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Characterizations of Clifford semigroups and t-Putcha semigroups by their quasi-ideals

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Abstract. There are bi-ideals of semigroups which are not quasi-ideals. In spite of this fact, here we show that a semigroup S is quasi-simple if and only if it is bi-simple, equivalently t-simple. Main results of this article are several equivalent characterizations of the Clifford semigroups and the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals. A semigroup S is a Clifford semigroup if and only if every quasi-ideal of S is a semiprime ideal, whereas S is a semilattice of t-Archimedean semigroups if and only if $\sqrt{Q}$ is an ideal for every quasi-ideal Q of S.

1. Introduction

In 1952, R.A. Good and D.R. Hughes [3] first defined the notion of bi-ideals of a semigroup. The notion of quasi-ideals in rings and semigroups was introduced and developed by Otto Steinfeld [12], [13], [14], [15]. Different classes of semigroups has been characterized by using bi-ideals and quasi-ideals by many authors [7], [8], [9], [10]. Later on different classes of semigroups has been characterized by using minimal and maximal left-ideals, bi-ideals and quasi-ideals by many authors [1], [17], [4], [2], [9], [6]. Here we characterize the Clifford semigroups and the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals.

There are several characterizations for a semigroup S equivalent to be a Clifford semigroup and t-Putcha semigroup by their bi-ideals. Every quasi-ideal of a semigroup is a bi-ideal but the converse is not true. So if a semigroup S is bi-simple or equivalently t-simple then it is quasi-simple. Here we have a strange observation that every quasi-simple semigroup is also t-simple and thus quasi-simplicity and t-simplicity becomes equivalent in semigroups. Therefore we hope that it may turns out to be the case that the semigroups which are semilattices of groups or t-Archimedean semigroups will be characterized by their quasi-ideals. We show that a semigroup S is a semilattice of t-Archimedean semigroups.

Some elementary results together with prerequisites have been discussed in Section 2. In Section 3 we have studied semilattice of quasi-simple semigroups.

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2. Preliminaries

A nonempty subset L of a semigroup S is called a *left ideal* of S if $SL \subseteq L$. The *right ideals* are defined dually. A subset I of S is called an *ideal* of S if it is both a left and a right ideal of S. For an element $a \in S$ the *principal left ideal* (*right ideal*) of S generated by $\{a\}$ is given by $Sa \cup \{a\}$ $(aS \cup \{a\})$ and are denoted by $L(a)$ and $R(a)$ respectively. A semigroup S is called *simple* (*left-simple, right-simple*) if it does not contain any proper ideal (left-ideal, right-ideal), and S is called *t-simple* if it is both left simple and right simple.

A nonempty subset Q is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. It follows that every quasi-ideal Q of S is a subsemigroup. Every nonempty intersection of a left ideal and a right ideal is a quasi ideal of S. Suppose Q is a quasi-ideal of S. Then $L = SQ \cup Q$ is a left ideal and $R = QS \cup Q$ is a right ideal of S such that $Q = L \cap R$. Thus a nonempty subset Q of S is a quasi-ideal if and only if it is an intersection of a left ideal and a right ideal. For $a \in S$, let $Q(a)$ be the quasi-ideal generated by $\{a\}$.

A semigroup S is called *quasi-simple* if it has no proper quasi-ideal.

The *Green's relations* $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on a semigroup S are defined by, for $a, b \in S$,

$$a\mathcal{L}b \text{ if } L(a) = L(b), \quad a\mathcal{R}b \text{ if } R(a) = R(b) \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Now we have the following theorem (cf. [9]).

Theorem 2.1. *Let S be a semigroup. Then $\mathcal{H}$ can be given as follows: for $a, b \in S$,*

$$a\mathcal{H}b \iff Q(a) = Q(b).$$

A nonempty subset A of S is called *semiprime* if for all $x \in S$ such that $x^2 \in A$ one has $x \in A$, and *completely prime* (resp. *semiprimary*) if for all $x, y \in S$ such that $xy \in A$ one has $x \in A$ or $y \in A$ (resp. $x^n \in A$ or $y^n \in A$ for some $n \in \mathbb{N}$). A subsemigroup F of S is called a *filter* of S if for any $a, b \in S$, $ab \in F \Rightarrow a, b \in F$. Let $N(a)$ be the filter generated by $\{a\}$. Define an equivalence relation $\mathcal{N}$ on S by: for $a, b \in S$,

$$a\mathcal{N}b \text{ if } N(a) = N(b).$$

The following lemma (proved in [9]) plays a crucial role in the main theorems of this article.

Lemma 2.2. *Let S be a semigroup. Then $\mathcal{N}$ is the least semilattice congruence on S.*

3. Semilattice of groups

In this section we characterize the semigroups which are semilattices (chains) of groups.

