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Characterizations of ordered k-regular semirings by closure operations

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Abstract. We introduce relations on the set of all closure operations on ordered semirings and then we characterize regular ordered semirings and ordered k-regular semirings using these relations.

1. Introduction

In 1936, J. von Neumann [14] called a ring $(S, +, \cdot)$ to be regular if $(S, \cdot)$ is regular. S. Bourne [3] has defined a semiring $(S, +, \cdot)$ to be regular if for all $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$ which is different from Neumann regularity in general but both are equivalent in rings. In 1996, M. R. Adhikari, M. K. Sen and H. J. Weinert [1] have renamed the Bourne regularity of semirings to be a k-regularity.

In 1958, M. Henricksen [6] introduced the notion of k-ideals in a semiring. M. K. Sen and P. Mukhopadhyay [13] showed that k-regular semirings were characterized by k-ideals. A. K. Bhuniya and K. Jana [2] have shown that k-regular semirings and intra k-regular semirings can be characterized by k-bi-ideals where these semirings are additive semilattices. Subsequently, K. Jana [7, 8] continued to consider additive semilattice semirings and investigated some properties of quasi k-ideals in k-regular semirings and intra k-regular semirings, k-bi-ideals and quasi k-ideals in k-Clifford semirings. For more information about k-regular semirings and k-ideals in semirings, the reader may refer e.g., [2, 7, 8, 11].

A. P. Gan and Y. L. Jiang [5] investigated some properties of ordered ideals in ordered semirings. S. Patchakhieo and B. Pibaljommee [11] introduced the notions of an ordered k-regular semiring and an ordered k-ideal in an ordered semiring and characterized ordered k-regular semirings by their ordered k-ideals.

In 1970, B. Pondělíček [12] investigated a relation on the set of all closure operations on a semigroup and characterized a regular semigroup by this relation. After that T. Changphas [4] generalized Pondělíček's relation to an ordered semigroup.

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In this paper, we investigate a relationship between ordered semirings and closure operations on the ordered semirings. Moreover, we introduce relations on the set of all closure operations on ordered semirings and characterize regular ordered semirings and ordered k-regular semirings using these relations.

2. Preliminaries

In this section, we recall notions of an ordered semiring, an ordered ideal in an ordered semiring and notions of closure operations.

Let S be a nonempty set and $+$ and $\cdot$ be binary operations on S, named addition and multiplication, respectively. Then $(S,+,\cdot)$ is called a *semiring* if the following conditions are satisfied:

1. $(S,+)$ is a commutative semigroup;
2. $(S,\cdot)$ is a semigroup;
3. both operations are connected by the distributive laws, namely, $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$ for all $a,b,c \in S$.

A semiring $(S,+,\cdot)$ is said to have a *zero element* if there exists an element $0 \in S$ such that $0+x = x = x+0$ and $0\cdot x = 0 = x\cdot 0$ for all $x \in S$. In particular, a semiring $(S,+,\cdot)$ is called *commutative* if $(S,\cdot)$ is a commutative semigroup, and called a *ring* if $(S,+)$ is a commutative group.

Definition 2.1. Let $(S,+,\cdot)$ be a semiring and $\emptyset \neq A \subseteq S$. Then A is called a *left* (*right*) *ideal* if the following conditions are satisfied:

1. $x+y \in A$ for all $x,y \in A$;
2. $SA \subseteq A$ $(AS \subseteq A)$.

We call A an *ideal* if it is both left ideal and right ideal of S.

Definition 2.2. Let $(S,\leqslant)$ be a partially ordered set. Then $(S,+,\cdot,\leqslant)$ is called an *ordered semiring* if the following conditions are satisfied:

1. $(S,+,\cdot)$ is a semiring;
2. if $a \leqslant b$ then $a+x \leqslant b+x$ and $x+a \leqslant x+b$;
3. if $a \leqslant b$ then $ax \leqslant bx$ and $xa \leqslant xb$

for all $a,b,x \in S$.

Instead of writing an ordered semiring $(S,+,\cdot,\leqslant)$, we denote S, for short, as an ordered semiring. Let A be a nonempty subset of S. We define

$$(A] = \{x \in S \mid x \leqslant a, \exists a \in A\}.$$

Definition 2.3. Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. Then A is called a *left* (*right*) *ordered ideal* if the following conditions are satisfied:

1. A is a left (right) ideal of S;
2. if $x \leqslant a$ for some $a \in A$ then $x \in A$.

We call A an *ordered ideal* if it is both left ordered ideal and right ordered ideal of S.

It is known, a result in [5], that if A is a left (right, two-sided) ideal of an ordered semiring S then $(A]$ is the smallest left ordered ideal (right ordered ideal, two-sided ordered ideal) containing A.

Now we recall the notion of a C-closure operation and some of its properties proved in [12].

Let S be a nonempty set and $Sub(S)$ be the set of all subsets of S. A mapping $\mathbf{U} : Sub(S) \to Sub(S)$ is called a *C-closure operation* on S if

1. $\mathbf{U}(\emptyset) = \emptyset$;
2. $A \subseteq B \Rightarrow \mathbf{U}(A) \subseteq \mathbf{U}(B)$;
3. $A \subseteq \mathbf{U}(A)$;
4. $\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A)$

for all $A, B \in Sub(S)$.

