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On the generalization of Brešer theorems

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Abstract. If S is a prime semiring with char $S \neq 2$ and $f : S \to S$ is an additive mapping which is skew-commuting on an ideal I of S, then $f(I) = 0$. We also prove that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime semiring. These statements are the generalization of Brešar's theorems.

1. Introduction

The notion of semiring was first introduced by H. S. Vandiver in 1934 [10]. An algebraic system $(S, +, \cdot)$ is called a *semiring* if $(S, \cdot)$ is a semigroup; $(S, +)$ is a commutative semigroup with 0 and distributive laws of multiplication over addition hold; furthermore, $0s = s0 = 0$ for all $s \in S$. A subsemiring I of S is called a *right ideal* of S if $s \in S$, $x \in I$ implies $xs \in I$. *Left ideals* are defined in a similar way. A subset which is both left and right ideal is called an *ideal*. An ideal I of a semiring S is called a *k-ideal* if $x + y \in I$, $x \in I$ implies $y \in I$. A proper ideal P of a semiring S is said to be *prime* if $AB \subset P$ implies $A \subset P$ or $B \subset P$ for any ideals A and B of S. A proper ideal P of a semiring S is called a *semiprime ideal* if $A^2 \subset P$ implies $A \subset P$ for every ideal A of S. A k-ideal I of a semiring S is semiprime ideal if and only if I is the intersection of all prime k-ideals of S containing it [9, Theorem 3.12]. A semiring S is prime if 0 is a prime ideal. A semiring S is semiprime if 0 is a semiprime ideal. For further details of semirings, we refer [2, 3, 4, 5, 6, 7]. An additive mapping $f : S \to S$ is said to be *skew-commuting* on a set $T \subseteq S$ if $f(s)s + sf(s) = 0$ for all $s \in T$.

In [1], M. Brešar proved that if S is a prime ring of characteristic not 2, and $f : S \to S$ is an additive mapping which is skew-commuting on an ideal I of S, then $f(I) = 0$. He also proved that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime rings. In this paper, we observe that these results still hold in the wider spectrum of semirings.

2. Preliminaries

One can easily prove the statement of following lemma.

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Lemma 2.1. *Let S be a semiring. If S has a nonzero nilpotent right ideal R, then it has a nonzero nilpotent ideal I containing R.* $\square$

Now, we extend Lemma 1.1 [7] and Lemma 1 [1] in the framework of semirings.

Lemma 2.2. *Let S be a semiring and $I \neq (0)$ a right ideal of S. If there exists a positive integer n such that $x^n = 0$ for all $x \in I$, then S has a nonzero nilpotent ideal.*

Proof. The proof is given by induction on n. For $n = 2$, we have $x^2 = 0$ for all $x \in I$. As $x + xs \in I$ for all $s \in S$, so we get $(x + xs)^2 = 0$. This implies $xsx = 0$. Multiply from right by $t \in S$ to get $xsxt = 0$, so we obtain $(xS)^2 = 0$. Now if $xS \neq 0$, then S has a nonzero nilpotent right ideal xS and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. When $xS = 0$, then $I^2 \subseteq IS = 0$. So S has a nonzero nilpotent right ideal I and hence has a nonzero nilpotent ideal.