Theorem 3.1. *The following conditions are equivalent on a semigroup S:*

(1) *S is a semilattice of groups;*

(2) *for all $a, b \in S$, $\quad ab, ba \in Q(a)$ and $a \in Q(a^2)$;*

(3) *for all $a \in S$, $\ Q(a)$ is a semiprime ideal of S;*

(4) *every quasi-ideal of S is a semiprime ideal of S;*

(5) *for all $a, b \in S$, $\ Q(ab) = Q(a) \cap Q(b)$;*

(6) *for all $a \in S$, $\ N(a) = \{x \in S | a \in Q(x)\}$;*

(7) *for every nonempty family $\{Q_\lambda | \lambda \in \Delta\}$ of quasi-ideals of S, $\bigcap_{\lambda \in \Delta} Q_\lambda$ is a semiprime ideal of S;*

(8) *$\mathcal{H} = \mathcal{N}$ is the least semilattice congruence of S such that each of its congruence classes is a group.*

Proof. (1) $\Rightarrow$ (2). Let S be a semilattice L of groups G_α, $(\alpha \in L)$. Consider $a, b \in S$. Then there are $\alpha, \beta \in L$ such that $a \in G_\alpha$, $\ b \in G_\beta$ and so aba, ab, ba are in $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. Since $G_{\alpha\beta}$ is a group, $ab \in Q(aba) \subseteq Q(a)$. Similarly, $ba \in Q(a)$. Also $a, a^2 \in G_\alpha$ implies that $a \in Q(a^2)$.

(2) $\Rightarrow$ (3). Let $a \in S$. Consider $q \in Q(a)$ and $s \in S$. Then $sq, qs \in Q(q) \subseteq Q(a)$ implies that $Q(a)$ is an ideal of S. Let $u \in S$ be such that $u^2 \in Q(a)$. Then $u \in Q(u^2) \subseteq Q(a)$. Thus $Q(a)$ is a semiprime ideal of S.

(3) $\Rightarrow$ (4). Follows similarly.

(4) $\Rightarrow$ (5). Let $a, b \in S$. Since $a \in Q(a)$ is an ideal of S, so $ab \in Q(a)$ and similarly, $ab \in Q(b)$. Then $ab \in Q(a) \cap Q(b)$ implies that $Q(ab) \subseteq Q(a) \cap Q(b)$. Let $x \in Q(a) \cap Q(b)$. Then $x \in R(a)$ implies that there exists $s_1 \in S$ such that $x = as_1$. Then $x^2 = (as_1)x = a(s_1x)$. Since $Q(a) \cap Q(b)$ is an ideal of S, so $s_1x \in Q(a) \cap Q(b)$ and hence $s_1x \in R(b)$. Then $s_1x = bs_2$ for some $s_2 \in S$. Then $x^2 = abs_2$ which implies that $x^2 \in R(ab)$. Similarly, $x^2 \in L(ab)$. Thus $x^2 \in Q(ab)$ which yields $x \in Q(ab)$. Then $Q(a) \cap Q(b) \subseteq Q(ab)$ and hence $Q(a) \cap Q(b) = Q(ab)$.

(5) $\Rightarrow$ (6). Let $F = \{x \in S \,|\, a \in Q(x)\}$. Consider $x, y \in F$. Then $a \in Q(x) \cap Q(y) = Q(xy)$ implies that $xy \in F$. Thus F is a subsemigroup of S. If for $x, y \in S$, $xy \in F$, then $a \in Q(xy) = Q(x) \cap Q(y)$ implies that $x, y \in F$. Thus F is a filter of S.

Let T be a filter of S containing a and $u \in F$. Then there exists $s \in S$ such that $a = s_1u$. Then $s_1u \in T$ implies that $u \in T$. Hence $F = N(a)$.

(6) $\Rightarrow$ (7). Let $Q = \bigcap_{\lambda \in \Delta} Q_\lambda$. Then Q is a quasi-ideal of S. Let $q \in Q$ and $s \in S$. Now $q \in N(qs)$ implies that $qs \in Q(q) \subseteq Q$. Similarly, $sq \in Q$. Let $a^2 \in Q$. Then $a^2 \in N(a)$ implies that $a \in Q(a^2) \subseteq Q$. Thus Q is a semiprime ideal of S.

(7) $\Rightarrow$ (4). Obvious.

