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Characterizing monomorphisms of actions on directed complete posets (S-dcpo)

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Abstract. Domain Theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called *domains*. It was introduced in the 1970s by Scott as a foundation for programming semantics and provides an abstract model of computation, and has grown into a respected field on the borderline between Mathematics and Computer Science.

In this paper we take domains as ordered algebraic structures and consider the actions of a partially ordered monoid which is itself a domain, on them. To study algebraic notions, in particular injectivity and flatness, in the categories so obtained, one needs to know the different kinds of monomorphisms, their properties and the relations between them. This is what we are going to discuss in this paper.

1. Introduction and preliminaries

Domain theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called domains. It was introduced in the 1970s by Scott as a foundation for programming semantics and provides an abstract model of computation using order structures and topology, and has grown into a respected field on the borderline between Mathematics and Computer Science [1].

Relationships between domain theory and logic were noted early on by Scott [10], and subsequently developed by many authors, including Smyth [11], Abramsky [2], and Zhang [12]. There has been much work on the use of domain logics as logics of types and of program correctness, with a focus on functional and imperative languages.

In this paper we take domains as ordered algebraic structures and consider the actions of a pomonoid which is itself a domain, on them. To study algebraic notions, in particular injectivity and flatness, in the categories so obtained, one needs to know the properties of different kinds of monomorphisms and the relations between them. This is what we are trying to do in the following.

First we recall some preliminaries needed in the sequel. The reader can find more details in [2, 4, 5, 6]. Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leqslant c$; and P is called *directed complete*,

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or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P. A dcpo which has a bottom element $\perp$ is said to be a *cpo*.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. A dcpo map $f : P \to Q$ between cpo's is called *strict* if $f(\perp) = \perp$. Thus we have the category **Dcpo** (**Cpo**) of all dcpo's (cpo's) with (strict) continuous maps between them.

A *po-monoid* S is a monoid with a partial order $\leqslant$ which is compatible with the binary operation (that is, for $s, t, s', t' \in S$, $s \leqslant t$ and $s' \leqslant t'$ imply $ss' \leqslant tt'$). Similarly, a *dcpo* (*cpo*)*-monoid* is a monoid which is also a dcpo (cpo) whose binary operation is a (strict) continuous map.

Recall that an (*right*) *S-act* or an *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with *action preserving* maps ($f : A \to B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S-act. An element $a \in A$ is called a *zero, fixed,* or *a trap element* if $as = a$, for all $s \in S$.

For a po-monoid S, an (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps between them is denoted by **Pos**-S.

Also, for a dcpo (cpo)-monoid S, an (*right*) *S-dcpo* (*S-cpo*) is a dcpo (cpo) A which is also an S-act whose action $\lambda : A \times S \to A$ is a (strict) continuous map.

Notice that in the definition of an S-cpo, the continuity of the action implies that it is also strict. This is because, since $\perp_S \leqslant e$ and the action is continuous, we have $\perp_A \perp_S \leqslant \perp_A e = \perp_A$ and so $\perp_A \perp_S \leqslant \perp_A$. Also, $\perp_A \leqslant \perp_A \perp_S$. Therefore, $\perp_A \perp_S = \perp_A$ and the action is strict. Also, note that the bottom element of an S-cpo in not necessarily a zero element. For example, consider the cpo-monoid $S = \{s, e\}$ where e is the identity element of S, $e \leqslant s$, and $ss = s$. Take the S-cpo $A = \{\perp_A, a\}$, where $\perp_A \leqslant a$, with the action $\perp_A s = a = as$. We see that $\perp_A$ is not a zero element.

A (possibly empty) subset B of an S-dcpo (S-cpo) A is called a *sub S-dcpo* (*sub S-cpo*) of A if B is both a sub dcpo (sub cpo) and a subact of A.

By an *S-dcpo map* (*S-cpo map*) between S-dcpo's (S-cpo's), we mean a map $f : A \to B$ which is both (strict) continuous and action preserving. We denote the category of all S-dcpo's (S-cpo's) and S-dcpo (S-cpo) maps between them by **Dcpo**-S (**Cpo**-S).

A separately (or semi-)cpo-monoid is a monoid which is also a cpo whose right and left translations $R_s : S \to S$, $t \rightsquigarrow ts$ and $L_s : S \to S$, $t \rightsquigarrow st$ are strict continuous.

Now, let S be a separately cpo-monoid. A separately S-cpo is a cpo A which is also an S-act with the action $A \times S \to A$ such that every $R_s : A \to A$, $a \rightsquigarrow as$ and $L_a : S \to A$, $s \rightsquigarrow as$, are strict continuous. The category of all separately

S-cpo's with action preserving strict continuous maps between them is denoted by **Sep-Cpo**-S.

