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The categories of actions of a dcpo-monoid on directed complete posets

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Abstract. In this paper, some categorical properties of the category **Cpo**-S of all S-cpo's; cpo's equipped with a compatible right action of a cpo-monoid S, with strict continuous action-preserving maps between them is considered. We also define and consider similarly, the category **Dcpo**-S of all S-dcpo's, and continuous action-preserving maps between them. In particular, we characterize products and coproducts in these categories. Also, epimorphisms and monomorphisms in **Dcpo**-S are studied. Further, we show that **Cpo**-S is not cartesian closed but **Dcpo**-S is cartesian closed.

1. Introduction and preliminaries

The category **Dcpo** of directed complete partial ordered sets plays an important role in Theoretical Computer Science, and specially in Domain Theory (see [1]). This category is complete and cocomplete. The completeness of **Dcpo** has been proved (in a constructive way) by Achim Jung ([1]) and it is stated there that to describe colimits is quite difficult. In [5], Fiech characterizes and describes colimits in **Dcpo**, but his construction is rather complicated. The cartesian closeness of **Dcpo** has also been proved by Achim Jung (see [7]). It is also shown that the category **Cpo** of directed complete partially ordered sets with bottom elements and strict continuous maps between them is monoidal closed, complete and cocomplete (see [1, 7]).

In this paper, we study some categorical properties of the categories **Dcpo**-S (and **Cpo**-S) of the actions of a dcpo(cpo)-monoid S on dcpo's (cpo's). In particular, we show that the category **Dcpo**-S is complete and cocomplete, and describe products and coproducts in these categories. Also, epimorphisms and monomorphisms in these categories are considered. Further, we show that **Cpo**-S is not cartesian closed but **Dcpo**-S is so.

Let us now give some preliminaries needed in the sequel.

Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. A nonempty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists

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$c \in D$ such that $a, b \leqslant c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P. A dcpo which has a bottom element $\perp$ is said to be a *cpo*.

A *dcpo map* or a *continuous map* $f\colon P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. A dcpo map $f\colon P \to Q$ between cpo's is called *strict* if $f(\perp) = \perp$. Thus we have the categories **Dcpo** (and **Cpo**) of all dcpo's (cpo's) with (strict) continuous maps between them.

The following lemmas are frequently used in this paper.

Lemma 1.1. [3, 7] *Let $\{A_i \colon i \in I\}$ be a family of dcpo's. Then the directed join of a directed subset $D \subseteq^d \prod_{i\in I} A_i$ is calculated as $\bigvee^d D = (\bigvee^d D_i)_{i\in I}$ where*

$$D_i = \{a \in A_i \colon \exists d = (d_k)_{k\in I} \in D,\ a = d_i\}$$

for all $i \in I$.

Lemma 1.2. [7] *Let P, Q, and R be dcpo's, and $f\colon P \times Q \to R$ be a function of two variables. Then f is continuous if and only if f is continuous in each variable; which means that for all $a \in P$, $b \in Q$, $f_a\colon Q \to R$ $(b \mapsto f(a,b))$ and $f_b\colon P \to R$ $(a \mapsto f(a,b))$ are continuous.*

Remark 1.3. The categories **Dcpo** and **Cpo** are both complete and cocomplete. In fact,

(i) The product of a family of dcpo's (cpo's) is their cartesian product, with componentwise order and ordinary projection maps. In particular, the terminal object of **Dcpo** (and **Cpo**) is the singleton poset $\{\theta\}$.

The equalizer of a pair $f, g\colon P \to Q$ of (strict) continuous maps is given by $E = \{x \in P : f(x) = g(x)\}$ with the order inherited from P.

Moreover, the pullback of (strict) continuous maps $f\colon P \to R$ and $g\colon Q \to R$ is the sub-dcpo $P = \{(a,b)\colon f(a) = g(b)\}$ of the product $P \times Q$ together with the restriction of projection maps.

(ii) The coproduct of a family of dcpo's is their disjoint union, with the order arising from each factor. In particular, the initial object of **Dcpo** is the empty poset.

The coproduct of a family of cpo's is their *coalesced sum*. Recall that *the coalesced sum* of the family $\{A_i \colon i \in I\}$ of cpo's is defined to be

$$\biguplus_{i\in I} A_i = \perp \oplus \dot{\bigcup}_{i\in I} (A_i \setminus \{\perp_{A_i}\}).$$

In particular, the initial object of **Cpo** is the singleton poset $\{\theta\}$.

Recall that a *po-monoid* is a monoid with a partial order $\leqslant$ which is compatible with the monoid operation: for $s, t, s', t' \in S$, $s \leqslant t$, $s' \leqslant t'$ imply $ss' \leqslant tt'$.

Similarly, a *dcpo* (*cpo*)-*monoid* is a monoid which is also a dcpo (cpo) whose binary operation is a (strict) continuous map.

Now, we recall the preliminary notions of the action of a (po)monoid on a set(poset). For more information, see [2, 4, 8].

Let S be a monoid. A (*right*) S-*act* or S-*set* is a set A equipped with an action $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act-**S denote the category of all S-acts with action-preserving maps (maps $f : A \to B$ with $f(as) = f(a)s$).

Also, recall that an element a of an S-act A is said to be a *zero element* if $as = a$ for all $s \in S$.

Let S be a po-monoid. A (*right*) S-*poset* is a poset A which is also an S-act whose action $\lambda\colon A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action-preserving monotone maps between them is denoted by **Pos-**S.

