, b \in T$. Let now F be a filter of M such that $x \in F$. Then $T \subseteq F$. Indeed: Let $a \in T$. Then $x \in M\Gamma a\Gamma M$, so $x = u\gamma a\rho v$ for some $u, v \in M$, $\gamma, \rho \in \Gamma$. Since $u, a\rho v \in M$, $u\gamma(a\rho v) \in F$ and F is a filter of M, we have $u \in F$ and $a\rho v \in F$. Since $a, v \in M$, $a\rho v \in F$ and F is a filter, we have $a \in F$ and $v \in F$, so $a \in F$.

(2) $\Longrightarrow$ (3). Let $(a,b) \in \mathcal{N}$. Then $a \in N(a) = N(b)$. Since $a \in N(b)$, by (2), we have $b \in M\Gamma a\Gamma M \subseteq a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M = I(a)$. Since $I(a)$ is an ideal of M containing b, we have $I(b) \subseteq I(a)$. Since $b \in N(a)$, by symmetry, we get $I(a) \subseteq I(b)$. Then $I(a) = I(b)$, and $(a,b) \in \mathcal{I}$. Thus we have $\mathcal{N} \subseteq \mathcal{I}$. On the other hand, by Lemma 2, $\mathcal{I} \subseteq \mathcal{N}$. Thus $\mathcal{N} = \mathcal{I}$.

(3) $\Longrightarrow$ (4). Let I be an ideal of M. If $y \in I$, then $y \in (y)_{\mathcal{N}} \subseteq \bigcup\limits_{x\in I}(x)_{\mathcal{N}}$. Let $y \in \bigcup\limits_{x\in I}(x)_{\mathcal{N}}$. Then $y \in (x)_{\mathcal{N}}$ for some $x \in I$. Then, by (3), $(y,x) \in \mathcal{N} = \mathcal{I}$, so $I(y) = I(x)$. Since $x \in I$ and $I(x)$ is the ideal of M generated by x, we have $I(x) \subseteq I$. Thus we have $y \in I(y) = I(x) \subseteq I$, and $y \in I$.

(4) $\Longrightarrow$ (5). Let $x \in M$. Since $\mathcal{N}$ is a semilattice congruence on M, $(x)_{\mathcal{N}}$ is a subsemigroup of M. Let I be an ideal of $(x)_{\mathcal{N}}$. Then $I = (x)_{\mathcal{N}}$. In fact: Let $y \in (x)_{\mathcal{N}}$. Take an element $z \in I$ and an element $\gamma \in \Gamma$ $(I, \Gamma \neq \emptyset)$. The set $M\Gamma z\gamma z\gamma z\Gamma M$ is an ideal of M. Indeed, it is a nonempty subset of M, and we have

$$M\Gamma(M\Gamma z\gamma z\gamma z\Gamma M) = (M\Gamma M)\Gamma z\gamma z\gamma z\Gamma M \subseteq M\Gamma z\gamma z\gamma z\Gamma M \text{ and}$$

$$(M\Gamma z\gamma z\gamma z\Gamma M)\Gamma M = M\Gamma z\gamma z\gamma z\Gamma(M\Gamma M) \subseteq M\Gamma z\gamma z\gamma z\Gamma M.$$

By hypothesis, we have $M\Gamma z\gamma z\gamma z\Gamma M = \bigcup\limits_{t\in M\Gamma z\gamma z\gamma z\Gamma M}(t)_{\mathcal{N}}$.

Since $z\gamma z\gamma z\gamma z\gamma z \in M\Gamma z\gamma z\gamma z\Gamma M$, we have $(z\gamma z\gamma z\gamma z\gamma z)_{\mathcal{N}} \subseteq M\Gamma z\gamma z\gamma z\Gamma M$. Since $(z\gamma z, z) \in \mathcal{N}$ and $z \in I \subseteq (x)_{\mathcal{N}}$, we have $(z\gamma z\gamma z\gamma z\gamma z)_{\mathcal{N}} = (z)_{\mathcal{N}} = (x)_{\mathcal{N}}$. Then $y \in (x)_{\mathcal{N}} \subseteq M\Gamma z\gamma z\gamma z\Gamma M$ and $y = a\delta z\gamma z\gamma z\xi b = (a\delta z)\gamma z\gamma(z\xi b)$ for some $a,b \in M$, $\delta,\xi \in \Gamma$.