Now suppose that Lemma is true for all positive integers less than n. Since $x^n = 0$ for all $x \in I$ for a fixed integer n and n is least such integer, therefore $x^{n-1} \neq 0$ and $(x^{n-1})^2 = 0$. Take $b = x^{n-1}$, then $b^2 = 0$. Let $B = bI$, then two cases arise. In the first case, let $B \neq (0)$. As $b + bs \in I$ for all $s \in S$, so we have $(b + bs)^n = 0$. On expansion, we arrive $(bs)^{n-1}b = 0$. This results in $(bs)^{n-1}B = (0)$. Let $T = \{x \in B \,|\, xB = 0\}$. It is easy to see that T is a k-ideal of B. Moreover, $y \in B$ implies that $y^{n-1} \in T$. Now let $y + T \in B/T$, then $(y + T)^{n-1} = y^{n-1} + T = T$. Hence by induction hypothesis B/T has a nilpotent ideal $U/T \neq T$. This yields $U \not\subset T$ and $(U/T)^k = U^k/T = T$ for some positive integer k. Since T is a k-ideal of B, therefore $U^k \subset T$ and hence $U^{k+1} \subset TU \subseteq TB = (0)$. As $U \not\subset T$ and U is an ideal of B, so we have $(0) \neq UB \subset U$ and $(UB)^{k+1} \subset U^{k+1} = (0)$. This implies that UB is a nonzero nilpotent right ideal of S and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. In the second case, when $B = x^{n-1}I = (0)$. Let $W = \{x \in I \,|\, xI = (0)\}$, then W is a k-ideal of I. If $W = I$, then $I^2 = (0)$ and so I is a nonzero nilpotent right ideal and hence, by Lemma 2.1, S has a nonzero nilpotent ideal. If $W \neq I$, then for each element $x \in I$, $x^{n-1} \in W$. Hence in I/W, each element $x + W$ satisfies $(x + W)^{n-1} = x^{n-1} + W = W$. So our induction hypothesis gives us a nilpotent ideal $V/W \neq W$, this means $V \not\subset W$ and $(V/W)^m = V^m/W = W$ for some positive integer m. Hence we have $V^m \subset W$ and $V^{m+1} \subset WV \subseteq WI = (0)$. Since $(0) \neq VI \subset V$, where V is ideal of I, so we have $(VI)^{m+1} \subset V^{m+1} = (0)$. This means that S has a nonzero nilpotent right ideal VI and hence again, in view of Lemma 2.1, S has a nonzero nilpotent ideal. $\square$

Lemma 2.3. *Let I be a nonzero ideal of a prime semiring S. If $I_n = \{x^n \,|\, x \in I\}$, then $I_n a = 0$ (or $aI_n = 0$) implies $a = 0$.*

Proof. Let $I_n a = 0$ and suppose on contrary $a \neq 0$. If $at = 0$ for all $t \in I$, then replacing t by st, where $s \in S$, we get $ast = 0$. As S is prime semiring, so we get $t = 0$ for all $t \in I$. This implies $I = 0$, which is not possible, hence $av \neq 0$ for some $v \in I$. As $avx \in I$ for all $x \in I$, so $(avx)^n a = 0$, this implies that $(avx)^{n+1} = 0$.

So we get right ideal $(av)S$ in which each element r satisfies $r^{n+1} = 0$. Hence, by Lemma 2.2, S has a nonzero nilpotent ideal but this is not possible in prime semiring, so we conclude $a = 0$. Similarly, we can prove the case when $aI_n = 0$. $\square$

One can also observe the following statements.

Lemma 2.4. *Let S be a semiring. If $a + b = 0$ and $a + c = 0$ for $a, b, c \in S$, then $b = c$.* $\square$

Lemma 2.5. *Let P be a prime ideal of semiring S and $ax \in P$ (or $xa \in P$) for all $x \in S$, then $a \in P$.* $\square$

3. Main results

Theorem 3.1. *Let S be a prime semiring of characteristic not 2. If an additive mapping $f : S \to S$ is skew-commuting on some ideal I of S, then $f(x) = 0$ for all $x \in I$.*

Proof. As f is skew-commuting on I, so we have

$$f(x)x + xf(x) = 0 \quad \forall x \in I. \tag{1}$$

Multiplying (1) from the right and left separately by x and applying Lemma 2.4, we get

$$f(x)x^2 = x^2 f(x). \tag{2}$$

Linearization of (1) yields

$$f(x)y + yf(x) + f(y)x + xf(y) = 0 \quad \forall x, y \in I. \tag{3}$$

Replacing y by x^2 in (3) and using (2), we get

$$2x^2 f(x) + f(x^2)x + xf(x^2) = 0. \tag{4}$$

After multiplying the last relation from right by x^2 and using (2), one can get the relation $2x^4 f(x) + f(x^2)x^3 + xf(x^2)x^2 = 0$. Now by adding $x^2 f(x^2)x + x^3 f(x^2)$ on both sides of this relation and using (1), we obtain

$$2x^4 f(x) = x^2 f(x^2)x + x^3 f(x^2). \tag{5}$$

Multiplying (4) by x^2 from left, the last relation reduces to $4x^4 f(x) = 0$. As S is of characteristic not 2, so we have