(6) $\Rightarrow$ (8). Let $a, b \in S$. Then $a\mathcal{H}b$ implies that $Q(a) = Q(b)$ and so $a \in N(b)$ and $b \in N(a)$. This implies that $N(a) = N(b)$, i.e., $a\mathcal{N}b$. Thus $\mathcal{H} \subseteq \mathcal{N}$. Similarly,

$\mathcal{N} \subseteq \mathcal{H}$. Hence $\mathcal{H} = \mathcal{N}$ is the least semilattice congruence on S. Then every $\mathcal{H}$-class is a group.

(8) $\Rightarrow$ (1). Obvious. □

In the following theorem we characterize the semigroups which are chains of groups.

Theorem 3.2. *The following conditions are equivalent on a semigroup S:*

(1) *S is a chain of groups;*

(2) *for all $a, b \in S$, $ab, ba \in Q(a)$; and $a \in Q(ab)$ or $b \in Q(ab)$;*

(3) *for all $a \in S$, $Q(a)$ is a completely prime ideal of S;*

(4) *every quasi-ideal of S is a completely prime ideal of S;*

(5) *for all $a, b \in S$, $Q(ab) = Q(a) \cap Q(b)$; and $Q(a) \subseteq Q(b)$ or $Q(b) \subseteq Q(a)$;*

(6) *for all $a, b \in S$, $N(a) = \{x \in S \mid a \in Q(x)\}$ and $N(ab) = N(a) \cup N(b)$;*

(7) *for every nonempty family $\{Q_\lambda \mid \lambda \in \Delta\}$ of quasi-ideals of S, $\bigcap_{\lambda \in \Delta} Q_\lambda$ is a completely prime ideal of S;*

(8) *$\mathcal{H} = \mathcal{N}$ is the least chain congruence on S such that each of its congruence classes is a group.*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of groups $G_\alpha (\alpha \in \mathcal{C})$. Then the first part follows from Theorem 3.1. For the second part, let $a \in G_\alpha, b \in G_\beta$, $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then a, $ab \in G_\alpha$ implies that $a\mathcal{H}ab$ and hence $a \in Q(ab)$. Similarly, $\alpha\beta = \beta$ implies that $b \in Q(ab)$. Thus either $a \in Q(ab)$ or $b \in Q(ab)$.

(2) $\Rightarrow$ (3). Let $a \in S$. Then $Q(a)$ is an ideal of S by Theorem 3.1. Consider $x, y \in S$ such that $xy \in Q(a)$. Now $x \in Q(xy)$ or $y \in Q(xy)$ implies that $x \in Q(a)$ or $y \in Q(a)$. Thus $Q(a)$ is a semiprime ideal of S.

(3) $\Rightarrow$ (4). Follows similarly.

(4) $\Rightarrow$ (5). Let $a, b \in S$. Then $Q(ab) = Q(a) \cap Q(b)$, by Theorem 3.1.

Again $a \in Q(ab)$ or $b \in Q(ab)$ implies that $Q(a) \subseteq Q(ab) \subseteq Q(b)$ or $Q(b) \subseteq Q(ab) \subseteq Q(a)$. Thus $Q(a) \subseteq Q(b)$ or $Q(b) \subseteq Q(a)$.

(5) $\Rightarrow$ (6). Let $a \in S$. Then $N(a) = \{x \in S \mid a \in Q(x)\}$, by Theorem 3.1. Let $a, b \in S$. Then, $N(a) \cap N(b) \subseteq N(ab)$. Let $x \in N(ab)$. Then $ab \in Q(x)$. Now we have $Q(ab) = Q(a)$ or $Q(ab) = Q(b)$ which implies that $Q(a) \subseteq Q(x)$ or $Q(b) \subseteq Q(x)$. Then $x \in N(a)$ or $x \in N(b)$. Thus $N(ab) \subseteq N(a)$ or $N(ab) \subseteq N(b)$. Then $N(ab) \subseteq N(a) \cup N(b)$. Hence $N(ab) = N(a) \cup N(b)$.

(6) $\Rightarrow$ (7). Let $Q = \bigcap_{\lambda \in \Delta} Q_\lambda$. In view of Theorem 3.1, we are only to show that Q is completely prime. For $a, b \in S$, if $ab \in Q$, then $ab \in N(ab) = N(a) \cup N(b)$ implies that $a \in Q(ab) \subseteq Q$ or $b \in Q(ab) \subseteq Q$, i.e., $a \in Q$ or $b \in Q$. Thus Q is a completely prime ideal of S.

(7) $\Rightarrow$ (4). Obvious.

(6) $\Rightarrow$ (8). In view of Theorem 3.1, we are only to show that $\mathcal{N}$ is a chain congruence on S. Let $a,b \in S$. Then $ab \in N(ab) = N(a) \cup N(b)$. Thus $ab \in N(a)$ or $ab \in N(b)$, i.e., $N(ab) \subseteq N(a) \subseteq N(a) \cup N(b) = N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(a) \cup N(b) = N(ab)$. Then $N(ab) = N(a)$ or $N(ab) = N(b)$. Then $ab\mathcal{N}a$ or $ab\mathcal{N}b$.

