Quasigroups and Related Systems **24** (2016), 151 − 156

On locally maximal product-free sets in 2-groups of coclass 1

Chimere S. Anabanti

Abstract. This paper is in two parts: first, we classify the 2-groups of coclass 1 that contain locally maximal product-free sets of size 4, then give a classification of the filled 2-groups of coclass 1.

1. Introduction

Let S be a product-free set in a finite group G. Then S is *locally maximal* in G if S is not properly contained in any other product-free set in G, and S is said to *fill* G if $G^* \subseteq S \sqcup SS$, where $G^* = G \setminus \{1\}$. We call G a *filled group* if every locally maximal product-free set in G fills G.

Street and Whitehead [6] classified the abelian filled groups as one of C_3, C_5 or an elementary abelian 2-group. Recently, Anabanti and Hart [2] classified the filled groups of odd order as well as gave a characterisation of the filled nilpotent groups. In the latter direction, they proved that if G is a filled nilpotent group, then G is one of C_3, C_5 or a 2-group. One of the goals of this paper is the classification of filled 2-groups of coclass 1.

By a 2-*group of coclass* 1, we mean a group of order 2^n and nilpotency class $n-1$ for $n \geqslant 3$, and is one of the following:
(i) $D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{-1}\rangle$, $n \geqslant 3$ (Dihedral);
(ii) $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, xy = yx^{-1}\rangle$ for $n \geqslant 3$ (Generalised quaternion);
(iii) $QD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{2^{n-2}-1}\rangle$, $n \geqslant 4$ (Quasi-dihedral).

In 2006, Giudici and Hart [5] began the classification of groups containing locally maximal product-free sets (LMPFS for short) of small sizes. They classified all finite groups containing LMPFS of sizes 1 and 2, and some of size 3. The classification problem for size 3 was concluded in [1]. Dihedral groups containing LMPFS of size 4 were classified in [2]. Another goal of this paper is to classify groups of forms (ii) and (iii) that contain locally maximal product-free sets of size 4, continuing work in [1] and [5].

2010 Mathematics Subject Classification: 20D60, 20P05, 05E15
Keywords: Locally maximal, sum-free, product-free, 2-groups.

2. Preliminaries

Here, we gather together some useful results.

Lemma 2.1. [5, Lemma 3.1] *Suppose S is a product-free set in a finite group G. Then S is locally maximal if and only if $G = T(S) \cup \sqrt{S}$, where $T(S) = S \cup SS \cup SS^{-1} \cup S^{-1}S$ and $\sqrt{S} = \{x \in G : x^2 \in S\}$.*

Proposition 2.2. [2, Proposition 1.3] *Each product-free set of size $\frac{|G|}{2}$ in a finite group G is the non-trivial coset of a subgroup of index 2. Furthermore such sets are locally maximal and fill G.*

Lemma 2.3. [6, Lemma 1] *Let N be a normal subgroup of a finite group G. If G is filled, then G/N is filled.*

Theorem 2.4. [2, Propositions 2.8 and 4.8]
(*a*) *The only filled dihedral 2-groups are D_4 and D_8.*
(*b*) *No generalised quaternion group is filled.*

3. Main results

For a subset S of a 2-group of coclass 1, we write $A(S)$ for $S \cap \langle x \rangle$, and $B(S)$ for $S \cap \langle x \rangle y$. Given $a \in \mathbb{N}$, we write $[0, a]$ for $\{0, 1, \dots, a\}$.

Proposition 3.1. *Let S be a LMPFS of size $m \geqslant 2$ in a generalised quaternion group G. If $x^{2^{n-2}} \notin S$, then $|G| \leqslant 2(|B(S)| + 4|A(S)||B(S)|)$.*

Proof. Let $A = A(S)$ and $B = B(S)$. By Lemma 2.1, $|G| = 2|B(T(S) \cup \sqrt{S})|$; so to bound $|G|$, we count only the possible elements of $B(S \cup SS \cup S^{-1}S \cup SS^{-1} \cup \sqrt{S})$, and double the result. As $x^{2^{n-2}} \notin S$, we have $B(\sqrt{S}) = \emptyset$. But $B(SS) = AB \cup BA$, $B(SS^{-1}) = BA^{-1} \cup AB^{-1}$ and $B(S^{-1}S) = B^{-1}A \cup A^{-1}B$. By the relations in a generalised quaternion group, $AB = BA^{-1}$ and $BA = A^{-1}B$.