Finally, let S be a monoid with identity e. By a cpo S-act, we mean an S-act in the category **Cpo**. In other words, a pair $(A; (\lambda_s)_{s\in S})$ is called a cpo S-act if A is a cpo, and each $\lambda_s : A \to A$, $a \rightsquigarrow as$, is a cpo map, called an action, such that for all $s, t \in S$, and $a \in A$, denoting $\lambda_s(a)$ by as we have:

(1) $a(st) = (as)t$;

(2) $ae = a$.

By a cpo S-act map between cpo S-acts, we mean a cpo map which is also action preserving. The category of all cpo S-acts with cpo S-act maps between them is denoted by $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Definition 1.1. A morphism $h : A \to B$ in **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) is called *order-embedding* provided that for all $x, y \in A$, $h(x) \leqslant h(y)$ if and only if $x \leqslant y$.

In this paper, first we characterize different kinds of monomorphisms namely regular, strict, strong and extremal in [3], in the categories **Dcpo**-S, **Cpo**-S, **Sep-Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ and see that they are the same as order-embeddings. Then, we study the relation of monomorphisms with one-one morphisms and see that in the categories **Dcpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, **Dcpo** and **Cpo**, monomorphisms are exactly one-one morphisms. Also, we show that under some conditions the same result is true for the category **Cpo**-S. In the last section we consider some categorical properties of monomorphisms and regular monomorphisms, in the mentioned categories, the properties such as factorization properties of morphisms and some categorical properties related to limits and colimits.

2. Characterization of monomorphisms

In this section we characterize different kinds of monomorphisms in categories **Dcpo**-S, **Cpo**-S, **Sep-Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, also we study their relation with one-one morphisms. First, we recall some related definitions from [3].

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. Then, the pair $(\mathcal{E}, \mathcal{M})$ is called a *factorization structure for morphisms* in $\mathcal{C}$ and $\mathcal{C}$ is called $(\mathcal{E}, \mathcal{M})$-structured provided that:

(1) each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms,

(2) $\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-*factorizations* (*of morphisms*); that is, each morphism f in $\mathcal{C}$ has a factorization $f = he$, with $e \in \mathcal{E}$ and $h \in \mathcal{M}$, and

(3) $\mathcal{C}$ has the unique $(\mathcal{E}, \mathcal{M})$-*diagonalization property*; that is, for each commutative square

$$\begin{array}{ccc} \mathrm{A} & \xrightarrow{e} & B \\ f\big\downarrow & & \big\downarrow g \\ C & \xrightarrow{h} & D \end{array}$$

with $e \in \mathcal{E}$ and $h \in \mathcal{M}$, there exists a unique *diagonal*; that is, a morphism $d : B \to C$ such that $de = f$ and $hd = g$.

Definition 2.2. A monomorphism $h : A \to B$ in a category $\mathcal{C}$ is called:

(1) *regular* if it is an equalizer of a pair of morphisms;

(2) *strict* if it has the universal property that given any morphism $h' : A' \to B$ such that $rh = sh$ implies $rh' = sh'$, for all $r, s : B \to C$, there exists a unique morphism $\bar{h} : A' \to A$ with $h' = h\bar{h}$;

(3) *strong* provided that $\mathcal{C}$ has the unique (Epi,$\{h\}$)-diagonalization property (Epi is the class of all epimorphisms);

(4) *extremal* provided that if $h = me$, where e is an epimorphism, then e is an isomorphism.

2.1. Monomorphisms and order-embeddings

In this subsection, we characterize different kinds of monomorphisms such as regular, strict, strong and extremal in **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 2.3. Notice that order-embeddings are one-one, and hence monomorphisms in the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. But the converse is not necessarily true. For example, take $S = \{e, s\}$ where $s \leqslant e$ and $s^2 = s$. Then S is a dcpo (cpo, separately cpo)-monoid. Now, take $A = \{\perp, a, a'\}$ with the order $\perp \leqslant a, a'$, $a \parallel a'$, and define the action on A as follows: $\perp$ is a zero element and $as = a's = \perp$. Also, take B to be the three element chain $\mathbf{3} = \{0, 1, 2\}$ with $0 \leqslant 1 \leqslant 2$, and define the action on B as follows: 0 is a zero element and $1s = 2s = 0$. Now, define $h : A \to B$ as $h(\perp) = 0$, $h(a) = 1$, $h(a') = 2$. Then h is one-one and hence a monomorphism in these categories, but it is not an order-embedding.

Theorem 2.4. *A monomorphism $h : A \to B$ in* **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S *and* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is regular if and only if it is order-embedding.*

Proof. Let $h : A \to B$ be a regular monomorphism in **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). Then h is the equalizer of morphisms $g_1, g_2 : B \to C$. Note that, the equalizer of g_1 and g_2 in these categories is $E = \{b \in B : g_1(b) = g_2(b)\}$ with order and action inherited from B (see also [7], [8], [9]). Hence there exists an isomorphism between E and A, so h is an order-embedding.