Let $x \in S$. We define $\mathbf{U}(x) = \mathbf{U}(\{x\})$. We denote by

$$\mathcal{F}(\mathbf{U}) = \{A \subseteq S \mid \mathbf{U}(A) = A\}$$

the set of all closed sets with respect to the operation $\mathbf{U}$ and by $\mathcal{C}(S)$ the set of all C-closure operations on S. Define a relation $\leqslant$ on $\mathcal{C}(S)$ by

$$\mathbf{U} \leqslant \mathbf{V} \iff \mathbf{U}(A) \subseteq \mathbf{V}(A) \text{ for all } A \in Sub(S).$$

We define a C-closure operation $\mathbf{I}$ on S by

$$\mathbf{I}(A) = \begin{cases} S, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and for any $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$ we denote by $\mathbf{U} \wedge \mathbf{V}$ and $\mathbf{U} \vee \mathbf{V}$ the infimum and the supremum, respectively, of $\mathbf{U}$ and $\mathbf{V}$ in $\mathcal{C}(S)$. It is known that for any $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

1. $\mathbf{U} \leqslant \mathbf{I}$,
2. $\mathbf{U} \leqslant \mathbf{V} \iff \mathcal{F}(\mathbf{V}) \subseteq \mathcal{F}(\mathbf{U})$,
3. $\mathbf{U} \vee \mathbf{V}, \mathbf{U} \wedge \mathbf{V}$ exist and
   (a) $\mathcal{F}(\mathbf{U} \vee \mathbf{V}) = \mathcal{F}(\mathbf{U}) \cap \mathcal{F}(\mathbf{V})$,
   (b) $\mathcal{F}(\mathbf{U} \wedge \mathbf{V}) = \{A \cap B \mid A \in \mathcal{F}(\mathbf{U}), B \in \mathcal{F}(\mathbf{V})\}$.

3. Regular ordered semirings

In this section, we define a relation on the set of all closure operations on an ordered semiring and characterizes a regular ordered semiring by the relation.

Let S be an ordered semiring and $\emptyset \neq A \subseteq S$. We denote by $\Sigma_{fin}A$ the set of all finite sums of elements of A. We define a relation ρ on $\mathcal{C}(S)$ by letting $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

$$\mathbf{U}\rho\mathbf{V} \iff A \cap B = (\Sigma_{fin}AB]$$

for all nonempty set $A \in \mathcal{F}(\mathbf{U})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{V})$.

The following lemma is easy to prove using the definition of ρ.

Lemma 3.1. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}' \in \mathcal{C}(S)$ such that $\mathbf{U}\rho\mathbf{V}$. If $\mathbf{U} \leqslant \mathbf{U}'$ and $\mathbf{V} \leqslant \mathbf{V}'$ then $\mathbf{U}'\rho\mathbf{V}'$.*

Let S be an ordered semiring. Then we define mappings $\mathbf{L}$ and $\mathbf{R}$ on $Sub(S)$ by letting $A \subseteq S$,

$$\mathbf{L}(A) = \begin{cases} (\Sigma_{fin}A + \Sigma_{fin}SA], & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and

$$\mathbf{R}(A) = \begin{cases} (\Sigma_{fin}A + \Sigma_{fin}AS], & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset. \end{cases}$$

It is easy to show that $\mathbf{L}$ and $\mathbf{R}$ are closure operations on $Sub(S)$.

Now, we show that $\mathcal{F}(\mathbf{L})$ is the set of all left ordered ideals of S (including the empty set). Let A is a left ordered ideal of S. Then we obtain $A \subseteq \mathbf{L}(A) = (\Sigma_{fin}A + \Sigma_{fin}SA] \subseteq (A] = A$. Hence, $A \in \mathcal{F}(\mathbf{L})$. Conversely, let $\emptyset \neq A \in \mathcal{F}(\mathbf{L})$. Then $A = \mathbf{L}(A) = (\Sigma_{fin}A + \Sigma_{fin}SA]$. Hence, A is a left ordered ideal of S. Similarly, we have $\mathcal{F}(\mathbf{R})$ is the set of all right ordered ideals of S (including the empty set).

The following lemma can be proved straightforward.

Lemma 3.2. *Let S be an ordered semiring and A be a nonempty subset of S. Then $\Sigma_{fin}(AS] \subseteq (\Sigma_{fin}AS] = \Sigma_{fin}(\Sigma_{fin}AS]$ and $\Sigma_{fin}(SA] \subseteq (\Sigma_{fin}SA] = \Sigma_{fin}(\Sigma_{fin}SA]$.*

Theorem 3.3. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$. Then $\mathbf{U}\rho\mathbf{V}$ if and only if $\mathbf{R} \leqslant \mathbf{U}, \mathbf{L} \leqslant \mathbf{V}$ and $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$ for all $x \in S$.*