Remark 1.4. Recall (see [2]) that:

(i) The product in the category of S-posets is the cartesian product with the componentwise action and order. In particular, the terminal S-poset is the singleton S-poset.

Also, recall that the equalizer of a pair $f, g\colon A \to B$ of S-poset maps is given by $E = \{a \in A\colon f(a) = g(a)\}$ with action and order inherited from A.

The pullback of S-poset maps $f\colon A \to C$ and $g\colon B \to C$ is the sub-S-poset $P = \{(a, b)\colon f(a) = g(b)\}$ of $A \times B$.

(ii) The coproduct is the disjoint union with the usual action and order. In particular, the initial S-poset is the empty set.

Finally, we introduce the notion which we work on in this paper.

Definition 1.5. Let S be a (cpo) dcpo-monoid. By a (*right*) S-*dcpo* (S-*cpo*) we mean a dcpo (cpo) A which is also an S-act whose action $\lambda\colon A \times S \to A$ is (strict) continuous, where $A \times S$ is considered as a dcpo with componentwise order.

By an S-*dcpo map* (S-*cpo map*) between S-dcpo's (S-cpo's), we mean a map $f\colon A \to B$ which is both (strict) continuous and action-preserving.

We denote the categories of all S-dcpo's (S-cpo's) and S-dcpo (S-cpo) maps between them by **Dcpo**-S (**Cpo**-S) .

Remark 1.6. (1) In the definition of an S-cpo, we can omit the property that the action is strict. Notice that $\perp_{A\times S} = (\perp_A, \perp_S)$, and the action being strict means that $\perp_A\perp_S = \perp_A$. But, assumig that there is a continuous (monotone) action on a cpo A, the fact that $\perp_S \leqslant 1$ implies $\perp_A\perp_S \leqslant \perp_A 1 = \perp_A$. Also, since $\perp_A$ is the bottom element in A, we have $\perp_A \leqslant \perp_A\perp_S$. Thus, $\perp_A\perp_S = \perp_A$ as required.

(2) Note that, by Lemma 1.2, the action $\lambda\colon A \times S \to A$ is continuous if and only if each $\lambda_a\colon S \to A$, $s \mapsto as$, and $\lambda_s\colon A \to A$, $a \mapsto as$, is continuous.

(3) Notice that the above note is not true for strictness. For example, consider the pomonoid $S = \{0 < 1\}$ with the binary operation max. It is clear that max is

strict continuous, so S is a cpo-monoid and hence an S-cpo. But the continuous map $\lambda_1 : S \to S$, $t \mapsto \max(t, 1)$ is not strict, because $\max(0, 1) = 1 \neq 0 = \perp_S$.

2. Limits and coproduts in Cpo-S and Dcpo-S

In this section, we give the description of products, equalizers, terminal object and pullback in the categories **Dcpo**-S and **Cpo**-S. We also, find coproducts in these two categories.

Remark 2.1. In both the categories **Dcpo**-S and **Cpo**-S, the terminal object is the one element object.

Proposition 2.2. *The product of a family of S-dcpo's (S-cpo's) is their cartesian product with componentwise action and order.*

Proof. Let $\{A_i \colon i \in I\}$ be a family of S-dcpo's (S-cpo's). Let $A = \prod_{i\in I} A_i$. First we see that A with componentwise action and order is a S-dcpo (S-cpo). By Remark 1.4, A is an S-poset. Also, the action on A is continuous. Applying Lemma 1.2, it is enough to check the continuity of the action in each component. Let $D \subseteq^d A$ and $s \in S$. We show that $(\bigvee^d D)s = \bigvee_{x\in D} xs$. By Lemma 1.1, $\bigvee^d D = (\bigvee^d D_i)_{i\in I}$, where $D_i = \{a \in A_i \colon \exists (d_k)_{k\in I} \in D,\ d_i = a\}$ is a directed subset of A_i, for all $i \in I$. Then we have $(\bigvee^d D)s = (\bigvee^d D_i)_{i\in I} s = ((\bigvee^d D_i)s)_{i\in I} = (\bigvee^d D_i s)_{i\in I}$, where the latter equality is because the action on each A_i is continuous. Now, we see that $(\bigvee^d D_i s)_{i\in I} = \bigvee^d_{x\in D} xs$. First, notice that $(\bigvee^d D_i s)_{i\in I}$ is an upper bound of the set $\{xs \colon\ x \in D\}$, since for $x = (d_i)_{i\in I} \in D$, we have $d_i \in D_i$, for all $i \in I$, and so $xs = (d_i s)_{i\in I} \leqslant ((\bigvee^d D_i)s)_{i\in I} = (\bigvee^d D_i s)_{i\in I}$. Secondly, if $c = (c_i)_{i\in I}$ is any upper bound of the set $\{xs \colon\ x \in D\}$, then for $i \in I$ and $a \in D_i$, taking $x = (d_i)_{i\in I}$ with $d_i = a$, we have $as = d_i s \leqslant c_i$. Thus $(\bigvee^d D_i s)_{i\in I} \leqslant c$, as required. Similarly, the action on A is continuous in the second component; that is for $T \subseteq^d S$ and $a = (a_i)_{i\in I} \in A$, $a(\bigvee^d T) = \bigvee^d_{t\in T} at$. Consequently, $A = \prod_{i\in I} A_i$ with the componentwise order and action is an S-dcpo (S-cpo). Also, the projection maps $p_i : A \to A_i$ are S-dcpo (S-cpo) maps, since by Remark 1.3 they are (strict) continuous, also they are action-preserving (see [8]). To see the universal property of products, notice that for every S-dcpo (S-cpo) B with S-dcpo (S-cpo) maps $f_i \colon B \to A_i$, $i \in I$, the unique S-poset map $f : B \to A$ given by $f(b) = (f_i(b))_{i\in I}$, $b \in B$ which exists by the universal property of products in **Pos**-S (see Remark 1.4), and satisfies $p_i \circ f = f_i$, for all $i \in I$, is a (strict) continuous map. This is because, $f(\perp_B) = (f_i(\perp_B))_{i\in I} = (\perp_{A_i})_{i\in I} = \perp_A$. Also, it is straightforward to see that for $D \subseteq^d B$, $f(\bigvee^d D) = \bigvee^d f(D)$. □