Conversely, let $h : A \to B$ be an order-embedding in one of the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S or $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. In each category, we define two morphisms whose equalizer is h.

(i). In **Dcpo**-S, consider the disjoint union $(B\times\{1\})\cup(B\times\{2\})$ of B with itself, which is the coproduct $B \sqcup B$ by Theorem 2.4 of [7]. Take B' to be the quotient $(B \sqcup B)/\theta(H)$, where $\theta(H)$ is the congruence generated by $H = \{((h(a), 1), (h(a), 2)) : a \in A\}$. Now, consider the natural epimorphism $q : B \sqcup B \to B'$ and the coproduct maps $g_1, g_2 : B \to B \sqcup B$. We prove later on that h is the equalizer of qg_1 and qg_2.

(ii). In **Cpo**-S, we consider the same S-dcpo B' as defined in (i). Since in this case h is strict, $h(\perp_A) = \perp_B$, and then $[(\perp_B, 1)]_{\theta(H)} = [(\perp_B, 2)]_{\theta(H)}$ is the bottom element of B'. So, B' is an S-cpo. Also qg_1 and qg_2 introduced in part (i) are strict, because $qg_1(\perp_B) = [(\perp_B, 1)]_{\theta(H)}$ and $qg_2(\perp_B) = [(\perp_B, 2)]_{\theta(H)}$. We will see that h is the equalizer of qg_1 and qg_2 in **Cpo**-S.

(iii). In **Sep**-**Cpo**-S, B is a separately S-cpo, and hence by Remark 3.3 of [8], B is also an S-cpo. So, from the discussion given in (ii), B' which introduced in part (i), is an S-cpo. Now again by applying Remark 3.3 of [8], we get that B' is a separately S-cpo. This is because, B is a separately S-cpo, and so for every $b \in B$ and $s \in S$ we have $b\perp_S = \perp_B$ and $\perp_B s = \perp_B$, therefore for every $b \in B$, $s \in S$, and $i = 1, 2$, we have $[(b, i)]\perp_S = [(b\perp_S, i)] = [(\perp_B, i)]$ and $[(\perp_B, i)]s = [(\perp_B s, i)] = [(\perp_B, i)]$. Also, similar to part (ii), qg_1 and qg_2 are strict continuous maps. We will see later on that h is the equalizer of qg_1 and qg_2.

(iv). In $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, similar to (i), take the coporoduct of B with itself (which is called the coalesced sum, see [9]), and apply the same argument to define q, g_1, g_2. We show that h is the equalizer of qg_1 and qg_2.

Now, we prove that h is the equalizer of qg_1 and qg_2 in all the above cases.

It is clear that $(qg_1)h = (qg_2)h$. Consider an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map $k : C \to B$ with $(qg_1)k = (qg_2)k$. Notice that $k(C) \subseteq h(A)$. This because, on the contrary if $x \in k(C) \setminus h(A)$, then since $x \notin h(A)$, we get $qg_1(x) \neq qg_2(x)$ but since $x \in k(C)$ and $(qg_1)k = (qg_2)k$, we have $qg_1(x) = qg_2(x)$ which is a contradiction. On the other hand, since h is an order-embedding, it is one-one, and so there exists a map $h' : B \to A$ such that $h'h = id_A$. Now we see that $k' = h'k : C \to A$ is the unique S-dcpo (S-cpo, separately S-cpo, cpo S-act) map with $hk' = k$. First, we prove that k' preserves the order. To see this, let $x, x' \in C$, $x \leqslant x'$. Then $k(x) \leqslant k(x')$. Since $k(C) \subseteq h(A)$, there exist $a, a' \in A$, $k(x) = h(a)$ and $k(x') = h(a')$. Therefore, $h(a) \leqslant h(a')$, and so $a \leqslant a'$ (since h is an order-embedding). Now, $h'h(a) \leqslant h'h(a')$ (since $h'h = id_A$) and hence $k'(x) = h'k(x) = h'h(a) \leqslant h'h(a') = h'k(x') = k'(x')$. Also, k' preserves the action. To show this, let $x \in C$ and $s \in S$, then $k'(xs) = h'k(xs) = h'(k(x)s) = h'(h(a)s) = h'(h(as)) = as$ where $k(x) = h(a), a \in A$. On the other hand, $k'(x)s = h'k(x)s = h'h(a)s = as$. To see that k' is continuous, let $D \subseteq^d C$. Then $k'(D) \subseteq^d A$, since k' is order-preserving. Also for each $d \in D$, there exists $a_d \in A$ with $k(d) = h(a_d)$ and $T = \{a_d : d \in D, h(a_d) = k(d)\} \subseteq^d A$. This is because, if $a_{d_1}, a_{d_2} \in T$, then $d_1, d_2 \in D \subseteq^d C$. Therefore, there exists $d_3 \in D$ with $d_1, d_2 \leqslant d_3$. Now, $k(d_1), k(d_2) \leqslant k(d_3)$ and so $h(a_{d_1}), h(a_{d_2}) \leqslant h(a_{d_3})$ for some $a_{d_3} \in A$, and hence $a_{d_1}, a_{d_2} \leqslant a_{d_3}$, since h is an order-embedding. Now, $k'(\bigvee^d D) = h'k(\bigvee^d D) = h'h(a) = a$ where $k(\bigvee^d D) = h(a), a \in A$. On the other hand, $\bigvee^d_{d\in D} k'(d) = \bigvee^d_{d\in D} h'k(d) = \bigvee^d_{d\in D} h'h(a_d) = \bigvee^d_{d\in D} a_d$. It is enough to prove that $\bigvee^d T = \bigvee^d_{d\in D} a_d = a$. For every $d \in D$, $a_d \leqslant a$, since $h(a_d) = k(d) \leqslant k(\bigvee^d D) = h(a)$ and h is an order-embedding. If $a' \in A$ is also an upper bound of T in A, then for every $d \in D$, $h(a_d) \leqslant h(a')$ and so $k(d) = h(a_d) \leqslant h(a')$ which implies $h(a) = k(\bigvee^d D) = \bigvee^d_{d\in D} k(d) \leqslant h(a')$. Thus $a \leqslant a'$, since h is an order-embedding. Therefore, $\bigvee^d T = a$. Notice that $hk' = k$ and k' is unique with this property. Also, in the case where h and k are strict, then so is k'. $\square$