We prove that $a\delta z$, $z\xi b \in (x)_{\mathcal{N}}$. Then, since I is an ideal of $(x)_{\mathcal{N}}$, we have $(a\delta z)\gamma z\gamma(z\xi b) \in (x)_{\mathcal{N}}\Gamma I\Gamma(x)_{\mathcal{N}} \subseteq I$, and $y \in I$. We have

$$\begin{aligned}
a\delta z \in (a\delta z)_{\mathcal{N}} &:= (a)_{\mathcal{N}}\delta(z)_{\mathcal{N}} = (a)_{\mathcal{N}}\delta(y)_{\mathcal{N}} \text{ (since } (z)_{\mathcal{N}} = (x)_{\mathcal{N}} = (y)_{\mathcal{N}})\\
&= (a)_{\mathcal{N}}\delta(a\delta z\gamma z\gamma z\xi b)_{\mathcal{N}}\\
&= (a)_{\mathcal{N}}\delta(a)_{\mathcal{N}}\delta(z\gamma z\gamma z\xi b)_{\mathcal{N}}\\
&= (a)_{\mathcal{N}}\delta(z\gamma z\gamma z\xi b)_{\mathcal{N}} \text{ (since } (a\delta a, a) \in \mathcal{N})\\
&= \Big(a\delta(z\gamma z\gamma z\xi b)\Big)_{\mathcal{N}}\\
&= (y)_{\mathcal{N}} = (x)_{\mathcal{N}}
\end{aligned}$$

and

$$\begin{aligned}
z\xi b \in (z\xi b)_{\mathcal{N}} &:= (z)_{\mathcal{N}}\xi(b)_{\mathcal{N}} = (y)_{\mathcal{N}}\xi(b)_{\mathcal{N}} = (a\delta z\gamma z\gamma z\xi b)_{\mathcal{N}}\xi(b)_{\mathcal{N}}\\
&= (a\delta z\gamma z\gamma z)_{\mathcal{N}}\xi(b)_{\mathcal{N}}\xi(b)_{\mathcal{N}}\\
&= (a\delta z\gamma z\gamma z)_{\mathcal{N}}\xi(b\xi b)_{\mathcal{N}}\\
&= (a\delta z\gamma z\gamma z)_{\mathcal{N}}\xi(b)_{\mathcal{N}}\\
&= (a\delta z\gamma z\gamma z\xi b)_{\mathcal{N}} = (y)_{\mathcal{N}} = (x)_{\mathcal{N}}.
\end{aligned}$$

(5) $\Longrightarrow$ (6). Since $\mathcal{N}$ is a semilattice congruence on M.

(6) $\Longrightarrow$ (7). Suppose σ be a semilattice congruence on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$. Let I be an ideal of M, $x \in M$ and $\gamma \in \Gamma$ such that $x\gamma x \in I$. The set $I \cap (x)_\sigma$ is an ideal of $(x)_\sigma$. In fact: Since $x\gamma x \in I$ and $x\gamma x \in (x)_\sigma$, the set $I \cap (x)_\sigma$ is a nonempty subset of $(x)_\sigma$ and, since $(x)_\sigma$ is a subsemigroup of M, we have

$$(x)_\sigma\Gamma(I \cap (x)_\sigma) \subseteq (x)_\sigma\Gamma I \cap (x)_\sigma\Gamma(x)_\sigma \subseteq M\Gamma I \cap (x)_\sigma \subseteq I \cap (x)_\sigma \text{ and}$$

$$(I \cap (x)_\sigma)\Gamma(x)_\sigma \subseteq I\Gamma(x)_\sigma \cap (x)_\sigma\Gamma(x)_\sigma \subseteq I\Gamma M \cap (x)_\sigma \subseteq I \cap (x)_\sigma.$$

Since $(x)_\sigma$ is a simple subsemigroup of M, we have $I \cap (x)_\sigma = (x)_\sigma$, and $x \in I$.

(7) $\Longrightarrow$ (1). Let $a \in M$ and $\gamma \in \Gamma$. Then $a \in M\Gamma a\gamma a\Gamma M$. Indeed: The set $M\Gamma a\gamma a\Gamma M$ is an ideal of M. This is because it is a nonempty subset of M and

$$M\Gamma(M\Gamma a\gamma a\Gamma M) = (M\Gamma M)\Gamma a\gamma a\Gamma M \subseteq M\Gamma a\gamma a\Gamma M,$$

$$(M\Gamma a\gamma a\Gamma M)\Gamma M = M\Gamma a\gamma a\Gamma(M\Gamma M) \subseteq M\Gamma a\gamma a\Gamma M.$$

By hypothesis, $M\Gamma a\gamma a\Gamma M$ is semiprime. Since $(a\gamma a)\gamma(a\gamma a) \in M\Gamma a\gamma a\Gamma M$, we have $a\gamma a \in M\Gamma a\gamma a\Gamma M$, and $a \in M\Gamma a\gamma a\Gamma M$. Thus M is intra-regular. □