Hence, $|B(T(S) \cup \sqrt{S})| \leqslant |B| + 4|A||B|$, and the result follows. □

A little modification to the proof of Proposition 3.1 gives the following:

Lemma 3.2. *If S is a LMPFS of size $m \geqslant 2$ in a generalised quaternion group G such that $A(S) = A(S)^{-1}$ and $x^{2^{n-2}} \notin S$, then $|G| \leqslant 2(|B(S)| + 2|A(S)||B(S)|)$.*

The next result is a complement of Proposition 3.1. We omit the proof since it is a consequence of the definition of the group in question.

Lemma 3.3. *Let G be a generalised quaternion group. If S is a LMPFS in G and contains the unique involution in $A(G)$, then $S \subseteq A(G)$ and S is locally maximal product-free in $A(G)$.*

In the light of Lemma 3.3, we need to study $A(G)$ more carefully. All cyclic groups containing LMPFS of sizes 1, 2 and 3 are known by the classification results in [1] and [5]. However, we cannot lay our hands on any literature that classified cyclic groups containing LMPFS of a given size $m \geqslant 1$; so we proceed in that direction. Our result (Corollary 3.5) addresses the question of Babai and Sós [3, p. 111] as well as Street and Whitehead [6, p. 226] on the minimal sizes of LMPFS in finite groups for the cyclic group case.

Proposition 3.4. *Let S be a LMPFS of size $m \geqslant 1$ in C_n. Then:*

(*i*) $|SS| \leqslant \frac{m(m+1)}{2}$,

(*ii*) $|SS^{-1}| \leqslant m^2 - m + 1$,

(*iii*) *if n is odd, then* $|\sqrt{S}| = m$,

(*iv*) *if n is even, then* $|\sqrt{S}| \leqslant 2m$.

Proof. Suppose $S = \{x_1, x_2, \dots, x_m\}$. For (*i*), observe that $SS \subseteq \{x_1x_1, \dots, x_1x_m\} \cup \{x_2x_2, \dots, x_2x_m\} \cup \dots \cup \{x_{m-1}x_{m-1}, x_{m-1}x_m\} \cup \{x_mx_m\}$. Hence, $|SS| \leqslant m + (m-1) + \dots + 2 + 1 = \frac{m(m+1)}{2}$. Case (*ii*) follows from $SS^{-1} \subseteq \{1, x_1x_2^{-1}, \dots, x_1x_{m-1}^{-1}, x_1x_m^{-1}\} \cup \{x_2x_1^{-1}, 1, \dots, x_2x_{m-1}^{-1}, x_2x_m^{-1}\} \cup \dots \cup \{x_mx_1^{-1}, x_mx_2^{-1}, \dots, x_mx_{m-1}^{-1}, 1\}$. For (*iii*) and (*iv*), define a homomorphism $\theta : C_n \to C_n$ by $\theta(x) = x^2 \ \forall x \in C_n$. If n is odd, then $\mathrm{Ker}(\theta) = \{1\}$, and if n is even, then $\mathrm{Ker}(\theta) = \{1, u\}$, where u is the unique involution in C_n. By the first isomorphism theorem, the latter case implies that each element of S has at most two square roots while the former case shows that every element of S has exactly one square root. □

Corollary 3.5. *If S is a LMPFS of size m in a cyclic group G, then $|G| \leqslant \frac{3m^2+3m+2}{2}$ or $\frac{3m^2+5m+2}{2}$ according as $|G|$ being odd or even.*

Proof. As G is abelian, $S^{-1}S = SS^{-1}$; hence by Lemma 2.1, $|G| \leqslant |S| + |SS| + |SS^{-1}| + |\sqrt{S}|$. The rest follows from Proposition 3.4. □

The bound in Corollary 3.5 is fairly tight. For instance, it says that the size of a cyclic group that can contain a LMPFS of size 1 is at most 4. Indeed, the singleton consisting of the unique involution in C_4 is an example.

Definition 3.6. Two LMPFS S and T in a group G are said to be *equivalent* if there is an automorphism of$G$ that takes one into the other.

For a finite group G, we write M_k for the set consisting of all locally maximal product-free sets of size $k \geqslant 1$ in G, S for the representatives of each equivalence class of M_k under the action of the automorphism groups of G, and N_k for the respective number of LMPFS in each orbit. Using GAP [4], we present our results in the Table below.