Definition 2.5. Recall from [4] that considering **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) as a concrete category over **Set**, a monomorphism h is said to be an *embedding over* **Set** if whenever g is a map between S-dcpo's (S-cpo's, separately S-cpo's, cpo S-acts) such that hg is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map, then g itself is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map.

As a consequence of Theorem 2.4 we have:

Corollary 2.6. *If $h : A \to B$ is a regular monomorphism in* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *then h is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) embedding over* **Set**.

Proof. Suppose that $h : A \to B$ is a regular monomorphism in **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). By Theorem 2.4, h is an order-embedding. Now, let $g : C \to A$ be a function between S-dcpo's (S-cpo's, separately S-cpo's, cpo S-acts) such that hg is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map. Then we prove that g is an S-dcpo (an S-cpo, a separately S-cpo, a cpo S-act) map. First, we show that g preserves the action. This is because, for $x \in C$ and $s \in S$,

$$h(g(xs)) = (hg)(xs) = ((hg)(x))s = (h(g(x)))s = h(g(x)s)$$

and so $g(xs) = g(x)s$, since h is one-one. Also, g preserves the order. To see this, let $x, x' \in C$ with $x \leqslant x'$. Then, $h(g(x)) \leqslant h(g(x'))$. Now, since h is an order-embedding we have $g(x) \leqslant g(x')$. Finally, g is continuous. To show this, let $D \subseteq^d C$. Then $g(D) \subseteq^d A$, since g preserves the order. Further,

$$h(g(\bigvee^d D)) = (hg)(\bigvee^d D) = \bigvee^d_{d\in D} (hg)(d) = \bigvee^d_{d\in D} h(g(d)) = h(\bigvee^d_{d\in D} g(d))$$

and so $g(\bigvee^d_{d\in D} D) = \bigvee^d_{d\in D} g(d)$. Also, $h(g(\perp_C)) = hg(\perp_C) = \perp_B = h(\perp_A)$ and $g(\perp_C) = \perp_A$. $\square$

Now, we will study the relation of different kinds of monomorphisms. First recall the following proposition.

Proposition 2.7. [3] *If the category $\mathcal{C}$ has equalizers and pushouts, also regular monomorphisms in $\mathcal{C}$ are closed under composition, then a monomorphism is regular if and only if it is extremal.* $\square$

Theorem 2.8. *For a monomorphism $h : A \to B$ in* **Dcpo**-S (**Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *the following are equivalent:*

(1) *h is regular,*
(2) *h is strict,*
(3) *h is strong,*
(4) *h is extremal.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are general category-theoretic results. For implication (4) $\Rightarrow$ (1), since these categories are complete and cocomplete (see [7], [8], [9]), and by Theorem 2.4, regular monomorphisms are exactly order-embeddings and hence they are closed under composition, applying Proposition 2.7, we get that any extremal monomorphism is regular. $\square$

Lemma 2.9. *If $h : A \to B$ is a morphism in* **Dcpo**-S (**Cpo**-S) *then $h' : A \to \langle h(A)\rangle$, to the sub S-dcpo (sub S-cpo) of B generated by $h(A)$, with $h'(a) = h(a)$ for all $a \in A$, is an epimorphism in* **Dcpo**-S (**Cpo**-S).