Proof. ($\Rightarrow$). Assume that $\mathbf{U}\rho\mathbf{V}$. First, we show that $\mathbf{R} \leqslant \mathbf{U}$. Let $A \in \mathcal{F}(\mathbf{U})$. It is clear that $S \in \mathcal{F}(\mathbf{V})$. By assumption, we have $A = A \cap S = (\Sigma_{fin}AS]$. By Lemma 3.2, we have $A \subseteq \mathbf{R}(A) = (\Sigma_{fin}A + \Sigma_{fin}AS] = (\Sigma_{fin}(\Sigma_{fin}AS] + \Sigma_{fin}AS] = ((\Sigma_{fin}AS] + \Sigma_{fin}AS] \subseteq ((\Sigma_{fin}AS]] = (\Sigma_{fin}AS] = A$. Hence, $\mathbf{R}(A) = A$. Thus, $A \in \mathcal{F}(\mathbf{R})$. It follows that $\mathbf{R} \leqslant \mathbf{U}$. Similarly, $\mathbf{L} \leqslant \mathbf{V}$. Let $x \in S$. Since $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$, we obtain $x \in \mathbf{U}(x) \cap \mathbf{V}(x)$. Since $\mathbf{U}(x) \in \mathcal{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathcal{F}(\mathbf{V})$, we obtain $\mathbf{U}(x) \cap \mathbf{V}(x) = (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$. Thus, $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$.

($\Leftarrow$). Assume that $\mathbf{R} \leqslant \mathbf{U}, \mathbf{L} \leqslant \mathbf{V}$ and $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]$ for all $x \in S$. We show that $\mathbf{U}\rho\mathbf{V}$. Let $A \in \mathcal{F}(\mathbf{U})\setminus\{\emptyset\}$ and $B \in \mathcal{F}(\mathbf{V})\setminus\{\emptyset\}$. By assumption, we have $A \in \mathcal{F}(\mathbf{R})$ and $B \in \mathcal{F}(\mathbf{L})$. Hence, $(\Sigma_{fin}AB] \subseteq (\Sigma_{fin}AS] \subseteq (\Sigma_{fin}A] \subseteq (A] = A$ and $(\Sigma_{fin}AB] \subseteq (\Sigma_{fin}SB] \subseteq (\Sigma_{fin}B] \subseteq (B] = B$. Thus, $(\Sigma_{fin}AB] \subseteq A \cap B$. Let $x \in A \cap B$. Then $\mathbf{U}(x) \subseteq \mathbf{U}(A) = A$ and $\mathbf{V}(x) \subseteq \mathbf{V}(B) = B$. By assumption, we have $x \in (\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)] \subseteq (\Sigma_{fin}AB]$. Hence, $A \cap B \subseteq (\Sigma_{fin}AB]$. Thus, $A \cap B = (\Sigma_{fin}AB]$. Therefore, $\mathbf{U}\rho\mathbf{V}$. □

As the notion of a regular ordered semigroup [9, 10], we define a notion of a regular ordered semiring as follows. An ordered semiring S is called *left* (*right*) *regular* if $a \in (Sa^2]$($a \in (a^2S]$) for all $a \in S$ and called *regular* if $a \in (aSa]$ for all $a \in S$. Similar to a result in ordered semigroups, we obtain the following theorem.

Theorem 3.4. *An ordered semiring S is ordered regular if and only if $A \cap B = (AB]$ for all right ordered ideal A and for all left ordered ideal B of S.*

Theorem 3.5. *An ordered semiring S is regular if and only if $\mathbf{R}\rho\mathbf{L}$.*

Proof. ($\Rightarrow$). Assume that S is regular. Let $a \in S$. By assumption, we have $a \in (aSa] \subseteq (\mathbf{R}(a)S\mathbf{L}(a)] \subseteq (\mathbf{R}(a)\mathbf{L}(a)] \subseteq (\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)]$. By Theorem 3.3, we obtain $\mathbf{R}\rho\mathbf{L}$.

($\Leftarrow$). Assume that $\mathbf{R}\rho\mathbf{L}$. Let $a \in S$. By Theorem 3.3, $a \in (\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)]$. Since $(\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)] \subseteq (aS]$ and $(\Sigma_{fin}\mathbf{R}(a)\mathbf{L}(a)] \subseteq (Sa]$, we get $a \in (aS] \cap (Sa]$. Since $(aS] \in \mathcal{F}(\mathbf{R}), (Sa] \in \mathcal{F}(\mathbf{L})$ and $\mathbf{R}\rho\mathbf{L}$, we obtain $a \in (\Sigma_{fin}(aS](Sa]]$. Then there exist $x_1, x_2, \ldots, x_n \in (aS]$ and $y_1, y_2, \ldots, y_n \in (Sa]$ for some $n \in \mathbb{N}$ such that $x \leqslant x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Since $x_i \in (aS]$ and $y_i \in (Sa]$ for all $i = 1, 2, \ldots, n$, there exist $s_i, r_i \in S$ such that $x_i \leqslant as_i$ and $y_i \leqslant r_ia$ for all $i = 1, 2, \ldots, n$. Hence, $x_iy_i \leqslant as_ir_ia$ for all $i = 1, 2, \ldots, n$. It follows that $a \leqslant as_1r_1a + as_2r_2a + \cdots + as_nr_na = a(s_1r_1 + s_2r_2 + \cdots + s_nr_n)a \in aSa$. Thus, $a \in (aSa]$. Therefore, S is regular. □

As a consequence of Theorem 3.4 and Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let S be an ordered semiring. Then $\mathbf{R}\rho\mathbf{L}$ if and only if $A \cap B = (AB]$ for all nonempty set $A \in \mathcal{F}(\mathbf{R})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{L})$.*

Theorem 3.7. *Let S be a commutative ordered semiring, A be a nonempty subset of S and $\mathbf{R}\rho\mathbf{L}$. Then A is an ordered ideal of S if and only if there exist $H \in \mathcal{F}(\mathbf{R})$ and $K \in \mathcal{F}(\mathbf{L})$ such that $A = (HK]$.*

Let S be an ordered semiring. We denote the C-closure operation $\mathbf{R} \vee \mathbf{L}$ on S by $\mathbf{M}$. Note that $\mathcal{F}(\mathbf{M})$ is the set of all ordered ideals of S (including empty set).