(8) $\Rightarrow$ (1). Obvious. □

4. Semilattice of t-Archimedean semigroups

In this section we characterize the semigroups which are semilattices of t-Archimedean semigroups by their quasi-ideals. Also in this section the semigroups which are chains of t-Archimedean semigroups are characterized.

Let A be a nonempty subset of a semigroup S. Then the *radical of* A in S is given by

$$\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N})\ x^n \in A\}.$$

A semigroup S is called *left* (*right*) *Archimedean* if for each $a \in S$, $S = \sqrt{Sa}$, ($S = \sqrt{aS}$) and *t-Archimedean semigroup* if it is both a left Archimedean semigroup and a right Archimedean semigroup. Thus a semigroup S is t-Archimedean if and only if for $a, b \in S$ there exist $n \in \mathbb{N}$ and $x_1, x_2 \in S$ such that $b^n = x_1 a$ and $b^n = a x_2$.

A semigroup S is called a *semilattice* (*chain*) of t-Archimedean semigroups if there exists a congruence ρ on S such that S/ρ is a semilattice (chain) and each ρ-class is a t-Archimedean semigroup.

Let S be a semigroup. Define a binary relation σ on S by : for $a, b \in S$,

$$a\sigma b \Longleftrightarrow b \in \sqrt{SaS} \Longleftrightarrow b^n \in SaS, \text{ for some } n \in \mathbb{N}.$$

Then $a^3 \in SaS$ shows that $a \in \sqrt{SaS}$, i.e., σ is reflexive. So the transitive closure $\rho = \sigma^*$ is reflexive and transitive and therefore the symmetric relation $\eta = \rho \cap \rho^{-1}$ is an equivalence relation. Thus the equivalence relation η is the least semilattice congruence on S.

Recall that for every $a \in S$, $Q(a) = L(a) \cap R(a)$. In general neither $L(a) = Sa$ nor $R(a) = aS$. Also, $Sa \cap aS$ is a quasi-ideal of S which may not contain a. But we have the following lemma.

Lemma 4.1. *Let S be a semigroup. Then $\sqrt{Q(a)} = \sqrt{Sa \cap aS} = \sqrt{Sa} \cap \sqrt{aS}$ for all $a \in S$.*

Lemma 4.2. *Let S be a semigroup such that for all $a, b \in S, ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then*

(1) *for all $a, b \in S, a \in Sb \cap bS \Rightarrow$ for every $r \in \mathbb{N}$ there are $n \in \mathbb{N}, x \in S$ such that $a^n = b^{2^r} x b^{2^r}$ and hence $a \in \sqrt{Q(b^{2^r})}$;*

(2) *for all* $a, b \in S, a \in \sqrt{Q(b)}$ *implies that* $\sqrt{Q(a)} \subseteq \sqrt{Q(b)}$*;*

(3) *the least semilattice congruence* η *on* S *is given by: for all* $a, b \in S$*,*

$$a\eta b \ \text{ if } \ b \in \sqrt{Q(a)} \ \text{ and } \ a \in \sqrt{Q(b)}.$$

Proof. (1). Let $a, b \in S$ with $a \in Sb \cap bS$. Then there exist $s_1, s_2 \in S$ such that $a = s_1 b = bs_2$. Also, there exist $n \in \mathbb{N}$ and $u_1, u_2 \in S$ such that $(bs_1)^n = u_1 b$ and $(s_2 b)^n = bu_2$. Then $a^{n+1} = s_1(bs_1)^n b = s_1 u_1 b^2$ and $a^{n+1} = b(s_2 b)^n s_2 = b^2 u_2 s_2$. Then $a^{2(n+1)} = b^2 u_2 s_2 s_1 u_1 b^2$ implies that the result is true for $r = 1$. Let for $k \in \mathbb{N}$, there is $p \in \mathbb{N}$ and $x \in S$ such that $a^p = b^{2^k} x b^{2^k}$. Then proceeding as above, we have $q \in \mathbb{N}$ and $y \in S$ such that $a^q = b^{2^{k+1}} y b^{2^{k+1}}$. Thus the result follows by the principle of Mathematical induction.

The last part follows by Lemma 4.1.