Remark 2.3. (i) It is clear that the initial object in the category **Dcpo**-S is the empty set.

(ii) The category **Cpo**-S has initial object if the identity of the cpo-monoid S is its bottom element. In fact S is the initial object. Since, for every S-cpo A the map $\lambda_\perp\colon S \to A$, defined by $\lambda_\perp(s) = \perp_A s$ is the unique S-cpo map from S to A. To show the uniqueness, let $\alpha\colon S \to A$ be an S-cpo map, then $\alpha(s) = \alpha(1s) = \alpha(1)s = \perp_A s = \lambda_\perp(s)$, for all $s \in S$. Thus, $\alpha = \lambda$.

Now, we consider coproducts.

Theorem 2.4. *The coproduct of a family of S-dcpo's is their disjoint union.*

Proof. Let $\{A_i\colon i \in I\}$ be a family of S-dcpo's. Let $A = \bigcup_{i\in I} A_i$ be the disjoint union of A_i, $i \in I$. By Remark 1.4, A with the order and the action inherited from A_i, $i \in I$; that is

$$x \leqslant y \text{ in } A \text{ if and only if } x \leqslant y \text{ in } A_i, \text{ for some } i \in I$$

and $a.s = as$ for $a \in A_i$, $s \in S$, is an S-poset. Applying Lemma 1.2, we see that this the action is also continuous. Therefore, A is an S-dcpo. Moreover, the injection maps $u_i\colon A_i \to A$, defined by $u_i = id_A|_{A_i}, i \in I$ are S-poset maps, by Remark 1.4, and they are continuous, by Remark 1.3. Finally, since A satisfies the universal property of coproducts in **Pos**-S, for every S-dcpo B and S-dcpo maps $f_i\colon A_i \to B$, $i \in I$, the mapping $f\colon A \to B$ given by $f(a) = f_i(a)$ for $a \in A_i$, is the unique S-poset map with $f \circ u_i = f_i$, for all $i \in I$. This map is also continuous, because if D is a directed subset of A then by the definition of the order on A, $D \subseteq^d A_i$ for some $i \in I$, and $\bigvee^d_A D = \bigvee^d_{A_i} D$. Thus $f(\bigvee^d D) = f_i(\bigvee^d D) = \bigvee^d f_i(D) = \bigvee^d f(D)$. $\square$

To describe the coproduct in **Cpo**-S, using the coalesced sum of cpo's, we need the following lemma.

Lemma 2.5. *The coalesced sum of a family of S-cpo's in which the bottom element is a zero element is an S-cpo.*

Proof. Let $\{A_i\colon i \in I\}$ be a family of S-cpo's. By Remark 1.3, the coalesced sum $A = \biguplus_{i\in I} A_i$ is a cpo. Define the action on A as:

$$a \cdot s = \begin{cases} as & \text{if} \quad as \neq \perp_{A_i} \\ \perp_A & \text{if} \quad as = \perp_{A_i} \end{cases}$$

for $a \in A_i$, $i \in I$, $s \in S$, and $\perp_A \cdot s = \perp_A$. In particular, $\perp_A \cdot 1 = \perp_A$. We see that also for $a \neq \perp_A$, $a \cdot 1 = a$, because, for some $i \in I$, $a \in A_i$, and so $a \cdot 1 = a1 = a$. Also, $a\cdot(st) = (a\cdot s)\cdot t$, for $a \in A$, $s, t \in S$. This is because, $\perp_A \cdot (st) = (\perp_A \cdot s) \cdot t$, by the definition, and for $a \neq \perp_A$, $a \in A_i$ for some $i \in A$. If $a(st) \neq \perp_{A_i}$, then $as \neq \perp_{A_i}$, (otherwise since $\perp_{A_i}$ is a zero element, $a(st) = (as)t = \perp_{A_i} t = \perp_{A_i}$); also $(as)t = a(st) \neq \perp_{A_i}$. So $(a\dot{s}) \cdot t = (as) \cdot t = (as)t = a(st) = a \cdot (st)$. Secondly, if $a(st) = \perp_{A_i}$, then $a \cdot (st) = \perp_A$. Now, if $as = \perp_{A_i}$ then $a \cdot s = \perp_A$ and so $(a \cdot s) \cdot t = \perp_A \cdot t = \perp_A$. Also, if $as \neq \perp_{A_i}$ then $a \cdot s = as$, and since $(as)t = a(st) = \perp_{A_i}$, $(a \cdot s) \cdot t = \perp_A$. Thus $(a \cdot s) \cdot t = (a \cdot s) \cdot t = \perp_A$, as required.