Theorem 3.8. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{L}\rho\mathbf{L}$;

(ii) $\mathbf{L}\rho\mathbf{M}$;

(iii) *S is left regular and* $\mathbf{R} \leqslant \mathbf{L}$.

Proof. $(i) \Rightarrow (ii)$. Since $\mathbf{L}\rho\mathbf{L}$ and by Lemma 3.1, we obtain $\mathbf{L}\rho\mathbf{M}$.

$(ii) \Rightarrow (iii)$. Assume that $\mathbf{L}\rho\mathbf{M}$. By Theorem 3.3, we have $\mathbf{R} \leqslant \mathbf{L}$. It follows that $\mathbf{L} = \mathbf{M}$. For any $x \in S$, we get

$$\begin{aligned} x \in (\Sigma_{fin}\mathbf{L}(x)\mathbf{M}(x)] &= (\Sigma_{fin}\mathbf{L}(x)\mathbf{L}(x)] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}xSx + \Sigma_{fin}SxSx] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{R}(x)x + \Sigma_{fin}S\mathbf{R}(x)x] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{L}(x)x + \Sigma_{fin}S\mathbf{L}(x)x] \\ &\subseteq (\mathbb{N}x^2 + \Sigma_{fin}Sx^2] \\ &= (\mathbb{N}x^2 + Sx^2]. \end{aligned}$$

Then there exist $k_1 \in \mathbb{N}, s \in S$ such that

$$x \leqslant k_1x^2 + sx^2. \tag{1}$$

Similarly, there exist $k_2 \in \mathbb{N}, r \in S$ such that $x^2 \leqslant k_2x^4 + rx^4$. Hence, $k_1x^2 \leqslant k_1k_2x^4 + k_1rx^4$. From (1), we have $x \leqslant k_1k_2x^4 + k_1rx^4 + sx^2$. This implies $x \in (Sx^2]$. Therefore, S is left regular.

$(iii) \Rightarrow (i)$. Assume that S is left regular and $\mathbf{R} \leqslant \mathbf{L}$. Then for any $x \in S$, we get $x \in (Sx^2] \subseteq (Sx\mathbf{L}(x)] \subseteq (\mathbf{L}(x)\mathbf{L}(x)] \subseteq (\Sigma_{fin}\mathbf{L}(x)\mathbf{L}(x)]$. By Theorem 3.3, it turns out $\mathbf{L}\rho\mathbf{L}$. □

Theorem 3.9. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{R}\rho\mathbf{R}$;

(ii) $\mathbf{M}\rho\mathbf{R}$;

(iii) *S is right regular and* $\mathbf{L} \leqslant \mathbf{R}$.

Proof. The proof of this theorem is similar to Theorem 3.8. □

An ordered semiring S is called *left simple* (*right simple, simple*) if S has no proper left (right, two-sided) ordered ideal.

Now we give characterizations of left simple, right simple and simple as the following theorem which is easy to verify.

Theorem 3.10. *Let S be an ordered semiring. Then*

(i) *S is left simple if and only if* $\mathbf{L} = \mathbf{I}$*;*

(ii) *S is right simple if and only if* $\mathbf{R} = \mathbf{I}$*;*

(iii) *S is simple if and only if* $\mathbf{M} = \mathbf{I}$.

4. Ordered k-regular semirings

In this section, we define a relation on the set of all closure operations on an ordered semiring S and characterizes an ordered k-regular semiring by the relation.

The *k-closure* of a nonempty subset A of an ordered semiring S is defined by

$$\overline{A} = \{x \in S \mid \exists a, b \in A,\ x + a \leqslant b\}.$$

Lemma 4.1. [11] *Let S be an ordered semiring and A be a nonempty subset of S. If $A + A \subseteq A$ then $A \subseteq \overline{(A]} = \overline{\overline{(A]}}$.*

Let A be a nonempty subset of S. We note that if A is closed under addition then $\overline{(A]}$ is also closed.

Definition 4.2. [11] A left (right, two-sided) ordered ideal A of an ordered semiring S is called a *left ordered k-ideal* (*right ordered k-ideal, ordered k-ideal*) if $\overline{A} = A$.

In [11], it is known that if A is a left (right, two-sided) ideal of S then $\overline{(A]}$ is the smallest left ordered k-ideal (right ordered k-ideal, ordered k-ideal) containing A.

Definition 4.3. [11] An ordered semiring S is called *left* (*right*) *ordered k-regular* if $a \in \overline{(Sa^2]}(a \in \overline{(a^2S]})$ for all $a \in S$ and called *ordered k-regular* if $a \in \overline{(aSa]}$ for all $a \in S$.