(2). For $a \in \sqrt{Q(b)}$, there are $n \in \mathbb{N}$ and $s_1, s_2 \in S$ such that $a^n = s_1 b = bs_2$. Let $x \in \sqrt{Q(a)}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in Sa \cap aS$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then, by (1), we find $p \in \mathbb{N}$ and $u \in S$ such that $x^p = a^{2^r} u a^{2^r}$ which implies that $x^p = a^n a^{2^r - n} u a^{2^r - n} a^n = bs_2 a^{2^r - n} u a^{2^r - n} s_1 b$. Then $x \in \sqrt{Q(b)}$, by the Lemma 4.1.

(3). Consider $a \in S$. Then $x \in \sqrt{Q(a)}$ implies that $x^n = s_1 a = as_2$ for some $n \in \mathbb{N}$ and $s_1, s_2 \in S$. Then $x^{n+n} = s_1 a^2 s_2$ implies that $x \in \sqrt{SaS}$. Thus $\sqrt{Q(a)} \subseteq \sqrt{SaS}$. Let $y \in \sqrt{SaS}$. Then there are $m \in \mathbb{N}$ and $t_1, t_2 \in S$ such that $y^m = t_1 a t_2$. Again $t_1 a t_2 \in \sqrt{St_1 a} \subseteq \sqrt{Sa}$ and $t_1 a t_2 \in \sqrt{at_2 S} \subseteq \sqrt{aS}$ implies that $y^m \in \sqrt{aS} \cap \sqrt{Sa} = \sqrt{Q(a)}$ and so $y \in \sqrt{Q(a)}$, by the Lemma 4.1. Thus $\sqrt{SaS} \subseteq \sqrt{Q(a)}$ and hence $\sqrt{Q(a)} = \sqrt{SaS}$.

Now for $a, b \in S$, $a\eta b$ implies that there are $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_m \in S$ such that $a\sigma c_1$, $c_1 \sigma c_2, \ldots, c_{n-1}\sigma c_n$, $c_n \sigma b$ and $b\sigma d_1$, $d_1 \sigma d_2, \ldots, d_{m-1}\sigma d_m$, $d_m \sigma a$. These give $c_1 \in \sqrt{Q(a)}$, $c_2 \in \sqrt{Q(c_1)}, \ldots, b \in \sqrt{Q(c_n)}$ and $d_1 \in \sqrt{Q(b)}$, $d_2 \in \sqrt{Q(d_1)}, \ldots, a \in \sqrt{Q(d_m)}$ so that $b \in \sqrt{Q(a)}$ and $a \in \sqrt{Q(b)}$, by (2). □

Recall that for $a, b \in S$,

$$a\mathcal{H}b \iff Q(a) = Q(b).$$

Let us define $\sqrt{\mathcal{H}}$, the radical of $\mathcal{H}$ on S by: for $a, b \in S$,

$$a\sqrt{\mathcal{H}}b \iff \sqrt{Q(a)} = \sqrt{Q(b)}.$$

Now we have the main theorem of this section:

Theorem 4.3. *The following conditions are equivalent on a semigroup* S*:*

(1) S *is a t-Putcha semigroup;*

(2) *for all* $a, b \in S$, $b \in SaS$ *implies* $b \in \sqrt{Q(a)}$*;*

(3) *for all* $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$;

(4) $\sqrt{Q}$ *is an ideal of* S *for every quasi-ideal* Q *of* S;

(5) $\sqrt{Q(a)}$ *is an ideal of* S, *for all* $a \in S$;

(6) $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$ *for all* $a \in S$;

(7) $\mathcal{N} = \sqrt{\mathcal{H}}$ *is the least semilattice congruence and the congruence classes are* t*-Archimedean semigroups.*

Proof. (1) $\Rightarrow$ (2). Let ρ be a semilattice congruence on S such that the ρ-classes $T_\alpha, \alpha \in S/\rho$ are t-Archimedean semigroups. Let $a, b \in S$ be such that $b \in SaS$. Then there are $s_1, s_2 \in$ such that $b = s_1 a s_2$. Now $s_1 a s_2 \rho a s_1 s_2 \rho s_1 s_2 a$ implies that $b, a s_1 s_2, s_1 s_2 a \in T_\alpha$ for some $\alpha \in S/\rho$. Since T_α is a t-Archimedean semigroup, there exist $n \in \mathbb{N}$ and $u_1, u_2 \in T_\alpha$ such that $b^n = a s_1 s_2 u_1$ and $b^n = u_2 s_1 s_2 a$. Thus $b \in \sqrt{Q(a)}$, by Lemma 4.1.