Proof. Let $h : A \to B$ be a morphism in **Dcpo**-S (**Cpo**-S). Then take $h' : A \to \langle h(A)\rangle$, to the sub S-dcpo (sub S-cpo) of B generated by $h(A)$, with $h'(a) = h(a)$ for all $a \in A$. To show that h' is an epimorphism, consider $g_1, g_2 : \langle h(A)\rangle \to C$ such that $g_1h' = g_2h'$. Since for all $D \subseteq h(A)$, $g_1(D) = g_2(D)$ and g_1 and g_2 are continuous, it is straightforward to show that $g_1(\langle h(A)\rangle) = g_2(\langle h(A)\rangle)$. Therefore, h' is an epimorphism in **Dcpo**-S (**Cpo**-S). $\square$

Remark 2.10. Notice that, if $h : A \to B$ is a morphism in the category **Dcpo**-S (**Cpo**-S), then $h(A)$ is not necessarily an S-dcpo (S-cpo). To see this, consider $A = (\mathbb{N})_\perp$ where the natural numbers $\mathbb{N}$ is considered with the discrete order and $\perp \leqslant n$, for all $n \in \mathbb{N}$. Also consider $B = (\mathbb{N}^\infty)_\perp$ where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ and the order on $\mathbb{N}$ is the usual one and $\perp \leqslant n \leqslant \infty$, for all $n \in \mathbb{N}$. It is straightforward to show that A and B with the identity action are S-dcpo's (S-cpo's). Now, define the map $h : A \to B$ by $h(\perp) = \perp$ and $h(n) = n$, for all $n \in \mathbb{N}$. We get h is a (strict) continuous map and $h(A) = (\mathbb{N})_\perp$ is not an S-dcpo (S-cpo). This is because $D = \mathbb{N}$ is a directed subset of $h(A)$ and $\bigvee^d D = \bigvee^d \mathbb{N} = \infty \notin h(A)$.

Lemma 2.11. *A monomorphism $h : A \to B$ in* **Cpo**-S *is order-embedding if it is extremal.*

Proof. Suppose that $h : A \to B$ is an extremal mono in **Cpo**-S and consider $h' : A \to \langle h(A)\rangle$, $h'(a) = h(a)$ for all $a \in A$. It is clear that $h = ih'$, where $i : \langle h(A)\rangle \hookrightarrow B$. Also by Lemma 2.9, h' is an epimorphism in **Cpo**-S. Hence, by the definition of extremal monomorphisms, h' is an isomorphism in **Cpo**-S, and consequently h is an order-embedding. $\square$

As a consequence of Lemma 2.11 and Theorem 2.4, we have:

Corollary 2.12. *For a monomorphism $h : A \to B$ in* **Cpo**-S, *the following are equivalent:*

(1) *h is regular,*
(2) *h is strict,*
(3) *h is strong,*
(4) *h is extremal.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are general category-theoretic results. For implication (4) $\Rightarrow$ (1), by Lemma 2.11 and Theorem 2.4, we get the result. $\square$

2.2. Monomorphisms and one-one morphisms

In this subsection, we study the relation between monomorphisms and one-one morphisms in the categories **Dcpo**-S, **Dcpo**, **Cpo**, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 2.13. Notice that in **Dcpo**-S, monomorphisms are exactly one-one morphisms (see [7]). Furthermore, in **Dcpo**, **Cpo**, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ by applying the adjoint pairs given in Corollary 2.5 and Theorem 3.4 of [6], Corollary 4.4 of [8] and Corollary 4.2 of [9] and the fact that right adjoints preserves limits, we get that monomorphisms are exactly one-one morphisms. In the category **Cpo**-S, whenever $\perp_S = e$ or $\top_S = e$, monomorphisms are exactly one-one morphisms (by the adjoint pairs given in Corollaries 3.2 and 3.7 of [6]).

Remark 2.14. In Remark 2.3, we see that in the categories **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, order-embeddings are monomorphisms, but the converse is not necessarily true. But it is clearly shown that in the ordered structures, if $h : A \to B$ is a monomorphism and A is a chain then we have h is an order-embedding.

Lemma 2.15. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *such that for every $a, a' \in A$ with $h(a) = h(a')$, we have $a \perp_S = a' \perp_S = \perp_A$, then h is one-one.*

Proof. Let $h : A \to B$ be a monomorphism in **Cpo**-S with the property mentioned in the hypothesis and $h(a) = h(a')$ for some $a, a' \in A$. Then $a = a'$. This is because, on the contrary if $a \neq a'$, then there exist S-cpo maps $g, k : S \to A$ given by $g(s) = as$ and $k(s) = a's$, for $s \in S$ where $hg = hk$ while $g \neq k$, which is a contradiction. Therefore, h is one-one. □

As a corollary of Lemma 2.15, we have:

Theorem 2.16. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *and for every $a \in A$, $a\perp_S = \perp_A$, then h is one-one.* □