$$x^4 f(x) = 0. \tag{6}$$

Using (2), we obtain

$$f(x)x^4 = 0. \tag{7}$$

Now multiplying (1) from right by $2x$ and applying the Lemma 2.4 to (4), we have $2xf(x)x = f(x^2)x + xf(x^2)$. By multiplying this from left and right by x simultaneously and using (2) and (6), we reach $xf(x^2)x^2 + x^2f(x^2)x = 0$. This, along with (5) and (6), becomes

$$x^3f(x^2) = xf(x^2)x^2. \tag{8}$$

Now (1) can be written as $f(x^2)x^2 + x^2f(x^2) = 0$. Multiplying it from left by x and using (8), we get $2x^3f(x^2) = 0$. This becomes

$$x^3f(x^2) = 0. \tag{9}$$

Similarly, we can prove

$$f(x^2)x^3 = 0. \tag{10}$$

Replace x by x^2 in (3) to get $f(x^2)y+yf(x^2)+f(y)x^2+x^2f(y) = 0$ for all $x, y \in I$. By multiplying this from right and left by x^3 simultaneously and using (9) and (10), we obtain

$$x^3f(y)x^5 + x^5f(y)x^3 = 0 \quad \forall x, y \in I. \tag{11}$$

Replace x by x^2 in last relation to get

$$x^6f(y)x^{10} + x^{10}f(y)x^6 = 0. \tag{12}$$

First multiplying (11) from left by x^3 and right by x^5, then the last relation, in view of Lemma 2.4, becomes $x^{10}f(y)x^6 = x^8f(y)x^8$. Similarly, we get $x^6f(y)x^{10} = x^8f(y)x^8$. So (12) becomes $x^8f(y)x^8 = 0$ for all $x, y \in I$. This can be written as

$$zf(y)z = 0 \quad \forall y \in I, \forall z \in I_8. \tag{13}$$

Replace y by z in (3) to get

$$f(x)z + zf(x) + f(z)x + xf(z) = 0 \quad \forall x \in I, \forall z \in I_8. \tag{14}$$

Multiplying last relation from right by z and using (13), we obtain

$$f(x)z^2 + f(z)xz + xf(z)z = 0. \tag{15}$$

Suppose $x \in I_8$, then (13) can be written as $xf(x)x = 0$. Left multiplying (1) by x and using this relation, we get $x^2f(x) = 0$ for all $x \in I_8$. Now multiplying (15) from left by x^2, using this relation and (13), we arrive $x^3f(z)z = 0$ for all $x, z \in I_8$. By Lemma 2.3, this reduces to $f(z)z = 0$, hence we have $zf(z) = 0$. In view of this, (15) reduces to

$$f(x)z^2 + f(z)xz = 0 \quad \forall x \in I, \forall z \in I_8. \tag{16}$$

Now replacing x by xz in last relation, we obtain $f(xz)z^2 + f(z)xz^2 = 0$, then multiplying (16) from right by z and using Lemma 2.4, we arrive

$$f(x)z^3 = f(xz)z^2 \quad \forall x \in I, z \in I_8. \tag{17}$$

Left multiplying (14) by z, where $z \in I_8$, using $zf(z) = 0$ and (13), we get $z^2 f(x) + zxf(z) = 0$. Replace x by xz in this relation and use $zf(z) = 0$ to have

$$z^2 f(xz) = 0 \quad \forall x \in I, z \in I_8. \tag{18}$$

As a special case of (3), we have

$$f(x)yz + yzf(x) + f(yz)x + xf(yz) = 0 \ \ \forall x, y \in I, z \in I_8.$$

Multiplying the last relation from left and right by z^2 simultaneously and using (13), (17) and (18), we get $z^2 f(x)yz^3 + z^2 xf(y)z^3 = 0$. Multiplying this relation from left by z, one can see $tf(x)yt + txf(y)t = 0$ for all $x, y \in I$ and all $t \in I_{24}$. Now replacing y by $ytf(s)$, where $s \in I$ and $t \in I_{24}$, in this relation and using (13), one can arrive $txf(ytf(s))t = 0$ for all $x, y, s \in I$ and $t \in I_{24}$. As S is a prime semiring, we get

$$f(ytf(s))t = 0 \tag{19}$$

Replacing y by $ytf(s)$ in (3), where $s \in I$, we obtain

$$f(x)ytf(s) + ytf(s)f(x) + f(ytf(s))x + xf(ytf(s)) = 0.$$

Multiplying the last equation from left by t, using (13) and (19), we have

$$ytf(s)f(x)t + f(ytf(s))xt = 0 \quad \forall x, y, s \in I, \forall t \in I_{24} \tag{20}$$