(2) $\Rightarrow$ (3). Let $a, b \in S$. Now $(ab)^2 = abab$ implies $(ab)^2 \in SaS \cap SbS$. Then $(ab)^2 \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (4). Let Q be a quasi-ideal of S and let $u \in \sqrt{Q}$ and $c \in S$. Then $u^n = q$ for some $n \in \mathbb{N}, q \in Q$. Also by (3), there is $m \in \mathbb{N}$ such that $(uc)^m \in Su$ and $(uc)^{m+1} \in uSu$. Consider $r \in \mathbb{N}$ such that $2^r > n$. Then by Lemma 4.2, there are $m_1 \in \mathbb{N}$ and $x \in S$ such that $(uc)^{m_1} = u^{2^r} x u^{2^r} = q u^{2^r - n} x u^{2^r - n} q$ which implies that $uc \in \sqrt{qS \cap Sq} = \sqrt{Q(q)} \subseteq \sqrt{Q}$, by Lemma 4.1. Similarly, $cu \in \sqrt{Q}$. Thus $\sqrt{Q}$ is an ideal of S.

(4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (3). Let $a, b \in S$. Then $\sqrt{Q(a)}$ and $\sqrt{Q(b)}$ are ideals of S. Then $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (6). Let $a \in S$ and $F = \{x \in S | a \in \sqrt{Q(x)}\}$. Consider $y, z \in F$. Then there exist $n \in \mathbb{N}, u_1, u_2 \in S$ such that $a^n = u_1 z$ and $a^n = y u_2$. Also, by (3), there are $m_1, m_2 \in \mathbb{N}, w_1, w_2 \in S$ such that $(u_2 u_1 z y)^{m_1} = z y w_1$ and $(z y u_2 u_1)^{m_2} = w_1 z y$. Now $a^{2n} = y u_2 u_1 z$ implies $a^{2n(m_1+1)} = (y u_2 u_1 z)^{m_1+1} = y (u_2 u_1 z y)^{m_1} u_2 u_1 z = (yz) y w_1 u_2 u_1 z$. Also, $a^{2n(m_2+1)} = y u_2 u_1 z w_2 z (yz)$. Thus $yz \in F$, by Lemma 4.1; and hence F is a subsemigroup of S.

Let $y, z \in S$ be such that $yz \in F$. Then $a \in \sqrt{Q(yz)} = \sqrt{yzS} \cap \sqrt{Syz} \subseteq \sqrt{yS} \cap \sqrt{Sz}$. Now, by (3), $yz \in \sqrt{Sy}$, and so $yz \in \sqrt{yS} \cap \sqrt{Sy} = \sqrt{Q(y)}$, by Lemma 4.1. Then $\sqrt{Q(yz)} \subseteq \sqrt{Q(y)}$, by Lemma 4.2. Thus $a \in \sqrt{Q(y)}$ and hence $y \in F$. Similarly, $z \in F$. Thus F is a filter that contains a. Let T be a filter of S containing a and $y \in F$. Then $a^m = sy$ for some $m \in \mathbb{N}, s \in S$. Now $a^m \in T$ implies $sy \in T$ and hence $y \in T$. Thus $F = N(a)$.

(6) $\Rightarrow$ (7). Consider $a, b \in S$. Then $ab \in N(ab)$ implies that $a, b \in N(ab)$. Then, by (6), $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$. If $a \mathcal{N} b$ then $N(a) = N(b)$ implies that $b \in \sqrt{Q(a)}$ and $a \in \sqrt{Q(b)}$. So, $\sqrt{Q(b)} \subseteq \sqrt{Q(a)}$ and $\sqrt{Q(a)} \subseteq \sqrt{Q(b)}$, by Lemma 4.2. Thus $a \sqrt{\mathcal{H}} b$ and hence $\mathcal{N} \subseteq \sqrt{\mathcal{H}}$. Similarly, $\sqrt{\mathcal{H}} \subseteq \mathcal{N}$. Hence $\mathcal{N} = \sqrt{\mathcal{H}}$ is the least semilattice congruence.

Let T be an $\mathcal{N}$-class in S. Since $\mathcal{N}$ is a semilattice congruence, T is a subsemigroup. Consider $a, b \in T$. Then $a^2\mathcal{N}b$ implies that $N(a^2) = N(b)$; and by (6) we have $b \in \sqrt{Q(a^2)}$. Thus there are $n \in \mathbb{N}$ and $s_1, s_2 \in S$ such that $b^n = s_1a^2$ and $b^n = a^2s_2$ which implies that $b^{n+1} = bs_1a^2$ and $b^{n+1} = a^2s_2b$. Since $\mathcal{N}$ is a semilattice congruence, $t_1 = bs_1a\mathcal{N}bs_1a^2\mathcal{N}b^{n+1}\mathcal{N}b$ and $t_2 = as_2b\mathcal{N}b$ which implies that $t_1 = bs_1a \in T$ and $t_2 = as_2b \in T$. Thus $b \in \sqrt{Ta} \cap \sqrt{aT}$ and hence T is a t-Archimedean semigroup.