Now, we show that the action is continuous. Notice that $D \subseteq^d A$ is directed if and only if $D \subseteq^d A_i$, for some $i \in I$, or $D = D' \cup \{\perp_A\}$, where $D' = \emptyset$ or D' is a directed subset of A_i, for some $i \in I$. This is because, if $D \subseteq^d A$ and $\perp_A \notin D$, and on the contrary, if there exist $d_1, d_2 \in D$ such that $d_1 \in A_i$ and $d_2 \in A_j$, $i \neq j$, then there exists $d_3 \in D$ such that $d_1 \leqslant d_3$ and $d_2 \leqslant d_3$. Also, by the definition of the order on A, $d_3 \in A_i \cap A_j = \emptyset$, which is a contradiction. So $D \subseteq^d A_i$, for some $i \in I$. Now, let $\perp_A \in D$. We show that $D' = D - \{\perp_A\}$ is a directed subset of A_i, for some $i \in I$. On the contrary, let there exist ${d_1}', {d_2}' \in D'$ such that ${d_1}' \in A_i$ and ${d_2}' \in A_j$, $i \neq j$. Since D is directed, there exists $d_3 \in D$ such that ${d_1}' \leqslant d_3$ and ${d_2}' \leqslant d_3$. By the definition of the order on A, $d_3 \in A_i \cap A_j = \emptyset$, which is a contradiction. So $D' \subseteq^d A_i$, for some $i \in I$. Now, applying Lemma 1.2, we show that the action is continuous. Let $D \subseteq^d \biguplus_{i\in I} A_i$ and $s \in S$. By the above discussion, two cases may occur:

Case (i): $D \subseteq^d A_i$, for some $i \in I$.

Subcase (i1): If $(\bigvee^d D)s \neq \perp_{A_i}$, then we have $(\bigvee^d D) \cdot s = (\bigvee^d D)s = \bigvee^d_{x\in D} xs$, where the last equality is because A_i is an S-cpo. Now we claim that

$$\bigvee_{x\in D}^{d} xs = \bigvee_{x\in D}^{d} x \cdot s \qquad (*)$$

Let $K = \{x \in D \colon xs \neq \perp_{A_i}\}$. Then K satisfies:

(1) $K \neq \emptyset$, because otherwise $(\bigvee^d D)s = \bigvee^d_{x\in D} xs = \perp_{A_i}$, which is a contradiction.

(2) For all $x \in K$, $x \cdot s = xs$, by the definition of the action on A.

(3) For all $x \in K$ and $x' \in D \setminus K$, there exists $x'' \in K$ with $x \leqslant x''$ and $x' \leqslant x''$, since D is directed. But, then $xs \leqslant x''s$, and hence $x'' \in K$, since $x \in K$.

Now to prove $(*)$, first we see that $\bigvee^d_{x\in D} xs$ is an upper bound of the set $\{x \cdot s \colon x \in D\}$. Also for all $x \in K$, $x \cdot s = xs \leqslant \bigvee^d_{x\in D} xs$. For $x \in D \setminus K$, $x \cdot s = \perp_A \leqslant \bigvee^d_{x\in D} xs$, as required. Secondly, if c is an upper bound of the set $\{x \cdot s : x \in D\}$. For all $x \in K$, we have $x \cdot s = xs \leqslant c$. For $x \in D \setminus K$ and $x' \in K$ (which exists, since $K \neq \emptyset$), by (3) there exists $x'' \in K$ such that $x < x''$ and $x' \leqslant x''$. This gives $xs \leqslant x''s = x'' \cdot s \leqslant c$. Then for all $x \in D$, we have $xs \leqslant c$, and so $\bigvee^d_{x\in D} xs \leqslant c$, as required.

Subcase (i2): If $(\bigvee^d D)s = \perp_{A_i}$, then we again have $(\bigvee^d D)s = \bigvee^d_{x\in D} xs$. This is because, the action on A_i is continuous on the second component. Also, $(\bigvee^d D)s = \perp_{A_i}$ gives $xs = \perp_{A_i}$, for all $x \in D$. This is because, $\perp_{A_i} = (\bigvee^d D)s = \bigvee^d_{x\in D} xs$. Hence by the definition of the action on A, $(\bigvee^d D)\cdot s = \bigvee^d_{x\in D} x\cdot s = \perp_A$.

Case (ii): $D = D' \cup \perp_A$, where $D' \subseteq^d A_i$, for some $i \in I$.

By case (i), we have $(\bigvee^d D') \cdot s = \bigvee^d_{x'\in D'} x' \cdot s$. Also, we have $(\bigvee^d D) \cdot s = (\bigvee^d D') \cdot s = \bigvee^d_{x'\in D'} x' \cdot s = \bigvee^d_{x\in D} x \cdot s$, as required.

Now to prove that the action is continuous in the second component, let $T \subseteq^d S$ and $a \in A$. We show that $a \cdot \bigvee^d T = \bigvee^d_{t\in T} a \cdot t$. Consider the following two cases:

(a): If $a = \perp_A$, then by the definition of the action on A, $a \cdot \bigvee^d T = \bigvee^d_{t\in T} a \cdot t = \perp_A$.

