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# Gyrogroups and the Cauchy property

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**Abstract.** A gyrogroup is a nonassociative group-like structure. In this article, we extend the Cauchy property from groups to gyrogroups. The (weak) Cauchy property for finite gyrogroups states that if $p$ is a prime dividing the order of a gyrogroup $G$, then $G$ contains an element of order $p$. An application of a result in loop theory shows that gyrogroups of odd order as well as solvable gyrogroups satisfy the Cauchy property. Although gyrogroups of even order need not satisfy the Cauchy property, we prove that every gyrogroup of even order contains an element of order two. As an application, we prove that every group of order $nq$, where $n \in \mathbb{N}$ and $q$ is a prime with $n < q$, contains a unique characteristic subgroup of order $q$.

## 1. Introduction

Gyrogroups abound as an integral part of group theory. In fact, (i) every gyrogroup is extendable to a group, called the *gyrosemidirect product group* [23, Section 2.6]; (ii) every gyrogroup is a twisted subgroup of some group [6, 7, 11]; and (iii) a certain group with an automorphism of order two gives rise to a gyrogroup [6, 7, 12, 17]. Further, any group may be viewed as a gyrogroup with trivial gyroautomorphisms. It turns out that gyrogroups share remarkable analogies with groups. Several well-known results in group theory can be naturally extended to the case of gyrogroups such as the Lagrange theorem [18], the fundamental isomorphism theorems, the Cayley theorem [19], the orbit-stabilizer theorem, the class equation, and the Burnside lemma [16]. Moreover, some gyrocommutative gyrogroups admit scalar multiplication, turning themselves into gyrovector spaces, just as some abelian groups admit scalar multiplication, turning themselves into vector spaces. Remarkably, gyrovector spaces form the algebraic setting for analytic hyperbolic geometry, just as vector spaces form the algebraic setting for analytic Euclidean geometry, as evidenced, for instance, from [20, 21, 22, 23, 24, 25, 26, 27]. Thus, like the group notion, the notion of gyrogroups plays a universal computational role.

It is known in the literature that every group satisfies the *Cauchy property*, that is, if $p$ is a prime dividing the order of a group $\Gamma$, then $\Gamma$ contains an element of order $p$. This is the familiar Cauchy theorem in abstract algebra. Cauchy's

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theorem leads to a better understanding of the structure of a finite group. For instance, using Cauchy's theorem, one can prove that every group of order $2p$, where $p$ is a prime, is isomorphic to the cyclic group or the dihedral group of order $2p$ [8]. Furthermore, the celebrated Sylow theorems are built on Cauchy's theorem, see for instance [4, p. 140] and [3, Section 9.2].

In [18] the authors extend the Cauchy property to the case of gyrogroups and prove that gyrogroups of order $pq$ and nongyrocommutative gyrogroups of order $pqr$ satisfy the (strong) Cauchy property, where $p, q$ and $r$ are primes. Unfortunately, there is no hope of extending Cauchy's theorem to all finite gyrogroups as Nagy proves the existence of a simple right Bol loop of exponent two and of order 96 [13, Corollary 3.7]. See also [2]. This loop gives rise to a gyrocommutative gyrogroup of order 96 in which every nonidentity element has order two. However, some classes of finite gyrogroups do satisfy the Cauchy property. As in the group case, the Cauchy property leads to a better understanding of the structure of a finite gyrogroup. For example, any gyrogroup of order $pq$, where $p$ and $q$ are distinct primes, is generated by two elements; one has order $p$ and the other has order $q$ [18, Theorem 6.10]. We will see shortly that any gyrogroup of order $pq$, where $p$ and $q$ are primes with $p < q$, contains a unique subgyrogroup of order $q$.

## 2. Preliminaries

For the basic theory of gyrogroups, the reader is referred to [15, 18, 19, 23]. For basic knowledge of loop theory, the reader is referred to [10, 14]. Subgyrogroups, gyrogroup homomorphisms, normal subgyrogroups, and quotient gyrogroups are studied in detail in [15, 18, 19].