(7) $\Rightarrow$ (1). Follows directly. □

Theorem 4.4. *The following conditions on a semigroup S are equivalent:*

(1) *S is a chain of t-Archimedean semigroups.*

(2) *S is a t-Putcha semigroup and for all $a, b \in S$, $b \in \sqrt{Q(a)}$ or $a \in \sqrt{Q(b)}$.*

(3) *For all $a, b \in S$, $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$ and $N(ab) = N(a) \cup N(b)$.*

(4) *$\mathcal{N} = \sqrt{\mathcal{H}}$ is the least chain congruence on S such that each of its congruence classes is t-Archimedean.*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of t-Archimedean semigroups $S_\alpha (\alpha \in \mathcal{C})$. Let $a, b \in S$. Then $a \in S_\alpha$ and $a \in S_\beta$ for some $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $a, ab \in S_\alpha$; and since S_α is a t-Archimedean semigroup, there exist $n \in \mathbb{N}$ and $x_1, x_2 \in S_\alpha$ such that $a^n = x_1ab$ and $a^n = abx_2$. Now, by Theorem 4.3, since S is a semilattice of t-Archimedean semigroup, there are $m \in \mathbb{N}$ and $s \in S$ such that $(abx_2)^m = bx_2s$. Then we have $a^{nm} = s_1b$ and $a^{nm} = bx_2s$ for some $s_1 \in S$ and hence $a \in \sqrt{Q(b)}$, by Lemma 4.1. If $\alpha\beta = \beta$, then $b, ab \in S_\beta$ and similarly as above we have $b \in \sqrt{Q(a)}$.

(2) $\Rightarrow$ (3). By Theorem 4.3, we have $N(a) = \{x \in S \mid a \in \sqrt{Q(x)}\}$, since S is a t-Putcha semigroup. Let $a, b \in S$. Then $ab \in N(ab)$ implies that $a \in N(ab)$ and $b \in N(ab)$, and hence $N(a) \cup N(b) \subseteq N(ab)$. Again, either $a \in \sqrt{Q(b)}$ or $b \in \sqrt{Q(a)}$. If $a \in \sqrt{Q(b)}$, then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n = bs$ and so $a^{n+1} = abs$. Since S is a semilattice of t-Archimedean semigroups, there exist $m \in \mathbb{N}$ and $t \in S$ such that $(abs)^m = tab$, by Theorem 4.3. Then we have $a^{(n+1)m} = tab$ and $a^{(n+1)m} = abt_1$ for some $t_1 \in S$. Then $a \in \sqrt{Q(ab)}$ which implies that $ab \in N(a)$. Thus $N(ab) \subseteq N(a)$. If $b \in \sqrt{Q(a)}$, then similarly we have $N(ab) \subseteq N(b)$, which shows that $N(ab) \subseteq N(a) \cup N(b)$. Hence $N(ab) = N(a) \cup N(b)$.

(3) $\Rightarrow$ (4). It follows by Theorem 4.3 that $\mathcal{N} = \sqrt{\mathcal{H}}$ is the least semilattice congruence on S and each $\mathcal{N}$-class is a t-Archimedean semigroup.

Now consider $a, b \in S$. Then $ab \in N(a) \cup N(b)$ shows that $ab \in N(a)$ or $ab \in N(b)$. Again $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab)$. Thus either $N(ab) \subseteq N(a) \subseteq N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(ab)$. i.e., either $a\mathcal{N}ab$ or $b\mathcal{N}ab$. Hence $\mathcal{N}$ is a chain congruence on S. Since every chain is a semilattice and $\mathcal{N}$ is the least semilattice congruence, it is the least chain congruence on S.

(4) $\Rightarrow$ (1). Trivial. □

Theorem 4.5. *The following conditions on a semigroup S are equivalent:*

(1) *S is a chain of t-Archimedean semigroups;*

(2) *$\sqrt{Q}$ is a completely prime ideal of S for every quasi-ideal Q of S;*

(3) *$\sqrt{Q(a)}$ is a completely prime ideal of S for every $a \in S$;*

(4) *$\sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)}$ for all $a, b \in S$ and every quasi-ideal of S is semiprimary .*