Theorem 2.17. *If $h : A \to B$ is a monomorphism in* **Cpo**-S *and $\perp_A$ is a zero element then h is one-one.*

Proof. Let $h : A \to B$ be a monomorphism in **Cpo**-S such that $\perp_A$ is a zero element. To see that h is a monomorphism in **Dcpo**-S, let $g_1, g_2 : D \to A$ be S-dcpo maps such that $hg_1 = hg_2$. Then, consider $D_\perp$ the S-cpo where $\perp$ is a zero element, and define $g'_i : D_\perp \to A$ for $i = 1, 2$ by

$$g'_i(d) = \begin{cases} g_i(d) & \text{if} \quad d \neq \perp \\ \perp_A & \text{if} \quad d = \perp \end{cases}$$

It is clear that g'_1 and g'_2 are S-cpo maps and $hg'_1 = hg'_2$. So $g'_1 = g'_2$, and hence $g_1 = g_2$. Therefore h is a monomorphism in **Dcpo**-S, and so h is one-one by Remark 2.13. □

As a consequence of Theorem 2.17 we get the following corollary.

Corollary 2.18. *If S is a cpo-monoid whose bottom element is a zero element or S is left zero as a semigroup, then in* **Cpo**-S *monomorphisms are exactly one-one morphisms.*

Proof. Let S be a cpo-monoid whose bottom element is a zero element and A be an S-cpo. Then $\perp_A$ is a zero element (for all $s \in S$, $\perp_A s = (\perp_A \perp_S)s = \perp_A(\perp_S s) = \perp_A \perp_S = \perp_A$). So by Theorem 2.17, in **Cpo**-S, monomorphisms are exactly one-one morphisms. In the case where S is left zero as a semigroup, since $\perp_S$ is zero element, the result follows similarly. $\square$

3. Monomorphisms and regular monomorphisms

We have divided this section into two subsections as follows:

3.1. Factorization properties of morphisms

Let $\mathcal{E}'$ be the class of order-embeddings in **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. Then, in the following theorem we show that **Dcpo**-S, **Cpo**-S, **Sep**-**Cpo**-S, and $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ have unique (Epi, $\mathcal{E}'$)-diagonalization property.

Corollary 3.1. **Dcpo**-S*,* **Cpo**-S*,* **Sep**-**Cpo**-S*, and* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *have unique* (*Epi,* $\mathcal{E}'$)*-diagonalization property.*

Proof. By Theorem 2.4, every order-embedding is a regular monomorphism in the mentioned categories and by Theorem 2.8 and Corollary 2.12 every regular monomorphism is a strong monomorphism. Now, by the definition of a strong monomorphism we get the result. $\square$

Theorem 3.2. **Dcpo**-S *and* **Cpo**-S *have* (*Epi, Mono*)*-factorization.*

Proof. Let $f : A \to B$ be a morphism in **Dcpo**-S (**Cpo**-S). Then, take $f' : A \to {<}f(A){>}$ by $f'(a) = f(a)$. So by Lemma 2.9, f' is an S-dcpo (S-cpo) epimorphism and $f = if'$, where $i : {<}f(A){>} \hookrightarrow B$ is an S-dcpo (S-cpo) monomorphism. $\square$

Remark 3.3. The factorization mentioned in Theorem 3.2, is not necessarily unique. To see this, consider $A = (\mathbb{N}^\infty)_\perp$, where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ has been considered with the discrete order, $\perp \leqslant n$ for all $n \in \mathbb{N}^\infty$ and the action on A is the identity action. Also consider $B = \perp \oplus \mathbb{N}^\infty \oplus \top$ where the order on $\mathbb{N}$ is the usual one, $\infty \parallel n$ for all $n \in \mathbb{N}$, and the action on B is the identity action. Define the map $f : A \to B$ as $f(\perp) = \perp$ and $f(n) = n$, for all $n \in \mathbb{N}^\infty$. It is straightforward to show that A and B are S-dcpo's (S-cpo's) and f is an S-dcpo (S-cpo) map. Furthermore, f is an epimorphism in **Dcpo**-S (**Cpo**-S). To prove this, let $g_1, g_2 : B \to D$ be S-dcpo (S-cpo) maps with $g_1 f = g_2 f$. Then, $g_1(n) = g_1(f(n)) = g_2(f(n)) = g_2(n)$, for all $n \in \mathbb{N}^\infty \cup \{\perp\}$. Also $g_1(\top) = g_1(\bigvee^d \mathbb{N}) =$