Let $G$ be a gyrogroup and let $a$ be an element of $G$. For $m \in \mathbb{Z}$, define recursively the following notation

$$0a = 0, \quad ma = a \oplus ((m-1)a),\, m \geq 1, \quad ma = (-m)(\ominus a),\, m < 0. \tag{1}$$

It can be shown that $(ma) \oplus (ka) = (m+k)a$ and $(mk)a = m(ka)$ for all $m, k \in \mathbb{Z}$. Hence, the *cyclic subgyrogroup generated by* $a$, written $\langle a \rangle$, forms a cyclic group with generator $a$ under the gyrogroup operation. In fact,

$$\langle a \rangle = \{ma \colon m \in \mathbb{Z}\}. \tag{2}$$

Further, the gyroautomorphism $\mathrm{gyr}\,[ma, ka]$ descends to the identity automorphism for all $m, k \in \mathbb{Z}$. The *order* of $a$, denoted by $|a|$, is defined to be the cardinality of $\langle a \rangle$ if $\langle a \rangle$ is finite. In this case, we will write $|a| < \infty$. If $\langle a \rangle$ is infinite, the order of $a$ is defined to be infinity, and we will write $|a| = \infty$. As in the theory of groups, if $|a| < \infty$, then $|a|$ is the smallest positive integer such that $|a|a = 0$. If $|a| = \infty$, then $|ma| = \infty$ for all $m \in \mathbb{Z} \setminus \{0\}$. Furthermore, if $G$ is a finite gyrogroup, then $|G|$ is divisible by $|a|$, see [18, Proposition 6.1].

As a consequence of the left cancellation law, the *left gyrotranslation by* $a$, defined by $L_a \colon x \mapsto a \oplus x$, $x \in G$, is a permutation of $G$ for all $a \in G$. Because gyrogroups are *left power alternative* [18, p. 288], that is, $L_a^m = L_{ma}$ for all $a \in G$, $m \in \mathbb{Z}$, the gyrogroup-theoretic order of $a$ and the group-theoretic order of $L_a$ coincide.

## 3. Main results

Throughout the remainder of the article, all gyrogroups are finite and $G$ denotes an arbitrary finite gyrogroup unless explicitly mentioned otherwise.

**Definition 3.1.** A gyrogroup $G$ has the *weak Cauchy property* if for every prime $p$ dividing the order of $G$, $G$ contains an element of order $p$.

**Definition 3.2.** A gyrogroup $G$ has the *strong Cauchy property* if every subgyrogroup of $G$ has the weak Cauchy property.

It is clear that any gyrogroup that satisfies the strong Cauchy property will automatically satisfy the weak Cauchy property as well. The Cauchy property is an invariant property of finite gyrogroups in the sense that if $G$ and $H$ are isomorphic gyrogroups, then $G$ has the weak (resp. strong) Cauchy property if and only if $H$ has the weak (resp. strong) Cauchy property [18, Corollary 6.6]. Therefore, the Cauchy property becomes important in classification of finite gyrogroups because not every gyrogroup has the Cauchy property. Further, the Cauchy property is a good example to see how information about a gyrogroup $G$ can be obtained from information on its normal subgyrogroup $N$ and on its quotient gyrogroup $G/N$, as shown in the following theorem.

**Theorem 3.3 (Corollary 6.8, [18]).** *Let $N$ be a normal subgyrogroup of $G$. If $N$ and $G/N$ have the weak (resp. strong) Cauchy property, then so has $G$.*

Using Theorem 3.3, one can show that finite solvable gyrogroups satisfy the strong Cauchy property. A (finite or infinite) gyrogroup $G$ is *solvable* if there exists a series $\{0\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ of subgyrogroups of $G$ such that $G_i \trianglelefteq G_{i+1}$ and the quotient gyrogroup $G_{i+1}/G_i$ is an abelian group for all $i$ with $0 \leq i \leq n-1$ (cf. [1, p. 116]).

**Theorem 3.4 (Proposition 46, [15]).** *Every solvable gyrogroup has the strong Cauchy property.*

*Proof.* The proof of the theorem can be done by induction on the number of subgyrogroups in a subnormal series using Theorem 3.3. $\square$

Recall that a loop $(L, \cdot)$ is a *left Bol loop* if it satisfies the *left Bol identity*:

$$a \cdot (b \cdot (a \cdot c)) = (a \cdot (b \cdot a)) \cdot c \tag{3}$$

for all $a, b, c \in L$. A loop $(L, \cdot)$ has the $\mathrm{A}_\ell$*-property* if the left inner mapping

$$\ell(a, b) := L_{a \cdot b}^{-1} \circ L_a \circ L_b$$

generated by $a$ and $b$ defines an automorphism of $L$ for all $a, b \in L$. Here, $L_a$ denotes the *left multiplication map* by $a$ defined by $L_a\colon x \mapsto a \cdot x$, $x \in L$. It is known in the literature that every gyrogroup forms a left Bol loop with the $\mathrm{A}_\ell$-property, where the gyroautomorphisms correspond to left inner mappings, and vice versa. In particular, the left loop property and the left Bol identity are equivalent, see for instance [10, Theorem 6.4].