G	$\|M_4\|$	S	N_4
C_8	1	$\{x, x^3, x^5, x^7\}$	1
C_{10}	2	$\{x, x^4, x^6, x^9\}$	2
C_{11}	5	$\{x, x^3, x^8, x^{10}\}$	5
C_{12}	9	$\{x, x^4, x^6, x^{11}\}, \{x, x^4, x^7, x^{10}\}, \{x^2, x^3, x^8, x^9\}, \{x^2, x^3, x^9, x^{10}\}$	$4, 2, 2, 1$
C_{13}	21	$\{x, x^3, x^5, x^{12}\}$, $\{x, x^3, x^{10}, x^{12}\}$, $\{x, x^5, x^8, x^{12}\}$	$12, 6, 3$
C_{14}	27	$\{x, x^3, x^8, x^{10}\}, \{x, x^3, x^8, x^{13}\}$, $\{x, x^4, x^6, x^{13}\}$, $\{x, x^4, x^7, x^{12}\}, \{x, x^6, x^8, x^{13}\}$	$6, 6, 6, 6, 3$
C_{15}	16	$\{x, x^3, x^5, x^7\}, \{x, x^3, x^7, x^{12}\}$	$8, 8$
C_{16}	37	$\{x, x^3, x^{10}, x^{12}\}$, $\{x, x^4, x^6, x^9\}$, $\{x, x^4, x^6, x^{15}\}$, $\{x, x^4, x^9, x^{14}\}$, $\{x, x^6, x^9, x^{14}\}$, $\{x, x^6, x^{10}, x^{14}\}$, $\{x^2, x^6, x^{10}, x^{14}\}$	$8, 4, 8, 4, 4, 8, 1$
C_{17}	48	$\{x, x^3, x^8, x^{13}\}, \{x, x^3, x^8, x^{14}\}, \{x, x^3, x^{11}, x^{13}\}$	$16, 16, 16$
C_{18}	54	$\{x, x^3, x^5, x^{12}\}, \{x, x^3, x^8, x^{14}\}, \{x, x^3, x^9, x^{14}\}$, $\{x, x^3, x^{12}, x^{14}\}, \{x, x^4, x^9, x^{16}\}, \{x, x^4, x^{10}, x^{17}\}$, $\{x, x^5, x^8, x^{12}\}, \{x, x^5, x^8, x^{17}\}, \{x, x^6, x^9, x^{16}\}$	$6, 6, 6, 6, 6, 6, 6, 6, 6$
C_{19}	36	$\{x, x^3, x^5, x^{13}\}, \{x, x^4, x^6, x^9\}$	$18, 18$
C_{20}	36	$\{x, x^3, x^{10}, x^{16}\}, \{x, x^3, x^{14}, x^{16}\}, \{x, x^4, x^{11}, x^{18}\}$, $\{x, x^5, x^{14}, x^{18}\}, \{x, x^6, x^8, x^{11}\}, \{x^2, x^5, x^{15}, x^{16}\}$	$8, 8, 4, 8, 4, 4$
C_{21}	34	$\{x, x^3, x^5, x^{15}\}, \{x, x^4, x^{10}, x^{17}\}, \{x, x^4, x^{14}, x^{16}\}$, $\{x, x^8, x^{12}, x^{18}\}$	$12, 12, 4, 6$
C_{22}	10	$\{x, x^4, x^{10}, x^{17}\}$	10
C_{24}	4	$\{x, x^6, x^{17}, x^{21}\}$	4

Table: LMPFS of size 4 in cyclic group G for $8 \leqslant |G| \leqslant 34$

In the light of Corollary 3.5 therefore, if a cyclic group G contains a LMPFS S of size 4, then both G and S are contained in Table. Proposition 2.2 clearly tells us that the LMPFS of size 4 in Q_8 are the non-trivial cosets of the subgroups of index 2. So we shall eliminate this from our investigation.

Proposition 3.7. *Let $G = Q_{2^n}$. If $|G| > 8$ and G contains a LMPFS of size 4, then $G = Q_{16}$. Moreover, up to automorphisms of Q_{16}, the only such set is $\{x, x^6, y, x^4y\}$.*