3. On some left regular Γ-semigroups

Again using filters, we characterize here the left regular Γ-semigroups M in which $x\Gamma M \subseteq M\Gamma x$ for every $x \in M$ and we prove that this type of Γ-semigroups are decomposable into left simple components. If $x\Gamma M \subseteq M\Gamma x$ for every $x \in M$, then $A\Gamma M \subseteq M\Gamma A$ for every $A \subseteq M$. Indeed: If $a \in A$, $\gamma \in \Gamma$ and $b \in M$, then $a\gamma b \in a\Gamma M \subseteq M\Gamma a \subseteq M\Gamma A$. Thus if A is a left ideal of M, then A is a right ideal of M as well. As a consequence, the left regular Γ-semigroups in which $x\Gamma M \subseteq M\Gamma x$ for every $x \in M$, are left regular and left duo. We also remark that the left regular Γ-semigroups are intra-regular. Indeed: Let $a \in M$. Since M is left regular, we have $a \in M\Gamma a\gamma a \subseteq M\Gamma(M\Gamma a\gamma a)\gamma a \subseteq M\Gamma a\gamma a\Gamma M$. The right regular Γ-semigroups are also intra-regular, and the right regular Γ-semigroups for which $M\Gamma x \subseteq x\Gamma M$ for every $x \in M$ are right regular and right duo, and decomposable into right simple subsemigroups.

Definition 4. (cf. [2]) A Γ-semigroup M is called *left* (resp. *right*) *regular* if $x \in M\Gamma x\gamma x$ (resp. $x \in x\gamma x\Gamma M$) for every $x \in M$ and every $\gamma \in \Gamma$.

Lemma 5. (cf. [1]) *If M is a Γ-semigroup, then $\mathcal{L} \subseteq \mathcal{N}$ and $\mathcal{R} \subseteq \mathcal{N}$.*

Theorem 6. *Let M be a Γ-semigroup. The following are equivalent:*

(1) *M is left regular and $x\Gamma M \subseteq M\Gamma x$ for every $x \in M$.*
(2) *$N(x) = \{y \in M \mid x \in M\Gamma y\}$ for every $x \in M$.*
(3) *$\mathcal{N} = \mathcal{L}$.*
(4) *For every left ideal L of M, we have $L = \bigcup_{x \in L} (x)_\mathcal{N}$.*

(5) $(x)_{\mathcal{N}}$ *is a left simple subsemigroup of* M *for every* $x \in M$.
(6) M *is a semilattice of left simple semigroups.*
(7) *Every left ideal of* M *is semiprime and two-sided.*

Proof. (1) $\Longrightarrow$ (2). Let $x \in M$ and $T := \{y \in M \mid x \in M\Gamma y\}$. The set T is a filter of M containing x. In fact: Take an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since M is left regular, we have

$$x \in M\Gamma x\gamma x = (M\Gamma x)\gamma x \subseteq (M\Gamma M)\gamma x \subseteq M\Gamma x,$$

then $x \in T$, and T is a nonempty subset of M. Let $a, b \in T$ and $\gamma \in \Gamma$. Then $a\gamma b \in T$. Indeed: Since $b, a \in T$, we have $x \in M\Gamma b$ and $x \in M\Gamma a$. Since M is left regular, we have

$$\begin{aligned} x \in M\Gamma x\gamma x &\subseteq M\Gamma(M\Gamma b)\gamma(M\Gamma a) = (M\Gamma M)\Gamma(b\gamma M\Gamma a) \\ &\subseteq M\Gamma(b\gamma M\Gamma a). \end{aligned}$$

We prove that $b\gamma M\Gamma a \subseteq M\Gamma a\gamma b$. Then we have

$$x \in M\Gamma(M\Gamma a\gamma b) = (M\Gamma M)\Gamma(a\gamma b) \subseteq M\Gamma(a\gamma b),$$

and $a\gamma b \in T$. Let now $b\gamma u\mu a \in b\gamma M\Gamma a$ for some $u \in M$, $\mu \in \Gamma$. Since M is left regular, we have

$$\begin{aligned} b\gamma u\mu a &\in M\Gamma(b\gamma u\mu a)\gamma(b\gamma u\mu a) = (M\Gamma b\gamma u)\mu(a\gamma b)\gamma(u\mu a) \\ &\subseteq M\Gamma\Big((a\gamma b)\Gamma M\Big) \\ &\subseteq M\Gamma(M\Gamma a\gamma b) \text{ (since } x\Gamma M \subseteq M\Gamma x \ \forall x \in M) \\ &\subseteq M\Gamma a\gamma b. \end{aligned}$$