By Glauberman's result [9], Foguel et al. prove that Cauchy's theorem holds for finite left Bol loops of *odd* order. More specifically, if $L$ is a left Bol loop of odd order and if $p$ is a prime dividing the order of $L$, then there exists an element $a$ of $L$ such that $|L_a| = p$ [5, Theorem 6.2]. With this result in hand, one can prove that gyrogroups of *odd* order satisfy the weak Cauchy property.

**Theorem 3.5 (Cauchy's theorem).** *Let $G$ be a gyrogroup of odd order. If $p$ is a prime dividing $|G|$, then $G$ has an element of order $p$. In other words, $G$ has the week Cauchy property.*

*Proof.* As noted above, $G$ is a left Bol loop of odd order. Hence, by Theorem 6.2 of [5], there is an element $a$ of $G$ such that $|L_a| = p$. Since $|a|$ equals $|L_a|$, the theorem follows. □

**Corollary 3.6.** *Every gyrogroup of odd order has the strong Cauchy property.*

*Proof.* Let $G$ be a gyrogroup of odd order and let $H$ be a subgyrogroup of $G$. By Lagrange's theorem for gyrogroups [18, Theorem 5.7], $|H|$ divides $|G|$ and so $H$ is a gyrogroup of odd order. It follows that $H$ has the week Cauchy property, which completes the proof. □

We have seen in Corollary 3.6 that any gyrogroup of odd order has the strong Cauchy property. Unfortunately, there is an example of a gyrogroup of *even* order that fails to satisfy the weak Cauchy property. In fact, by Corollary 3.7 of [13], there exists a simple *right* Bol loop of exponent two and of order 96, say $(L_N, \cdot)$. The *dual loop* of $L_N$, denoted by $\hat{L}_N$, consists of the underlying set $L_N$ with the dual operation

$$a * b := b \cdot a$$

for all $a, b \in L_N$. It is straightforward to check that $\hat{L}_N$ is a left Bol loop, that $L_N$ and $\hat{L}_N$ share the same identity, and that if $a \in L_N$, then the inverse of $a$ in $L_N$ and the inverse of $a$ in $\hat{L}_N$ are identical. Note that $a = a^{-1}$ for all $a \in \hat{L}_N$ since $\hat{L}_N$ is of exponent two. Hence,

$$(a * b)^{-1} = a * b = a^{-1} * b^{-1}$$

for all $a, b \in \hat{L}_N$. This shows that $\hat{L}_N$ is a *left* Bol loop satisfying the automorphic inverse property. Hence, $\hat{L}_N$ is a gyrocommutative gyrogroup by Theorem 6.6 of [10] and Theorem 3.2 of [23]. Since $a * a = 1$ for all $a \in \hat{L}_N$, every nonidentity element of $\hat{L}_N$ has order two. From this it is clear that $\hat{L}_N$ does not satisfy the weak Cauchy property. Nevertheless, any gyrogroup of *even* order does contain an element of order two, as shown in the following theorem.

**Theorem 3.7.** *If $G$ is a gyrogroup of even order, then $G$ contains an element of order two.*

*Proof.* We first show that $\{\{a, \ominus a\} : a \in G\}$ forms a *disjoint* partition of $G$. For each $a \in G$, set $C_a = \{a, \ominus a\}$. Clearly, $C_a \neq \emptyset$ for all $a \in G$ and $\bigcup_{a\in G} C_a = G$. We claim that $C_a \cap C_b \neq \emptyset$ implies $C_a = C_b$. In fact, if $x \in C_a \cap C_b$, then there are four possibilities:

(1) $x = a$ and $x = b$;

(2) $x = a$ and $x = \ominus b$;

(3) $x = \ominus a$ and $x = b$;

(4) $x = \ominus a$ and $x = \ominus b$.