(b): If $a \neq \perp_A$, then for some $i \in I$, $a \in A_i$. We have the following two situations:

(b1): If $a(\bigvee^d T) \neq \perp_{A_i}$ then we have $a \cdot (\bigvee^d T) = a(\bigvee^d T) = \bigvee^d_{t \in T} at$, where the last equality is true because A_i is an S-cpo. Now, we claim that

$$\bigvee_{t \in T}^{d} a \cdot t = \bigvee_{t \in T}^{d} at \qquad (**).$$

Let $L = \{t \in T : at \neq \perp_{A_i}\}$. Then one can prove (in a similar way to the set K in the above discussion) that L satisfies:

(1) $L \neq \emptyset$.

(2) For all $t \in L$, $a \cdot t = at$.

(3) For all $t \in L$ and $t' \in T \setminus L$, there exists $t'' \in L$ with $t \leqslant t''$ and $t' \leqslant t''$.

Now to prove $(**)$, we see that first $\bigvee^d_{t \in T} at$ is an upper bound of the set $\{a \cdot t \,:\, t \in T\}$. Also for all $t \in L$, $a \cdot t = at \leqslant \bigvee^d_{t \in T} at$. For $t \in T \setminus L$, $a \cdot t = \perp_A \leqslant \bigvee^d_{t \in T} at$, as required. Secondly, if c is an upper bound of the set $\{a \cdot t : t \in T\}$, then for all $t \in L$, we have $at = a \cdot t \leqslant c$. Now, by (3) and in the same way of Subcase (i1), for $t \in T \setminus L$ there exists $t'' \in L$ such that $at \leqslant at'' \leqslant c$. Then for all $t \in T$, we have $at \leqslant c$, and so $\bigvee^d_{t \in T} at \leqslant c$. Therefore, $(**)$ has been proved.

(b2): If $a(\bigvee^d T) = \perp_{A_i}$, we show that $a \cdot (\bigvee^d T) = \bigvee^d_{t \in T} a \cdot t$. Since A_i is an S-cpo, we have $\bigvee^d_{t \in T} at = a(\bigvee^d T) = \perp_{A_i}$. So for all $t \in T$, $at = \perp_{A_i}$. Then by the definition of the action on A, $a \cdot (\bigvee^d T) = \bigvee^d_{t \in T} a \cdot t = \perp_A$.

Therefore, the action on A is continuous, and so $A = \biguplus_{i \in I} A_i$ is an S-cpo. $\square$

Theorem 2.6. *Let $\{A_i :\ i \in I\}$ be a family of S-cpo's whose bottom elements are zero elements. Then their coproduct exists in* **Cpo-**S.

Proof. Let $A = \biguplus_{i \in I} A_i$. By Proposition 2.5, A is an S-cpo and by Remark 1.3, the injections $u_i \colon A_i \to A$, $i \in I$, defined by

$$u_i(x) = \begin{cases} x & \text{if} \quad x \neq \perp_{A_i} \\ \perp_A & \text{if} \quad x = \perp_{A_i} \end{cases}$$

are cpo maps. Also we show that $u_i \colon A_i \to A$, $i \in I$ are action-preserving. First notice that $u_i(\perp_{A_i s}) = u_i(\perp_{A_i}) = \perp_A = \perp_A \cdot s = u_i(\perp_{A_i}) \cdot s$. Now, let $\perp_{A_i} \neq x \in A_i$ and $s \in S$. If $xs = \perp_{A_i}$, then by the definition of the action on A, $x \cdot s = \perp_A$, and so $u_i(xs) = \perp_A = x \cdot s = u_i(x) \cdot s$. If $xs \neq \perp_{A_i}$, then $x \cdot s = xs$, and so $u_i(xs) = xs = x \cdot s = u_i(x) \cdot s$. Moreover for every S-cpo B with S-cpo maps $f_i \colon A_i \to B$, $i \in I$, the unique cpo map $f \colon A \to B$ given by

$$f(a) = \begin{cases} f_i(a) & \text{if} \quad a \in A_i \\ \perp_B & \text{if} \quad x = \perp_A \end{cases}$$

which exists by the universal property of coproducts in **Cpo**, and satisfies $f \circ u_i = f_i$ for all $i \in I$, is action-preserving. First notice that since each f_i is action-preserving and $\perp_{A_i}$ is a zero element, $f_i(\perp_{A_i}) = \perp_B$ is a zero element. Now, $f(\perp_A \cdot s) = f(\perp_A) = \perp_B = \perp_B s = f(\perp_A)s$, for all $s \in S$. Also, for $a \neq \perp_A$, we have $a \in A_i$, for some $i \in I$. Therefore, if $as = \perp_{A_i}$ we get $a \cdot s = \perp_A$, and so $f(a \cdot s) = f(\perp_A) = \perp_B = f_i(\perp_{A_i}) = f_i(as) = f_i(a)s = f(a)s$. If $as \neq \perp_A$, then we have $a \cdot s = as$, and so $f(a \cdot s) = f(as) = f_i(as) = f_i(a)s = f(a)s$. □

Corollary 2.7. *Let S be a cpo-monoid in which the identity element is the top element. Then* **Cpo**-S *has all coproducts.*

Proof. By Theorem 2.6, it is enough to show that the bottom element of every S-cpo A is a zero element. For all $s \in S$, we have $s \leqslant 1$, and so $\perp_A s \leqslant \perp_A 1 = \perp_A$. But, $\perp_A$ is the bottom element of A and so $\perp_A s = \perp_A$. □

Theorem 2.8. *Pullbacks and equalizers exist in the categories* **Cpo**-S *and* **Dcpo**-S.