Proof. Let S be a LMPFS of size 4 in G. We conclude from Lemma 3.3 and deductions from Corollary 3.5 that no such S exist if $x^{2^{n-2}} \in S$. So, suppose $x^{2^{n-2}} \notin S$. In Proposition 3.1, if $|B(S)| = 0$ or 4, then $|G| < 16$, contrary to our assumption that $|G| > 8$. If $|B(S)| = 1$ or 3, we get $|G| < 32$; so $|G| = 16$, and by direct computation, no such S exists. Finally, if $|B(S)| = 2$, then $|G| < 64$. It can easily be seen using dynamics of Lemma 2.1 that S cannot be contained in Q_{32}, and hence the only possibility is that $S \subseteq Q_{16}$. Also elements of $A(S)$ cannot have same order, and that if $B(S) = \{x^i y, x^j y\}$, then i and j must have same parity. Thus, the only possibilities for S are $S_1 := \{x, x^6, y, x^4y\}$, $S_2 := \{x, x^6, xy, x^5y\}$, $S_3 := \{x, x^6, x^3y, x^7y\}$, $S_4 := \{x, x^6, x^2y, x^6y\}$, $S_5 := \{x^2, x^7, y, x^4y\}$, $S_6 := \{x^2, x^7, xy, x^5y\}$, $S_7 := \{x^2, x^7, x^3y, x^7y\}$, $S_8 := \{x^2, x^7, x^2y, x^6y\}$, $S_9 := \{x^2, x^3, y, x^4y\}$, $S_{10} := \{x^5, x^6, y, x^4y\}$, $S_{11} := \{x^2, x^3, xy, x^5y\}$, $S_{12} := \{x^2, x^3, x^3y, x^7y\}$, $S_{13} := \{x^2, x^3, x^2y, x^6y\}$, $S_{14} := \{x^5, x^6, xy, x^5y\}$, $S_{15} := \{x^5, x^6, x^3y, x^7y\}$ and $S_{16} := \{x^5, x^6, x^2y, x^6y\}$. The result follows from the fact that the automorphism ϕ_i takes S_1 into S_i for $1 \leqslant i \leqslant 16$, where $\phi_1 : x \mapsto x, y \mapsto y$, $\phi_2 : x \mapsto x, y \mapsto xy$, $\phi_3 : x \mapsto x, y \mapsto x^3y$, $\phi_4 : x \mapsto x, y \mapsto x^2y$, $\phi_5 : x \mapsto x^7, y \mapsto y$, $\phi_6 : x \mapsto x^7, y \mapsto xy$, $\phi_7 : x \mapsto x^7, y \mapsto x^3y$, $\phi_8 : x \mapsto x^7, y \mapsto x^2y$, $\phi_9 : x \mapsto x^3, y \mapsto y$, $\phi_{10} : x \mapsto x^5, y \mapsto y$, $\phi_{11} : x \mapsto x^3, y \mapsto xy$, $\phi_{12} : x \mapsto x^3, y \mapsto x^3y$, $\phi_{13} : x \mapsto x^3, y \mapsto x^2y$, $\phi_{14} : x \mapsto x^5, y \mapsto xy$, $\phi_{15} : x \mapsto x^5, y \mapsto x^3y$ and $\phi_{16} : x \mapsto x^5, y \mapsto x^2y$. □

Proposition 3.8. *Let S be a LMPFS of size $m \geqslant 4$ in a quasi-dihedral group G. If $x^{2^{n-2}} \notin S$, then $|G| \leqslant 2(|B(S)| + 6|A(S)||B(S)|)$.*

Proof. Similar to the proof of Proposition 3.1. □

Lemma 3.9. *No LMPFS of size 4 in a quasi-dihedral group G contains the unique involution in $A(G)$.*

Proof. Let S be a LMPFS of size 4 in a quasi-dihedral group G such that $x^{2^{n-2}} \in S$. First observe that S must contain elements from both $A(G)$ and $B(G)$; so we have the following three cases: (I) $|A(S)| = 1$ and $|B(S)| = 3$; (II) $|A(S)| = 2$ and $|B(S)| = 2$; (III) $|A(S)| = 3$ and $|B(S)| = 1$. As S is product-free in G, it cannot contain elements of the form $\{x^{2l+1}y,\ l \geqslant 0\}$; otherwise $(x^{2l+1}y)^2 = x^{2^{n-2}} \in S$. For Case I, let $S := \{x^{2^{n-2}}, x^{2i}y, x^{2j}y, x^{2k}y\}$ for $0 \leqslant i,j,k \leqslant 2^{n-2}-1$. Then $A(T(S)) = A(S \cup SS \cup S^{-1}S \cup SS^{-1}) = A(S \cup SS)$. But $A(S \cup SS)$ cannot yield an element of the form x^{2l+1}; so we can only rely on $A(\sqrt{S})$ for such element. Observe that $\sqrt{x^{2i}y} = \sqrt{x^{2j}y} = \sqrt{x^{2k}y} = \emptyset$, and from Proposition 3.4(iv), $|A(\sqrt{x^{2^{n-2}}})| \leqslant 2$. In particular, $A(\sqrt{x^{2^{n-2}}}) = \{x^{2^{n-3}}, x^{3(2^{n-3})}\}$. Hence, there is no element of the form x^{2l+1} in $A(T(S) \cup \sqrt{S})$; a fallacy! as the number of such element in $A(QD_{2^n})$ is 2^{n-2}. Thus, no such S exist. For Case II, let $S := \{x^{2^{n-2}}, x^r, x^{2j}y, x^{2k}y\}$. If r is even, then the number of elements of the form x^{2l+1} in $A(\sqrt{S})$, $A(SS)$ and $A(SS^{-1})$ are at most 2, 0 and 0 respectively; so no such S exist. If r is odd, then the number of elements of the form x^{2l+1} in $A(\sqrt{S})$, $A(SS)$ and $A(SS^{-1})$ are at most 0, 1 and 1 respectively; again, no such S exist. Case III is similar. □