Theorem 4.4. [11] *An ordered semiring S is ordered k-regular if and only if $A \cap B = \overline{(AB]}$ for all right ordered k-ideal A and for all left ordered k-ideal B of S.*

Let S be an ordered semiring. We define a relation β on $\mathcal{C}(S)$ by letting $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$,

$$\mathbf{U}\beta\mathbf{V} \iff A \cap B = \overline{(\Sigma_{fin} AB]}$$

for all nonempty set $A \in \mathcal{F}(\mathbf{U})$ and for all nonempty set $B \in \mathcal{F}(\mathbf{V})$.

By the definition of β, we have the following lemma.

Lemma 4.5. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}' \in \mathcal{C}(S)$ such that $\mathbf{U}\beta\mathbf{V}$. If $\mathbf{U} \leqslant \mathbf{U}'$ and $\mathbf{V} \leqslant \mathbf{V}'$ then $\mathbf{U}'\beta\mathbf{V}'$.*

Lemma 4.6. [11] *Let S be an ordered semiring and A be a nonempty subset of S. Then*

(*i*) $\overline{(\Sigma_{fin} A + \Sigma_{fin} SA]}$ *is the smallest left ordered k-ideal of S containing A;*

(*ii*) $\overline{(\Sigma_{fin} A + \Sigma_{fin} AS]}$ *is the smallest right ordered k-ideal of S containing A.*

Let $(S, +, \cdot, \leqslant)$ be an ordered semiring. Then we define mappings $\mathbf{L}_k$ and $\mathbf{R}_k$ on $Sub(S)$ by letting $A \subseteq S$,

$$\mathbf{L}_k(A) = \begin{cases} \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]}, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset, \end{cases}$$

and

$$\mathbf{R}_k(A) = \begin{cases} \overline{(\Sigma_{fin}A + \Sigma_{fin}AS]}, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset. \end{cases}$$

It is easy to show that $\mathbf{L}_k$ and $\mathbf{R}_k$ are closure operations on $Sub(S)$ and if $A \neq \emptyset$ then $\mathbf{L}_k(A)$ and $\mathbf{R}_k(A)$ are left ordered k-ideal and right ordered k-ideal of S, respectively.

Now, we show that $\mathcal{F}(\mathbf{L}_k)$ is the set of all left ordered k-ideals of S (including the empty set). Let A be a left ordered k-ideal of S. Then we obtain $A \subseteq \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} \subseteq \overline{(A]} = A$. Hence, $A \in \mathcal{F}(\mathbf{L}_k)$. Conversely, let $A \in \mathcal{F}(\mathbf{L}_k) \setminus \{\emptyset\}$. By Lemma 4.6, we get $A = \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]}$ is a left ordered k-ideal of S. Similarly, we have $\mathcal{F}(\mathbf{R}_k)$ is the set of all right ordered k-ideals of S (including the empty set).

Lemma 4.7. *Let S be an ordered semiring and A be a nonempty subset of S. Then $\Sigma_{fin}\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]} = \Sigma_{fin}\overline{(\Sigma_{fin}AS]}$ and $\Sigma_{fin}\overline{(SA]} \subseteq \overline{(\Sigma_{fin}SA]} = \Sigma_{fin}\overline{(\Sigma_{fin}SA]}$.*

Proof. Since $\Sigma_{fin}AS$ is closed under addition, then $\overline{(\Sigma_{fin}AS]}$ is also closed. Since $\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]}$ and $\overline{(\Sigma_{fin}AS]}$ is closed under addition, we have $\Sigma_{fin}\overline{(AS]} \subseteq \overline{(\Sigma_{fin}AS]}$ and $\Sigma_{fin}\overline{(\Sigma_{fin}AS]} \subseteq \overline{(\Sigma_{fin}AS]}$. Hence, $\overline{(\Sigma_{fin}AS]} = \Sigma_{fin}\overline{(\Sigma_{fin}AS]}$. Similarly, we have $\Sigma_{fin}\overline{(SA]} \subseteq \overline{(\Sigma_{fin}SA]} = \Sigma_{fin}\overline{(\Sigma_{fin}SA]}$. $\square$

For any element a of an ordered semiring S, $\mathbb{N}a$ means $\{na \mid n \in \mathbb{N}\}$.

As a consequence of definitions of $\mathbf{L}_k$ and $\mathbf{R}_k$, we have the following lemma.