Proof. Let $f, g \colon A \to B$ be S-cpo (S-dcpo) maps. Then

$$E = \{x \in A \colon f(x) = g(x)\}$$

is a sub S-cpo (S-dcpo) of A, and the inclusion map satisfies $f \circ i = g \circ i$. Also, if $e : K \to L$ is an S-cpo (S-dcpo) map with $f \circ e = g \circ e$ then the map $\gamma : K \to E$ given by $\gamma(x) = e(x)$ is the unique S-cpo (S-dcpo) map such that $i \circ \gamma = e$.

Also, it is easily seen that the pullback of S-cpo (S-dcpo) maps $f \colon A \to C$ and $g \colon B \to C$ is the sub-S-cpo (S-dcpo) $P = \{(a, b) \colon f(a) = g(b)\}$ of $A \times B$, together with the restricted projection maps. □

As a consequence of Theorems 2.2 and 2.8, we get the following result.

Proposition 2.9. *The categories* **Cpo**-S *and* **Dcpo**-S *are complete.*

3. Cocompleteness and cartesian closedness

In this section, we consider some other categorical properties of **Cpo**-S and **Dcpo**-S. We show that monomorphism in **Dcpo**-S are exactly one-one S-dcpo maps, while epimorphisms are not necessarily onto S-dcpo maps. Also, we prove that **Dcpo**-S is a cocomplete category. Further, it is proved that **Dcpo**-S is cartesian closed while **Cpo**-S is not so, and hence it is neither a topos nor a quasitopos (see [9]).

Lemma 3.1. *A morphism in* **Dcpo**-S *is a monomorphism if and only if it is one-one.*

Proof. Let $h\colon A \to B$ be a monomorphism in **Dcpo**-S, and $h(a) = h(a')$. Consider the S-dcpo maps $f, g\colon S \to A$ given by $f(s) = as$ and $g(s) = a's$, for $s \in S$. Then, $h \circ f = h \circ g$ and so $f = g$. Thus, $a = a'$. □

In the following we show that the category **Dcpo**-S is cocomplete.

Recall that an object C of a category $\mathcal{C}$ is called a *coseparator* if the functor $hom(-, C) : \mathcal{C}^{op} \to \mathbf{Set}$ is faithful; in other words, for each distinct arrows $f, g\colon A \to B$ there exists an arrow $h\colon B \to C$ such that $h \circ f \neq h \circ g$.

Also, recall from [6], Theorem 23.14 that a complete well-powered category which has a coseparator, is cocomplete. Therefore, we show that **Dcpo**-S has a coseparator and is well-powered.

Proposition 3.2. *The forgetful functor* $U_1\colon \mathbf{Dcpo}\text{-}S \to \mathbf{Dcpo}$ *has a right adjoint.*

Proof. We define the cofree functor $K_1\colon \mathbf{Dcpo} \to \mathbf{Dcpo}\text{-}S$ as $K_1(P) = P^{(S)}$, where $P^{(S)}$ is the set of all dcpo maps from S to P. We give it the pointwise order and the action by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in P^{(S)}$. Then, $P^{(S)}$ is an S-dcpo. We know that $P^{(S)}$ is a dcpo (see [7]). Now, we show that the action defined above is a continuous map. Applying Lemma 1.2, let $F \subseteq^d P^{(S)}$. Then

$$((\bigvee^d F)s)(t) = (\bigvee^d F)(st) = \bigvee^d_{f \in F} f(st) = \bigvee^d_{f \in F} (fs)(t) = (\bigvee^d_{f \in F} fs)(t)$$

so we have $(\bigvee^d F)s = \bigvee^d (Fs)$. Now, assume that $T \subseteq^d S$ and $f \in P^{(S)}$, then

$$\begin{aligned}(f(\bigvee^d T))(s) &= f((\bigvee^d T)s) = f(\bigvee^d_{t\in T} ts)\\ &= \bigvee^d_{t\in T} f(ts) = \bigvee^d_{t\in T} (ft)(s) = (\bigvee^d_{t\in T} ft)(s)\end{aligned}$$

and so $f(\bigvee^d T) = \bigvee^d_{t\in T} ft$, as required. Consequently $P^{(S)}$ is an S-dcpo.

Now, consider the cofree map (the counit of the adjunction) $\sigma : P^{(S)} \to P$, given by $\sigma(f) = f(1)$. We show that it is continuous. Let $F \subseteq^d P^{(S)}$. Then

$$\sigma(\bigvee^d_{f\in F} f) = (\bigvee^d_{f\in F} f)(1) = \bigvee^d_{f\in F} f(1) = \bigvee^d_{f\in F} \sigma(f).$$

To see the universal property, let $\alpha\colon A \to P$ be a continuous map from an S-dcpo A. Then the unique S-poset map $\overline{\alpha}\colon A \to P^{(S)}$ given by $\overline{\alpha}(a)(s) = \alpha(as)$ and satisfying $\sigma \circ \overline{\alpha} = \alpha$ (see [2]) is continuous. To show this, let $D \subseteq^d A$ and $s \in S$. Then

$$\begin{aligned}\overline{\alpha}(\bigvee^d D)(s) &= \alpha((\bigvee^d D)s) = \alpha(\bigvee^d_{x\in D} xs)\\ &= \bigvee^d_{x\in D} \alpha(xs) = \bigvee^d_{x\in D} \overline{\alpha}(x)(s) = (\bigvee^d_{x\in D} \overline{\alpha}(x))(s)\end{aligned}$$

as required. □

Notice that the forgetful functor $U\colon \mathbf{Cpo}\text{-}S \to \mathbf{Cpo}$ does not necessarily have a right adjoint. This is because, U does not preserve initial object in general. For example, let S be the 2-element chain $\{1, a\}$ with $1 < a$, and $aa = a$, $1a = a = a1$. Then S is an S-cpo, and it is the initial object of $\mathbf{Cpo}\text{-}S$ (see Remark 2.3), whereas the initial object in the category $\mathbf{Cpo}$ is the singleton cpo.