Putting ry for y in last relation, where $r \in S$, leads to

$$rytf(s)f(x)t + f(rytf(s))xt = 0.$$

Multiplying (20) from left by r and using Lemma 2.4, we obtain $f(ryf(s))xt = rf(ytf(s))xt$. Again multiplying this from left by z, we obtain $zf(ryf(s))xt = zrf(ytf(s))xt$ for all $x, y, s \in I$, $z \in I_8$, $t \in I_{24}$, $r \in S$. Replace x by zx in this relation and use (13) to get $zrf(ytf(s))zxt = 0$. Due to primeness of S, this becomes $f(ytf(s))zxt = 0$. Again by primeness of S, we get $f(ytf(s))z = 0$. In view of Lemma 2.3, we have

$$f(ytf(s)) = 0. \tag{21}$$

Now suppose $f(s) \neq 0$ for some $s \in I$, otherwise theorem is proved. By Lemma 2.3, $tf(s) \neq 0$ for some $t \in I_{24}$. As $I \neq 0$, therefore for some $x \in I$, $a = xtf(s) \neq 0$. Thus $L = Sa$ is a nonzero left ideal of S contained in I. Hence using (21), we get $f(L) = 0$. Now, using (3), we have $f(x)t + tf(x) = 0$ for all $t \in L$ and $x \in I$. Substituting st for t, where $s \in S$, gives $f(x)st + stf(x) = 0$. Now by replacing s by $x^4 s$ and using (7), we have $x^4 stf(x) = 0$. As S is a prime semiring, so we get $tf(x) = 0$. This implies that $f(x)t = 0$ and hence $f(x) = 0$ for all $x \in I$. This completes the proof. □

Theorem 3.2. *Let S be a 2-torsion free semiprime semiring. If an additive mapping $f : S \to S$ is skew-commuting on S, then $f = 0$.*

Proof. As S is a semiprime semiring, there exits a collection of prime k-ideals τ such that $\cap\tau = 0$. Let $\tau_1 = \{P \in \tau | charS/P \neq 2\}$ and $\tau_2 = \{P \in \tau | charS/P = 2\}$. Let $x \in \cap\tau_1$, then $2x \in (\cap\tau_1) \cap (\cap\tau_2) = \cap\tau = 0$, since S is 2-torsion free, so $x = 0$. Hence $\cap\tau_1 = 0$. The theorem will be complete if we prove $f(x) \in \cap\tau_1$ for all $x \in S$. Take a prime k-ideal $P \in \tau_1$. Linearize $f(x)x + xf(x) = 0$ to get $f(x)y+yf(x)+f(y)x+xf(y) = 0$ for all $x, y \in S$. This implies $f(p)x+xf(p) \in P$ for all $p \in P$, $x \in S$, so we get $xf(p)x+x^2f(p) \in P$ and $f(p)x^2+xf(p)x \in P$. This gives $2xf(p)x + x^2f(p) + f(p)x^2 \in P$. As P is k-ideal and $x^2f(p) + f(p)x^2 \in P$, so we have $2xf(p)x \in P$. As char $S/P \neq 2$, so by Lemma 2.5, we obtain $xf(p)x \in P$ for all $p \in P, x \in S$. Since the k-ideal P is prime, therefore, in view of Lemma 2.6, $f(p) \in P$ for every $p \in P$. Now define a mapping F on S/P by $F(x + P) = f(x)+P$. It can be seen that F is additive and skew-commuting on prime semiring S/P. Hence $F = 0$ by Theorem 3.1. This gives $f(x) \in P$ for all $x \in S$. Hence $f(x) \in \cap\tau_1 = 0$. This completes the proof. □

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