$\bigvee^d_{n\in\mathbb{N}} g_1(n) = \bigvee^d_{n\in\mathbb{N}} g_2(n) = g_2(\bigvee^d \mathbb{N}) = g_2(\top)$, since $g_1(n) = g_2(n)$, for all $n \in \mathbb{N}$. Therefore, $g_1(\top) = g_2(\top)$ and so $g_1 = g_2$. Hence, f is an epimorphism and it has the factorization $f = id_B f$. Now, let $C =\perp \oplus((\mathbb{N} \oplus \top) \cup \{\infty\})$ where the order on $\mathbb{N}$ is the natural one, $n \leqslant \top$ for all $n \in \mathbb{N}$, $\infty \parallel n$ for all $n \in (\mathbb{N} \oplus \top)$, and the action on C is the identity action. Then define $f' : A \to C$ by $f'(\perp) =\perp$ and $f'(n) = n$, for all $n \in \mathbb{N} \cup \{\infty\}$. It is clear that f' is an S-dcpo (S-cpo) map. Also f' is an epimorphism in **Dcpo**-S (**Cpo**-S) (the proof of the fact that f' is an epimorphism is similar to proof of the fact that f is an epimorphism) and $f = if'$ where i is an inclusion map from C to B. Hence, we have two factorizations for f, which are not equal.

Theorem 3.4. *The category* **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *has neither* (*Onto,Mono*)*-diagonalization property nor* (*Epi,Mono*)*-diagonalization property.*

Proof. Suppose that $A = \{\perp_A, a_1, a_2, a_3\}$ where $\perp_A$ is the bottom element, $a_2 \leqslant a_3$ and $a_1 \parallel a_2, a_3$, $B = \{\perp_B, b_1, b_2\}$ where the order on B is $\perp_B \leqslant b_1 \leqslant b_2$, $C = \{\perp_C, c_1, c_2\}$ where $\perp_C$ is the bottom element and $c_1 \parallel c_2$ and $D = \{\perp_D, d_1, d_2, d_3\}$ where $\perp_D$ is the bottom element, $d_1 \parallel d_2$ and $d_1, d_2 \leqslant d_3$. It is clear that A, B, C and D with the identity action are S-dcpo's (S-cpo's, cpo S-acts). Now, define $e : A \to B$ as $e(\perp_A) =\perp_B$, $e(a_1) = b_1$ and $e(a_3) = e(a_2) = b_2$, $f : A \to C$ as $f(\perp_A) =\perp_C$, $f(a_1) = c_1$ and $f(a_2) = f(a_3) = c_2$, $h : C \to D$ as $h(\perp_C) =\perp_D$, $h(c_1) = d_1$ and $h(c_2) = d_3$, $g : B \to D$ as $g(\perp_B) =\perp_D$, $g(b_1) = d_1$ and $g(b_2) = d_3$. It is straightforward to show that e, g, f and h are S-dcpo (S-cpo, cpo S-act) maps and $ge = hf$, but if there exists an S-dcpo (an S-cpo, a cpo S-act) map $k : B \to C$, such that $ke = f$ and $hk = g$, then $k(b_1) = k(e(a_1)) = f(a_1) = c_1$ and $k(b_2) = k(e(a_2)) = f(a_2) = c_2$ but $c_1 \nleqslant c_2$, which is a contradiction (because k is an order-preserving and $b_1 \leqslant b_2$). So **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) does not have (Onto,Mono)-diagonalization property. Also **Dcpo**-S (**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) does not have (Epi,Mono)-diagonalization property. □

3.2. Limits and colimits

The following theorem is easily proved, and it is in fact a corollary of the next result.

Theorem 3.5. *In* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *we have:*

(1) *The class of monomorphisms is closed under products;*
(2) *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a family of monomorphisms. Then their product morphism* $f : A \to \prod B_\alpha$ *is also a monomorphism.* □

Theorem 3.6. *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a source of monomorphisms in the categories* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). *Then the morphism* $f : A \to limB_\alpha$ (*existing by the universal property of limits*) *is also a monomorphism.*

Proof. Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of monomorphisms in one of the categories mentioned in the hypothesis. To prove that $f : A \to limB_\alpha$ is a monomorphism, let $g_1, g_2 : C \to A$ be such that $fg_1 = fg_2$. Then, $fg_1(c) = fg_2(c)$ for all $c \in C$. Also for all $c \in C$ and $\alpha \in I$, $\pi_\alpha(fg_1(c)) = f_\alpha(g_1(c)) = f_\alpha(g_2(c)) = \pi_\alpha(fg_2(c))$, where $\pi_\alpha : limB_\alpha \to B_\alpha$ is a limit morphism. Hence, $f_\alpha g_1 = f_\alpha g_2$ for all $\alpha \in I$, and since f_α is a monomorphism, we have $g_1 = g_2$. □

Proposition 3.7. *In* **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *we have:*
(1) *The class of regular monomorphisms is closed under products;*
(2) *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a family of regular monomorphisms. Then their product morphism* $f : A \to \prod B_\alpha$ *is also a regular monomorphism.*

Proof. We just prove (1) in **Dcpo**-S and the rest are proved similarly.