Let $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. Then $a, b \in T$. Indeed: Since $a\gamma b \in T$, we have $x \in M\Gamma a\gamma b \subseteq (M\Gamma M)\Gamma b \subseteq M\Gamma b$, so $b \in T$. By hypothesis, $a\gamma b \in a\Gamma M \subseteq M\Gamma a$. Then $x \in M\Gamma a\gamma b \subseteq M\Gamma(M\Gamma a) \subseteq M\Gamma a$, so $a \in T$. Let now F be a filter of M such that $x \in F$. Then $T \subseteq F$. Indeed: Let $a \in T$. Then $x \in M\Gamma a$, that is $x = u\rho a$ for some $u \in M$, $\rho \in \Gamma$. Since $u \in M$, $\rho \in \Gamma$, $u\rho a \in F$ and F is a filter of M, we have $u \in F$ and $a \in F$, then $a \in F$.

(2) $\Longrightarrow$ (3). Let $(a, b) \in \mathcal{N}$. Then $a \in N(a) = N(b)$. Since $a \in N(b)$, by (2), we have $b \in M\Gamma a \subseteq a \cup M\Gamma a = L(a)$, so $L(b) \subseteq L(a)$. Since $b \in N(a)$, by symmetry, we get $L(a) \subseteq L(b)$. Then we have $L(a) = L(b)$, and $(a, b) \in \mathcal{L}$. By Lemma 5, $\mathcal{L} \subseteq \mathcal{N}$, so $\mathcal{L} = \mathcal{N}$.

(3) $\Longrightarrow$ (4). Let L be a left ideal of M. If $y \in L$, then $y \in (y)_{\mathcal{N}} \subseteq \bigcup\limits_{x \in L} (x)_{\mathcal{N}}$. Let $y \in \bigcup\limits_{x \in L} (x)_{\mathcal{N}}$. Then $y \in (x)_{\mathcal{N}}$ for some $x \in L$. Then, by (3), $(y, x) \in \mathcal{N} = \mathcal{L}$, so $L(y) = L(x)$. Since $x \in L$ and $L(x)$ is the left ideal of M generated by x, we have $L(x) \subseteq L$. Then $y \in L(y) = L(x) \subseteq L$, so $y \in L$.

$(4) \Longrightarrow (5)$. Let L be a left ideal of $(x)_{\mathcal{N}}$. Then $L = (x)_{\mathcal{N}}$. In fact: Let $y \in (x)_{\mathcal{N}}$. Take an element $z \in L$ and an element $\gamma \in \Gamma$ $(L, \Gamma \neq \emptyset)$. Since $M\Gamma z\gamma z$ is a left ideal of M, by hypothesis, we have $M\Gamma z\gamma z = \bigcup\limits_{t \in M\Gamma z\gamma z} (t)_{\mathcal{N}}$. Since $z\gamma z\gamma z \in M\Gamma z\gamma z$, we have $(z\gamma z\gamma z)_{\mathcal{N}} \subseteq M\Gamma z\gamma z$. Since $(z\gamma z, z) \in \mathcal{N}$ and $z \in L \subseteq (x)_{\mathcal{N}}$, we have $(z\gamma z\gamma z)_{\mathcal{N}} = (z)_{\mathcal{N}} = (x)_{\mathcal{N}}$. Then $y \in (x)_{\mathcal{N}} \subseteq M\Gamma z\gamma z$, thus $y = a\mu z\gamma z$ for some $a \in M$ and $\mu \in \Gamma$. We prove that $a\mu z \in (x)_{\mathcal{N}}$. Then, since L is a left ideal of $(x)_{\mathcal{N}}$, we have $(a\mu z)\gamma z \in (x)_{\mathcal{N}}\Gamma L \subseteq L$, and $y \in L$. We have

$$\begin{aligned} a\mu z \in (a\mu z)_{\mathcal{N}} &= (a)_{\mathcal{N}}\mu(z)_{\mathcal{N}} = (a)_{\mathcal{N}}\mu(y)_{\mathcal{N}} \text{ (since } (z)_{\mathcal{N}} = (x)_{\mathcal{N}} = (y)_{\mathcal{N}}) \\ &= (a)_{\mathcal{N}}\mu(a\mu z\gamma z)_{\mathcal{N}} = (a)_{\mathcal{N}}\mu(a)_{\mathcal{N}}\mu(z\gamma z)_{\mathcal{N}} \\ &= (a)_{\mathcal{N}}\mu(z\gamma z)_{\mathcal{N}} = (a\mu z\gamma z)_{\mathcal{N}} \\ &= (y)_{\mathcal{N}} = (x)_{\mathcal{N}}. \end{aligned}$$

$(5) \Longrightarrow (6)$. Since $\mathcal{N}$ is a semilattice congruence on M.