Lemma 4.8. *Let S be an ordered semiring and $a \in S$. Then $\mathbf{L}_k(a) = \overline{(\mathbb{N}a + Sa]}$ and $\mathbf{R}_k(a) = \overline{(\mathbb{N}a + aS]}$.*

Theorem 4.9. *Let S be an ordered semiring and $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$. Then $\mathbf{U}\beta\mathbf{V}$ if and only if $\mathbf{R}_k \leqslant \mathbf{U}$, $\mathbf{L}_k \leqslant \mathbf{V}$ and $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$.*

Proof. ($\Rightarrow$). Assume that $\mathbf{U}\beta\mathbf{V}$. We first show that $\mathbf{R}_k \leqslant \mathbf{U}$. Let $A \in \mathcal{F}(\mathbf{U}) \setminus \{\emptyset\}$. It is clear that $S \in \mathcal{F}(\mathbf{V})$. By assumption, $A = A \cap S = \overline{(\Sigma_{fin}AS]}$. By Lemma 4.7, we have $A \subseteq \mathbf{R}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}AS]} = \overline{(\Sigma_{fin}\overline{(\Sigma_{fin}AS]} + \Sigma_{fin}AS]} = \overline{(\overline{(\Sigma_{fin}AS]} + \Sigma_{fin}AS]} \subseteq \overline{(\overline{(\Sigma_{fin}AS]}]} = \overline{(\Sigma_{fin}AS]} = A$. Hence, $\mathbf{R}_k(A) = A$. Thus, $A \in \mathcal{F}(\mathbf{R}_k)$. It follows that $\mathbf{R}_k \leqslant \mathbf{U}$. Similarly, $\mathbf{L}_k \leqslant \mathbf{V}$. Since $\mathbf{U}, \mathbf{V} \in \mathcal{C}(S)$, we obtain $x \in \mathbf{U}(x) \cap \mathbf{V}(x)$ for all $x \in S$. Since $\mathbf{U}(x) \in \mathcal{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathcal{F}(\mathbf{V})$, we obtain $\mathbf{U}(x) \cap \mathbf{V}(x) = \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$. Thus, $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$.

($\Leftarrow$). Assume that $\mathbf{R}_k \leqslant \mathbf{U}, \mathbf{L}_k \leqslant \mathbf{V}$ and $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]}$ for all $x \in S$. We show that $\mathbf{U}\beta\mathbf{V}$. Let $A \in \mathcal{F}(\mathbf{U}) \setminus \{\emptyset\}$ and $B \in \mathcal{F}(\mathbf{V}) \setminus \{\emptyset\}$. By assumption, $A \in \mathcal{F}(\mathbf{R}_k)$ and $B \in \mathcal{F}(\mathbf{L}_k)$. We obtain $\overline{(\Sigma_{fin}AB]} \subseteq \overline{(\Sigma_{fin}AS]} \subseteq \overline{(\Sigma_{fin}A]} \subseteq \overline{(A]} = A$ and $\overline{(\Sigma_{fin}AB]} \subseteq \overline{(\Sigma_{fin}SB]} \subseteq \overline{(\Sigma_{fin}B]} \subseteq \overline{(B]} = B$. Hence, $\overline{(\Sigma_{fin}AB]} \subseteq A \cap B$. Let $x \in A \cap B$. Then $\mathbf{U}(x) \subseteq A$ and $\mathbf{V}(x) \subseteq B$. By assumption, we have $x \in \overline{(\Sigma_{fin}\mathbf{U}(x)\mathbf{V}(x)]} \subseteq \overline{(\Sigma_{fin}AB]}$. Hence, $A \cap B \subseteq \overline{(\Sigma_{fin}AB]}$. Thus, $A \cap B = \overline{(\Sigma_{fin}AB]}$. Therefore, $\mathbf{U}\beta\mathbf{V}$. □

The following theorem gives a characterization of an ordered k-regular semiring by closure operations.

Theorem 4.10. *An ordered semiring S is ordered k-regular if and only if $\mathbf{R}_k\beta\mathbf{L}_k$.*

Proof. ($\Rightarrow$). Assume that S is ordered k-regular. Let $a \in S$. Then we have $a \in \overline{(aSa]} \subseteq \overline{(\mathbf{R}_k(a)S\mathbf{L}_k(a)]} \subseteq \overline{(\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]}$. By Theorem 4.9, we obtain $\mathbf{R}_k\beta\mathbf{L}_k$.

($\Leftarrow$). Assume that $\mathbf{R}_k\beta\mathbf{L}_k$. Let $a \in S$. Then $a \in \overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]}$ by Theorem 4.9. Since $\overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(aS]}$ and $\overline{(\Sigma_{fin}\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(Sa]}$, we get $a \in \overline{(aS]} \cap \overline{(Sa]}$. Since $\overline{(aS]} \in \mathcal{F}(\mathbf{R}_k), \overline{(Sa]} \in \mathcal{F}(\mathbf{L}_k)$ and $\mathbf{R}_k\beta\mathbf{L}_k$, we obtain $a \in \overline{(\Sigma_{fin}\overline{(aS]}\;\overline{(Sa]}]}$. There exist $x, x' \in (\Sigma_{fin}\overline{(aS]}\;\overline{(Sa]}]$ such that $a + x \leqslant x'$. But $x, x' \in (\Sigma_{fin}\overline{(aS]}\;\overline{(Sa]}]$, so there exist $x_1, x_2, \ldots, x_n, x'_1, \ldots, x'_m \in \overline{(aS]}$, $y_1, y_2, \ldots, y_n, y'_1, \ldots, y'_m \in \overline{(Sa]}$ such that $x \leqslant \Sigma_{i=1}^{n} x_i y_i$ and $x' \leqslant \Sigma_{j=1}^{m} x'_j y'_j$. For each $1 \leqslant i \leqslant n$, we get