Let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of regular monomorphisms in **Dcpo**-S. We show that $f = \prod f_\alpha : \prod A_\alpha \to \prod B_\alpha$ where $f((a_\alpha)_{\alpha\in I}) = (f_\alpha(a_\alpha))_{\alpha \in I}$ is an order-embedding and so by Theorem 2.4, it is a regular monomorphism. Suppose that $f((a_\alpha)_{\alpha\in I}) \leqslant f((a'_\alpha)_{\alpha\in I})$ for $(a_\alpha)_{\alpha\in I}$, $(a'_\alpha)_{\alpha\in I} \in \prod A_\alpha$. We have $f((a_\alpha)_{\alpha\in I}) \leqslant f((a'_\alpha)_{\alpha\in I})$ if and only if $(f_\alpha(a_\alpha))_{\alpha\in I} \leqslant (f_\alpha(a'_\alpha))_{\alpha\in I}$ if and only if $f_\alpha(a_\alpha) \leqslant f_\alpha(a'_\alpha)$, for all $\alpha \in I$ if and only if $a_\alpha \leqslant a'_\alpha$, for all $\alpha \in I$ (since each f_α is an order-embedding) if and only if $(a_\alpha)_{\alpha\in I} \leqslant (a'_\alpha)_{\alpha\in I}$. So f is a regular monomorphism. □

Theorem 3.8. *Let* $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ *be a source of regular monomorphisms in* **Dcpo**-S (**Cpo**-S, **Sep-Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$). *Then the morphism* $f : A \to limB_\alpha$ (*existing by the universal property of limits*) *is also a regular monomorphism.*

Proof. Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of regular monomorphisms in one of the categories mentioned in the hypothesis. To prove that $f : A \to limB_\alpha$ is a regular monomorphism, by Theorem 2.4 it is enough to show that f is an order-embedding. To see this, let $f(a) \leqslant f(a')$ where $a, a' \in A$. We have $f_\alpha(a) = \pi_\alpha(f(a)) \leqslant \pi_\alpha(f(a')) = f_\alpha(a')$, for all $\alpha \in I$ ($\pi_\alpha : limB_\alpha \to B_\alpha$ is a limit morphism). So $a \leqslant a'$, because by Theorem 2.4, f_α is an order-embedding, for every $\alpha \in I$ and hence f is an order-embedding and also it is a regular monomorphism. □

Proposition 3.9. *In* **Dcpo**-S, **Sep-Cpo**-S *and* $\mathbf{Cpo}_{\mathbf{Act-S}}$, *the class of monomorphisms and regular monomorphisms are closed under coproducts.*

Proof. Assume that $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ is a family of monomorphisms and $\coprod f_\alpha : \coprod A_\alpha \to \coprod B_\alpha$ is the coproduct morphism. We show that $\coprod f_\alpha$ defined by $(\coprod f_\alpha)(a, \alpha) = (f_\alpha(a), \alpha)$, $a \in A_\alpha$, $\alpha \in I$, is a monomorphism. By Remark 2.13, it is enough to show that $\coprod f_\alpha$ is one-one. To see this, let $(\coprod f_\alpha)(a, \alpha) = (\coprod f_\alpha)(a', \alpha')$ where $a \in A_\alpha, a' \in A_{\alpha'}, \alpha, \alpha' \in I$. Therefore, $(f_\alpha(a), \alpha) = (f_\alpha(a'), \alpha')$ and so $\alpha = \alpha'$ and $f_\alpha(a) = f_\alpha(a')$. Since f_α is one-one we have $a = a'$. Consequently, $(a, \alpha) = (a', \alpha) = (a', \alpha')$. Now, suppose that $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ is a family of regular monomorphisms. We show that $\coprod f_\alpha$ is a regular monomorphism. By Theorem 2.4, it is enough to show that $\coprod f_\alpha$ is an order-embedding. To prove this, let $(\coprod f_\alpha)(a, \alpha) \leqslant (\coprod f_\alpha)(a', \alpha')$

where $a \in A_\alpha, a' \in A_{\alpha'}, \alpha, \alpha' \in I$. Therefore, $(f_\alpha(a), \alpha) \leqslant (f_\alpha(a'), \alpha')$. But this is impossible except $\alpha = \alpha'$ and then $f_\alpha(a) \leqslant f_\alpha(a')$. Since f_α is order-embedding, we have $a \leqslant a'$. Consequently, $(a, \alpha) \leqslant (a', \alpha) = (a', \alpha')$. $\square$

Recall that a class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the final theorem, we see that the class of order-embeddings satisfying this property.

Theorem 3.10. *The class of order-embeddings in* **Dcpo**-S (**Cpo**-S, **Sep**-**Cpo**-S, $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$) *is pullback stable.*

Proof. By Proposition 11.18 of [3], the class of regular monomorphisms is pullback stable. Therefore by Theorem 2.4, we get the result. $\square$

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