Corollary 3.3. *The category* $\mathbf{Dcpo}\text{-}S$ *has a coseparator.*

Proof. We show that for each dcpo P with $|P| \geq 2$ and non discrete order, the cofree object $P^{(S)}$ described in Proposition 3.2 is a coseparator.

Let $f, g\colon A \to B$ be S-dcpo maps with $f \neq g$. We should define an S-dcpo map $h : B \to P^{(S)}$ with $h \circ f \neq h \circ g$. To this end, we define a dcpo map $k\colon B \to P$ such that $k \circ f \neq k \circ g$.

Since $f \neq g$, there exists $a \in A$ with $f(a) \neq g(a)$. We consider three cases

$$(1)\ \ f(a) < g(a) \qquad (2)\ \ g(a) < f(a) \qquad (3)\ \ f(a) \parallel g(a)$$

Let $f(a) < g(a)$. Take $B' = \{b \in B \mid\ b \leqslant f(a)\}$. Define $k : B \to P$ by

$$k(b) = \begin{cases} x & \text{if } b \in B' \\ y & \text{otherwise} \end{cases}$$

where $x, y \in P$ and $x < y$ (such x, y exist since $|P| \geq 2$ and the order on P is not discrete). First we show that k is order-preserving, and hence it take directed subsets to directed ones. Let $b_1, b_2 \in B$ with $b_1 \leqslant b_2$. If $b_1 \in B'$, then for the case where $b_2 \in B'$, $x = k(b_1) = k(b_2)$; and for the case where $b_2 \notin B'$, $x = k(b_1) < y = k(b_2)$. Also, if $b_1 \notin B'$ then $b_2 \notin B'$, and so $k(b_1) = k(b_2) = y$. To prove the continuity of k, let $D \subseteq^d B$. Notice that $\bigvee^d D \in B'$ if and only if $D \subseteq B'$. Now, if $\bigvee^d D \in B'$, then $D \subseteq B'$ and so $k(\bigvee^d D) = x = \bigvee^d_{z\in D} k(z)$. Also, if $\bigvee^d D \notin B'$ then $k(\bigvee^d D) = y$, and $D \nsubseteq B'$. Thus $D \setminus B' \neq \emptyset$, and

$$\bigvee_{z\in D}^{d} k(z) = \bigvee_{z\in (D\setminus B')\cup(B'\cap D)}^{d} k(z) = y \vee x = y$$

as required. Finally, since $P^{(S)}$ is the cofree S-dcpo on P, there exists a unique S-dcpo map $h : B \to P^{(S)}$ such that $\sigma \circ h = k$, where σ is the cofree map defined in the above proposition. This gives that $h \circ f \neq h \circ g$, and so $P^{(S)}$ is a coseparator.

The cases (2) and (3) are proved similarly. $\square$

Lemma 3.4. *The category* $\mathbf{Dcpo}\text{-}S$ *is well-powered.*

Proof. We should prove that the class of isomorphic subobjects of any S-dcpo is a set. Let B be an S-dcpo and A be a suboject of B; that is there exists a monomorphism $f\colon A \to B$. By Lemma 3.1, f is one-one and so A is isomorphic to a subset of B. Hence the class of isomorphic subobjects of B is a subset of the powerset of B, and therefore is a set. $\square$

Theorem 3.5. *The category* **Dcpo**-S *is cocomplete.*

Proof. By Theorem 23.14 of [6], Corollary 3.3, Lemma 3.4, and Proposition 2.9, **Dcpo**-S is cocomplete. □

The following example shows that epimorphisms in the categories **Dcpo**-S and **Cpo**-S are not necessarily surjective.

Example 3.6. Let S be an arbitrary dcpo(cpo)-monoid. Take A to be the dcpo(cpo) $\perp \oplus \mathbb{N}$ in which the order on $\mathbb{N}$ is discrete and $B = \perp \oplus \mathbb{N} \oplus \top$ in which the order on $\mathbb{N}$ is the usual order. Then both of A and B with the trivial action are S-dcpo's (cpo's). Let $h\colon A \to B$ be the inclusion map. Then h clearly preserves the action. Also, h is (strict) continuous. To see this, let $D \subseteq^d \perp \oplus \mathbb{N}$. Then $D = \{\perp\}$, or there exists $n \in \mathbb{N}$ such that $D = \{\perp, n\}$, or there exists $n \in \mathbb{N}$ such that $D = \{n\}$. If $D = \{\perp, n\}$ for some $n \in \mathbb{N}$, then

$$h(\bigvee^d D) = h(n) = n = \bigvee^d \{\perp, n\} = \bigvee^d \{h(\perp), h(n)\} = \bigvee^d h(D).$$

This is clearly true for other kinds of D. Now we claim that h is an S-dcpo(cpo) map which is an epimorphism but is not surjective. The latter is because $\top$ is not in the image of h. To show that h is an epimorphism, let $f_1, f_2\colon B \to P$ be S-dcpo(cpo) maps with $f_1 \circ h = f_2 \circ h$, and P be an S-dcpo(cpo). Then $f_1(\perp) = f_1(h(\perp)) = f_2(h(\perp)) = f_2(\perp)$ and $f_1(n) = f_1(h(n)) = f_2(h(n)) = f_2(n)$, for all $n \in \mathbb{N}$. Also

$$f_1(\top) = f_1(\bigvee^d \mathbb{N}) = \bigvee^d_{n\in\mathbb{N}} f_1(n) = \bigvee^d_{n\in\mathbb{N}} f_2(n) = f(\bigvee^d \mathbb{N}) = f_2(\top).$$

Therefore, $f_1 = f_2$, and so h is an epimorphism.