$$x_i + u_i \leqslant u'_i, \tag{2}$$

$$y_i + v_i \leqslant v'_i, \tag{3}$$

where $u_i \leqslant as_i, u'_i \leqslant as'_i, v_i \leqslant t_i a, v'_i \leqslant t'_i a$ for some $s_i, s'_i, t_i, t'_i \in S$. From (2), we have $x_i y_i + u_i y_i \leqslant u'_i y_i$. From (3), we have $u_i y_i + u_i v_i \leqslant u_i v'_i$ and $u'_i y_i + u'_i v_i \leqslant u'_i v'_i$. Hence, $x_i y_i + u_i y_i + u_i v_i + u'_i v_i \leqslant u'_i y_i + u_i v_i + u'_i v_i$. Then we get $u_i y_i + u_i v_i + u'_i v_i \leqslant u_i v'_i + u'_i v_i \leqslant as_i t'_i a + as'_i t_i a = a(s_i t'_i + s'_i t_i)a \in aSa$ and $u'_i y_i + u_i v_i + u'_i v_i \leqslant u_i v_i + u'_i v'_i \leqslant as_i t_i a + as'_i t'_i a = a(s_i t_i + s'_i t'_i)a \in aSa$. It follows that $x_i y_i \in \overline{(aSa]}$. Hence, $\Sigma_{i=1}^{n} x_i y_i \in \overline{(aSa]}$. Similarly, we obtain $\Sigma_{j=1}^{m} x'_j y'_j \in \overline{(aSa]}$. Since $x \leqslant \Sigma_{i=1}^{n} x_i y_i$ and $x' \leqslant \Sigma_{j=1}^{m} x'_j y'_j$, we have $x, x' \in (\overline{(aSa]}] = \overline{(aSa]}$. Then there exist $c, c'd, d' \in (aSa]$ such that $x + c \leqslant d$ and $x' + c' \leqslant d'$. It follows that $a + x + c + c' \leqslant x' + c + c' \leqslant c + d' \in (aSa]$ and $x + c + c' \leqslant d + c' \in (aSa]$. Thus, $a \in \overline{(aSa]}$. Therefore, S is ordered k-regular. □

By Theorem 4.4 and Theorem 4.10, we have the following result.

Corollary 4.11. *Let S be an ordered semiring. Then $\mathbf{R}_k\beta\mathbf{L}_k$ if and only if $A \cap B = \overline{(AB]}$ for all nonempty set $A \in \mathcal{F}(\mathbf{R}_k)$ and for all nonempty set $B \in \mathcal{F}(\mathbf{L}_k)$.*

Example 4.12. Let $S = \{a, b, c\}$ with a partially ordered set $\leqslant$ be defined $a \leqslant b \leqslant c$. Define binary operations $+$ and $\cdot$ on S by the following tables.

$+$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

and

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

Then we have $(S, +, \cdot, \leqslant)$ is an ordered semiring. Moreover, $\overline{(\Sigma_{fin}\mathbf{R}(x)\mathbf{L}(x)]} = S$ for every $x \in S$. It follows that $x \in \overline{(\Sigma_{fin}\mathbf{R}(x)\mathbf{L}(x)]}$ for every $x \in S$. By Theorem 4.9 and Theorem 4.10, we obtain that S is an ordered k-regular semiring.

Theorem 4.13. *Let S be a commutative ordered semiring, A be a nonempty subset of S and $\mathbf{R}_k\beta\mathbf{L}_k$. Then A is an ordered k-ideal of S if and only if there exist $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$ such that $A = \overline{(HK]}$.*

Proof. $(\Rightarrow)$. Assume that A is an ordered k-ideal of S. Let $H = \mathbf{R}_k(A)$ and $K = \mathbf{L}_k(A)$. Then we have $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$. Since S is a commutative ordered semiring, $H = A = K$. Let $a \in A$. Since $\mathbf{R}_k\beta\mathbf{L}_k, a \in \overline{(\mathbf{R}_k(a)\mathbf{L}_k(a)]} \subseteq \overline{(\mathbf{R}_k(A)\mathbf{L}_k(A)]} = \overline{(HK]}$. Hence, $A \subseteq \overline{(HK]}$. Since $A^2 \subseteq A, \overline{(HK]} = \overline{(A^2]} \subseteq \overline{(A]} = A$. Therefore, $A = \overline{HK}$.

$(\Leftarrow)$. Assume that there exist $H \in \mathcal{F}(\mathbf{R}_k)$ and $K \in \mathcal{F}(\mathbf{L}_k)$ such that $A = \overline{(HK]}$. Since $\mathbf{R}_k\beta\mathbf{L}_k$, we have $H \cap K = \overline{(HK]}$. Since $A = \overline{(HK]} = H \cap K$ and S is commutative, A is an ordered ideal. Since $\overline{A} = \overline{\overline{(HK]}} = \overline{(HK]} = A$, A is an ordered k-ideal. $\square$

Let S be an ordered semiring. We denote the C-closure operation $\mathbf{R}_k \vee \mathbf{L}_k$ on S by $\mathbf{M}_k$. Note that $\mathcal{F}(\mathbf{M}_k)$ is the set of all ordered k-ideals of S (including empty set).