$(6) \Longrightarrow (7)$. Let σ be a semilattice congruence on M such that $(x)_\sigma$ is a left simple subsemigroup of M for every $x \in M$. Let L be a left ideal of M and $x \in M$, $\gamma \in \Gamma$ such that $x\gamma x \in L$. The set $L \cap (x)_\sigma$ is a left ideal of $(x)_\sigma$. Indeed: The set $L \cap (x)_\sigma$ is a nonempty subset of $(x)_\sigma$ (since $x\gamma x \in L$ and $x\gamma x \in (x)_\sigma$) and

$$(x)_\sigma\Gamma(L \cap (x)_\sigma) \subseteq (x)_\sigma\Gamma L \cap (x)_\sigma\Gamma(x)_\sigma \subseteq M\Gamma L \cap (x)_\sigma \subseteq L \cap (x)_\sigma.$$

Since $(x)_\sigma$ is a left simple subsemigroup of M, we have $L \cap (x)_\sigma = (x)_\sigma$, then $x \in L$. Thus L is semiprime. Let now L be a left ideal of M. Then $L\Gamma M \subseteq L$. Indeed: Let $y \in L$, $\gamma \in \Gamma$ and $x \in M$. Since L is a left ideal of M, we have $x\gamma y \in M\Gamma L \subseteq L$. The set $L \cap (x\gamma y)_\sigma$ is a left ideal of $(x\gamma y)_\sigma$. Indeed:

$\emptyset \neq L \cap (x\gamma y)_\sigma \subseteq (x\gamma y)_\sigma$ (since $x\gamma y \in L$ and $x\gamma y \in (x\gamma y)_\sigma$) and

$$(x\gamma y)_\sigma\Gamma(L \cap (x\gamma y)_\sigma) \subseteq (x\gamma y)_\sigma\Gamma L \cap (x\gamma y)_\sigma\Gamma(x\gamma y)_\sigma \subseteq M\Gamma L \cap (x\gamma y)_\sigma.$$

Since $(x\gamma y)_\sigma$ is left simple, we have $L \cap (x\gamma y)_\sigma = (x\gamma y)_\sigma = (y\gamma x)_\sigma$, so $y\gamma x \in L$.

$(7) \Longrightarrow (1)$. Let $x \in M$ and $\gamma \in \Gamma$. Since $M\Gamma x\gamma x$ is a left ideal of M, by hypothesis it is semiprime. Since $(x\gamma x)\gamma(x\gamma x) \in M\Gamma x\gamma x$, we have $x\gamma x \in M\Gamma x\gamma x$, and $x \in M\Gamma x\gamma x$, thus M is left regular. Let now $x \in M$. Then $x\Gamma M \subseteq M\Gamma x$. Indeed: Since M is left regular, we have $x \in M\Gamma x\gamma x \subseteq (M\Gamma M)\Gamma x \subseteq M\Gamma x$, so $M\Gamma x$ is a nonempty subset of M. In addition, $M\Gamma(M\Gamma x) = (M\Gamma M)\Gamma x \subseteq M\Gamma x$, so $M\Gamma x$ is a left ideal of M. By hypothesis, $M\Gamma x$ is a right ideal of M as well. Since $M\Gamma x$ is an ideal of M containing x, we have $I(x) \subseteq M\Gamma x$. On the other hand, $x\Gamma M \subseteq x \cup M\Gamma x \cup x\Gamma M \cup M\Gamma x\Gamma M = I(x)$. Thus we obtain $x\Gamma M \subseteq M\Gamma x$. $\square$

The right analogue of Theorem 6 also holds, and we have the following:

Theorem 7. *Let M be a Γ-semigroup. The following are equivalent:*

(1) *M is right regular and $M\Gamma x \subseteq x\Gamma M$ for every $x \in M$.*

(2) *$N(x) = \{y \in M \mid x \in y\Gamma M\}$ for every $x \in M$.*

(3) $\mathcal{N} = \mathcal{R}$.
(4) *For every right ideal R of M, we have* $R = \bigcup\limits_{x \in R} (x)_{\mathcal{N}}$.
(5) $(x)_{\mathcal{N}}$ *is a right simple subsemigroup of M for every $x \in M$.*
(6) *M is a semilattice of right simple semigroups.*
(7) *Every right ideal of M is semiprime and two-sided.*

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