The proof of the next result is similar to that of Proposition 3.7 using Proposition 3.8 and Lemma 3.9.

Proposition 3.10. *Up to automorphisms of QD_{16}, the LMPFS of size 4 in QD_{16} are $\{x, x^6, y, x^4y\}$ and $\{x, x^6, x^3y, x^7y\}$. Furthermore, there is no LMPFS of size 4 in QD_{2^n} for $n > 4$.*

We are now in the position to address the second aim of this paper: classification of filled 2-groups of coclass 1.

Theorem 3.11. *The only filled* 2*-group of coclass* 1 *is* D_8.

Proof. By Theorem 2.4, we only show that no quasi-dihedral group is filled. Let $G = QD_{2^n}$, $n \geqslant 4$. Then $N := \langle x^8 \rangle$ is a normal subgroup of G whose quotient is of size 16. Suppose $|G| > 16$. Given $a_1 \in [0,7]$, $x^{a_1}N = x^{a_1+8a_2}N$ for $1 \leqslant a_2 \leqslant |N|-1$. Similarly, given $b_1 \in [0,7]$, $x^{b_1}yN = x^{b_1+8b_2}yN$ for $1 \leqslant b_2 \leqslant |N|-1$. Thus, $G/N = X \sqcup Y$, where $X = \{x^iN |\ 0 \leqslant i \leqslant 7\}$ and $Y = \{x^iyN |\ 0 \leqslant i \leqslant 7\}$. Clearly, $X \cong C_8 \cong A(D_{16})$. On the other hand, each element of Y has order 2 since for i even, $(x^iyN)(x^iyN) = NN = N$, and for i odd, $(x^iyN)(x^iyN) = x^{2^{n-2}}NN = N$. Hence, $G/N \cong D_{16}$. By Theorem 2.4(a) and Lemma 2.3 therefore, G is not a filled group. Now let $|G| = 16$. By Proposition 3.10, $S = \{x, x^6, y, x^4y\}$ is locally maximal in QD_{16}. However, S does not fill QD_{16} since $|A(QD_{16}^*)| = 7 > 6 = |A(S \sqcup SS)|$; so QD_{16} is not filled. □

We conclude this discussion with the following question:

Question 3.12. *Are there infinitely many non-abelian filled groups?*

Acknowledgements

This research is supported by a Birkbeck PhD scholarship. The author is grateful to Professor Sarah Hart for useful comments.

References

[1] **C.S. Anabanti and S.B. Hart**, *Groups containing small locally maximal product-free sets*, Intern. J. Combin. (2016), Article ID 8939182, 5 pages.

[2] **C.S. Anabanti and S.B. Hart**, *On a conjecture of Street and Whitehead on locally maximal product-free sets*, Australasian J. Combin. **63** (2015), 385 – 398.

[3] **L. Babai and V.T. Sós**, *Sidon sets in groups and induced subgraphs of Cayley graphs*, European J. Combin. **6** (1985), 101 – 114.

[4] **The GAP Group**, *GAP – Groups, Algorithms, and Programming, Version 4.7.7*; 2015, (`http://www.gap-system.org`).

[5] **M. Giudici and S. Hart**, *Small maximal sum-free sets*, Electronic J. Combin. **16** (2009), 1 – 17.

[6] **A.P. Street and E.G. Whitehead Jr.**, *Group Ramsey Theory*, J. Combinatorial Theory, Ser. A **17**, (1974), 219 – 226.

Received February 15, 2016

Department of Economics, Mathematics and Statistics, Birkbeck, University of London, WC1E 7HX London, United Kingdom
E-mail: c.anabanti@mail.bbk.ac.uk

Department of Mathematics, University of Nigeria, Nsukka, Nigeria
E-mail: chimere.anabanti@unn.edu.ng