Theorem 4.14. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{L}_k\beta\mathbf{L}_k$;

(ii) $\mathbf{L}_k\beta\mathbf{M}_k$;

(iii) *S is left ordered k-regular and $\mathbf{R}_k \leqslant \mathbf{L}_k$.*

Proof. $(i) \Rightarrow (ii)$. Since $\mathbf{L}_k\beta\mathbf{L}_k$ and by Lemma 4.5, we obtain $\mathbf{L}_k\beta\mathbf{M}_k$.

$(ii) \Rightarrow (iii)$. Assume that $\mathbf{L}_k\beta\mathbf{M}_k$. By Theorem 4.9, we have $\mathbf{R}_k \leqslant \mathbf{L}_k$. It follows that $\mathbf{L}_k = \mathbf{M}_k$. For any $x \in S$, we get

$$
\begin{aligned}
x \in \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{M}_k(x)]} &= \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{L}_k(x)]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}xSx + \Sigma_{fin}SxSx]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{R}_k(x)x + \Sigma_{fin}S\mathbf{R}_k(x)x]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2 + \Sigma_{fin}\mathbf{L}_k(x)x + \Sigma_{fin}S\mathbf{L}_k(x)x]} \\
&\subseteq \overline{(\mathbb{N}x^2 + \Sigma_{fin}Sx^2]} \\
&= \overline{(\mathbb{N}x^2 + Sx^2]}.
\end{aligned}
$$

Then there exist $y, z \in (\mathbb{N}x^2 + Sx^2]$ such that $x + y \leqslant z$. It follows that there exist $k_1, k_2 \in \mathbb{N}, s, t \in S$ such that $y \leqslant k_1x^2 + sx^2, z \leqslant k_2x^2 + tx^2$. Similarly, there exist $u, v \in (\mathbb{N}x^4 + Sx^4]$ such that $x^2 + u \leqslant v$. It follows that there exist $k_3, k_4 \in \mathbb{N}, q, r \in S$ such that $u \leqslant k_3x^4 + qx^4, v \leqslant k_4x^4 + rx^4$. Since $x^2 + u \leqslant v$, we obtain $k_1x^2 + k_1u \leqslant k_1v$ and $k_2x^2 + k_2u \leqslant k_2v$. Hence, $y + k_1u \leqslant k_1x^2 + sx^2 + k_1u \leqslant k_1v + sx^2 \leqslant k_1k_4x^4 + k_1rx^4 + sx^2$ and $z + k_2u \leqslant k_2x^2 + tx^2 + k_2u \leqslant k_2v + tx^2 \leqslant k_2k_4x^4 + k_2rx^4 + tx^2$. It turns out $y + k_1u + k_2u \leqslant k_1k_4x^4 + k_1rx^4 + sx^2 + k_2k_3x^4 + k_2qx^4 \in Sx^2$ and $z + k_1u + k_2u \leqslant k_2k_4x^4 + k_2rx^4 + tx^2 + k_1k_3x^4 + k_1qx^4 \in Sx^2$. Since $x + y \leqslant z$, we get $x + y + k_1u + k_2u \leqslant z + k_1u + k_2u$. This implies that $x \in \overline{(Sx^2]}$. Therefore, S is left ordered k-regular.

$(iii) \Rightarrow (i)$. Assume that S is left ordered k-regular and $\mathbf{R}_k \leqslant \mathbf{L}_k$. Then $x \in \overline{(Sx^2]} \subseteq \overline{(Sx\mathbf{L}_k(x)]} \subseteq \overline{(\mathbf{L}_k(x)\mathbf{L}_k(x)]} \subseteq \overline{(\Sigma_{fin}\mathbf{L}_k(x)\mathbf{L}_k(x)]}$ for all $x \in S$. By Theorem 4.9, it turns out $\mathbf{L}_k\beta\mathbf{L}_k$. □

Theorem 4.15. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) $\mathbf{R}_k\beta\mathbf{R}_k$;

(ii) $\mathbf{M}_k\beta\mathbf{R}_k$;

(iii) *S is right ordered k-regular and* $\mathbf{L}_k \leqslant \mathbf{R}_k$.

Proof. The proof of this theorem is similar to Theorem 4.14. □

An ordered semiring S is called *left k-simple* (*right k-simple, k-simple*) if S has no proper left (right, two-sided) ordered k-ideal.

Theorem 4.16. *Let S be an ordered semiring. Then*

(i) *S is left k-simple if and only if* $\mathbf{L}_k = \mathbf{I}$;

(ii) *S is right k-simple if and only if* $\mathbf{R}_k = \mathbf{I}$;

(iii) *S is k-simple if and only if* $\mathbf{M}_k = \mathbf{I}$.

Proof. (i). Assume that S is left k-simple. It is clear that $\mathbf{L}_k(\emptyset) = \emptyset = \mathbf{I}(\emptyset)$. Let A be a nonempty subset of S. Then we have $\mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} = S = \mathbf{I}(A)$. Hence, $\mathbf{L}_k = \mathbf{I}$. Conversely, if A is a left ordered k-ideal, then we obtain $S = \mathbf{I}(A) = \mathbf{L}_k(A) = \overline{(\Sigma_{fin}A + \Sigma_{fin}SA]} \subseteq \overline{(A]} = A \subseteq S$. Hence, $A = S$. Thus, S is left k-simple.

The proof of (ii) and (iii) are similar to (i). □

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