Proof. (1) $\Rightarrow$ (2). Let S be a chain $\mathcal{C}$ of t-archimedean semigroups $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. We take a quasi-ideal Q of S. Then $\sqrt{Q}$ is an ideal of S, by Theorem 4.3. Let $x, y \in S$ be such that $xy \in \sqrt{Q}$. Then there is $n \in \mathbb{N}$ such that $(xy)^n = u \in Q$. Suppose $x \in S_\alpha$ and $y \in S_\beta$,$\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$. Since S_α is t-Archimedean, so $x \in \sqrt{Q(u)} \subseteq \sqrt{Q}$. Similarly, if $\alpha\beta = \beta$, then $y \in \sqrt{Q}$. Hence $\sqrt{Q}$ is a completely prime ideal of S.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). Let $a, b \in S$. Then $\sqrt{Q(a)}$ and $\sqrt{Q(b)}$ are ideals of S and hence $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)}$. This implies $\sqrt{Q(ab)} \subseteq \sqrt{Q(a)} \cap \sqrt{Q(b)}$, by Lemma 4.2 and Theorem 4.3. Since $\sqrt{Q(ab)}$ is completely prime, so $a, b \in \sqrt{Q(ab)}$ which implies $\sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Q(ab)}$. Thus $\sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)}$.

Let Q be a quasi-ideal of S and $x, y \in S$ be such that $xy \in Q$. Then $xy \in \sqrt{Q(xy)}$ implies that $x \in \sqrt{Q(xy)}$ or $y \in \sqrt{Q(xy)}$. Thus $x^n \in \sqrt{Q(xy)} \subseteq Q$ or $y^n \in \sqrt{Q(xy)} \subseteq Q$ for some $n \in \mathbb{N}$. Hence Q is semiprimary.

(4) $\Rightarrow$ (1). Let $a, b \in S$. Then $ab \in \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$. Then by Theorem 4.3, S is a t-Putcha semigroup. Since $\sqrt{Q(ab)}$ is a semiprimary, $ab \in Q(ab)$ implies that $a \in \sqrt{Q(ab)} = \sqrt{Q(a)} \cap \sqrt{Q(b)} \subseteq \sqrt{Q(b)}$ or $b \in \sqrt{Q(ab)} \subseteq \sqrt{Q(a)}$. Thus S is a chain of t-Archimedean semigroups by Theorem 4.4. $\square$

References

[1] **Y. Cao and X. Xu**, *On minimal and maximal left ideals in ordered semigroups*, Semigroup Forum **60** (2000), 202 – 207.

[2] **R. Chinram and K. Tinpum**, *A note on minimal bi-ideal in ordered Γ-semigroups*, Intern. Math. Forum **4** (2009), 1 – 5.

[3] **R.H. Good and D.R. Hughes**, *Associated groups for semigroup*, Bull. Amer. Math. Soc. **58** (1952), 624 – 625.

[4] **A. Iampan**, *On bi-ideals of semigroups*, Lobachevskii J. Math.**29** (2008), 68 – 72.

[5] **R.D. Jagatap and Y.S. Pawar**,*Quasi-ideals and minimal quasi-ideals in Γ-semigroups*, Navi Sad J. Math. **39** (2009), no.2, 79 – 87.

[6] **C. Jirokul, R. Sripakorn and R. Chinram**, *Minimal quasi-ideals in Γ-semigroups*, Intern. Math. Forum **15** (2009), 7 – 11.

[7] **K.M. Kapp**, *On bi-ideals and quasi-ideals in semigroups*, Publ. Math. Debrecan **16** (1969), 179 – 185.

[8] **K.M. Kapp**, *Green's relations and quasi-ideals*, Czechoslovak Math. J. **19(94)** (1969), 80 – 85.

[9] **N. Kehayopulu**, *On minimal quasi-ideals in ordered semigroups*, Abstract AMS **15** (1994) no.6, 1 – 5.

[10] **Y. Kemprasit**, *Quasi-ideals and bi-ideals in semigroups and rings*, Proc. Intern. Conf. Algebra and its Appl. (2002), 30 – 46.

[11] **M.M. Miccoli**, *Bi-ideals in regular semigroups and in orthogroups*, Acta. Math. Hung. **47(1-2)** (1986), 3 – 6.

[12] **O. Steinfeld**, *Über die Quasiideale von Halbgruppen*, Publ. Math. Debrecan, **4** (1956), 262 – 275.

[13] **O. Steinfeld**, *Über die Quasiideale von Ringen*, Acta Sci. Math. Szeged **17** (1956), 170 – 180.

[14] **O. Steinfeld**, *Über die Quasiideale von Haibgruppen mit eigentlichem Suschke-witsch-Kern*, Acta Sci. Math. Szeged **18** (1957), 235 – 242.

[15] **O. Steinfeld**, *Quasi-ideals in rings and regular semigroups*, Akademiai Kiado, Budapest, 1978.

[16] **O. Steinfeld**, *On completely regular semigroups*, Semigroup Forum **21** (1980), 93 – 94.

[17] **X. Xu and J. Ma**, *A note on minimal bi-ideals in ordered semigroups*, Southeast Asian Bull. Math. **27** (2003), 149 – 154.

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