Finally, we consider cartesian closedness. Recall that a category $\mathcal{C}$ which has finite products, is called *cartesian closed* if, for every pair of objects A and B of $\mathcal{C}$, an object B^A and a morphism $ev\colon A \times B^A \to B$ exist with the universal property that for every morphism $f\colon A \times C \to B$ in $\mathcal{C}$, there exists a unique morphism $\hat{f}\colon C \to B^A$ such that $ev \circ (id_A \times \hat{f}) = f$. In this definition, the objects B^A are called *power objects or exponentials*, and ev is said to be *the evaluation map*, and $\hat{f}$ is called *the exponential map* associated to f.

Theorem 3.7. *The category* **Cpo**-S *is not necessarily cartesian closed.*

Proof. Let $S = \{1\}$, then the category **Cpo**-S is isomorphic to the category **Cpo** which is not cartesian closed (See [4]).

For an example in which S is not trivial, let S be the 2-element chain $\{1, a\}$ with identity 1, $1 < a$ and $aa = a$. Then S is an S-cpo, and by Remark 2.3, it is the initial object of **Cpo**-S. Then for a non trivial S-cpo A, the functor

$A\times -\colon \mathbf{Cpo}\text{-}S \to \mathbf{Cpo}\text{-}S$ does not have preserve the initial object (since $|A|\times 2 \neq 2$), and so does not have a right adjoint. Therefore, the category $\mathbf{Cpo}$-S is not cartesian closed. $\square$

In the following, we show that $\mathbf{Dcpo}$-S is cartesian closed.

Theorem 3.8. *The category* $\mathbf{Dcpo}$-S *is cartesian closed.*

Proof. By Proposition 2.9, $\mathbf{Dcpo}$-S has finite products. Given S-dcpo's A, B, we define the exponential object B^A to be $Hom(S\times A, B)$, the set of all S-dcpo maps from the product object $S\times A$ to B. This set is an S-dcpo with pointwise order, and action given by $(fs)(t,a) = f(st,a)$. The evaluation arrow $ev\colon A\times B^A \to B$ is defined by $ev(a,f) = f(1,a)$, is an S-dcpo map. It is an S-poset map (see [2]), to prove continuity, let $D \subseteq^d A$ and $f \in B^A$, then

$$ev(\bigvee^d D, f) = f(1, \bigvee^d D) = \bigvee^d_{x\in D} f(1,x) = \bigvee^d_{x\in D} ev(x,f)$$

since f is continuous. Also, for $F \subseteq^d B^A$ and $a\in A$, we have

$$ev(a, \bigvee^d F) = (\bigvee^d F)(1,a) = \bigvee^d_{f\in F} f(1,a) = \bigvee^d_{f\in F} ev(a,f)$$

To prove the universal property, take an S-dcpo C and an S-dcpo map $f\colon A\times C \to B$. Define the map $\hat{f}\colon C \to B^A$ by $\hat{f}(x)(s,a) = f(a, xs)$, for $x\in C$, $a\in A$, and $s\in S$. As in the case of S-sets (see [4]), it can be shown that $\hat{f}$ and $\hat{f}(x)$, for each $x\in C$, preserve the action. Also, we show that each $\hat{f}(x)$ is continuous. Let $T \subseteq^d S$ and $a\in A$. Then

$$\hat{f}(x)(\bigvee^d T, a) = f(a, x(\bigvee^d T)) = f(a, \bigvee^d_{t\in T} xt) = \bigvee^d_{t\in T} f(a, xt) = \bigvee^d_{t\in T} \hat{f}(x)(t,a)$$

Now, let $D \subseteq^d A$ and $s\in S$. Then

$$\hat{f}(x)(s, \bigvee^d D) = f(\bigvee^d D, xs) = \bigvee^d_{d\in D} f(d, xs) = \bigvee^d_{d\in D} \hat{f}(x)(s,d)$$

as required. Further, $\hat{f}$ is continuous, because for every $D \subseteq^d C$ and $(s,a)\in S\times A$, we have

$$\begin{aligned} \hat{f}(\textstyle\bigvee^d D)(s,a) &= f(a, (\textstyle\bigvee^d D)s) = f(a, \textstyle\bigvee^d_{x\in D} xs) \\ &= \textstyle\bigvee^d_{x\in D} f(a, xs) = \textstyle\bigvee^d_{x\in D} \hat{f}(x)(s,a) \end{aligned}$$

as required. $\square$

Remark 3.9. The above proof for the case where S is a one-element dcpo-monoid shows that the exponential object B^A in **Dcpo** is the set of all continuous maps from A into B, with pointwise order (for another proof of this fact, see [7]).

Open Problems:

1. *Is the category* **Cpo**-S *cocomplete? If yes, what is the description of coequalizers and pushouts?*

2. *For which class of semigroups* S*, the category* **cpo**-S *is cartesian closed?*

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