Each of (1)–(4) implies that $C_a = C_b$ since $\ominus(\ominus x) = x$. Note that $|C_a| = 1$ or 2. Note also that $|C_a| = 1$ if and only if $a = \ominus a$.

Set $m = |\{a \in G : |C_a| = 2\}|$ and $n = |\{a \in G : |C_a| = 1\}|$. Then $|G| = 2m + n$. Since 2 divides $|G|$, we have $2 \mid n$. Thus, $n \geq 2$ and so there must be a nonidentity element $c$ of $G$ such that $c = \ominus c$. Hence, $|c| = 2$. □

As a consequence of Theorem 3.7, every gyrocommutative gyrogroup of even order contains the nontrivial subgyrogroup of elements of order two together with the gyrogroup identity.

**Lemma 3.8.** *Let $G$ be a (finite or infinite) gyrocommutative gyrogroup. Then*

$$L^2_{a\oplus b} = L_a \circ L^2_b \circ L_a$$

*for all $a, b \in G$.*

*Proof.* Note that $L_a^{-1} = L_{\ominus a}$ for all $a \in G$. By (2.126) of [23], $\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[\ominus a, \ominus b]$. By (12) of [19] and Theorem 3.2 of [23],

$$L^{-1}_{a\oplus b} \circ L_a \circ L_b = L^{-1}_{\ominus a\ominus b} \circ L_{\ominus a} \circ L_{\ominus b} = L_{\ominus(\ominus a\ominus b)} \circ L_a^{-1} \circ L_b^{-1} = L_{a\oplus b} \circ L_a^{-1} \circ L_b^{-1},$$

which implies $L^2_{a\oplus b} = L_a \circ L^2_b \circ L_a$. □

**Theorem 3.9.** *If $G$ is a (finite or infinite) gyrocommutative gyrogroup, then*

$$G_2 := \{a \in G : 2a = 0\}$$

*forms a subgyrogroup of $G$.*

*Proof.* Clearly, $0 \in G_2$. Let $a, b \in G_2$. Then $a \oplus a = 2a = 0$, which implies $a = \ominus a$. Hence, $\ominus a \in G_2$. As in the proof of Proposition 3.10 of [18], $L_a^m = L_{ma}$ for all $a \in G$, $m \in \mathbb{Z}$. Hence, by Lemma 3.8,

$$L_{2(a\oplus b)} = L^2_{a\oplus b} = L_a \circ L_b^2 \circ L_a = L_a \circ L_{2b} \circ L_a = L_{2a} = \mathrm{id}_G.$$

It follows that $2(a \oplus b) = 0$ and so $a \oplus b \in G_2$. By the subgyrogroup criterion [19, Proposition 14], $G_2 \leqslant G$. $\square$

If $G$ is a finite gyrocommutative gyrogroup of *odd* order, then $G_2$ given in Theorem 3.9 is the trivial subgyrogroup of $G$. In fact, if $a$ is a nonidentity element of $G$, then $|a|$ divides $|G|$ by Proposition 6.1 of [18]. This implies $|a|$ is odd and hence $2a \neq 0$. In contrast, if $G$ is a finite gyrocommutative gyrogroup of *even* order, then $G_2$ is nontrivial since Theorem 3.7 ensures the existence of a nonidentity element of $G$ of order two.

# 4. Applications of Cauchy's theorem

Let $H$ be a subgyrogroup of a gyrogroup $G$. For each $a \in G$, the *right coset of $H$ by $a$*, denoted by $H \oplus a$, is defined as $H \oplus a = \{h \oplus a \colon h \in H\}$. As a consequence of the right cancellation law in a gyrogroup [23, Eq. (2.64)], the right gyrotranslation by $a$, $R_a$, is a bijection from $G$ to itself. Hence, the restriction of $R_a$ to $H$ is a bijection from $H$ to $H \oplus a$ and so $H$ and $H \oplus a$ have the same size. The following theorem shows that the *right* cosets of a cyclic subgyrogroup of $G$ forms a *disjoint* partition of $G$. In contrast, the *left* cosets of a cyclic subgyrogroup of $G$ need not partition $G$.

**Theorem 4.1.** *Let $G$ be a (finite or infinite) gyrogroup and let $a \in G$. The collection of right cosets of the cyclic subgyrogroup $\langle a \rangle$ in $G$ is a disjoint partition of $G$.*

*Proof.* Note that if $x \in G$, then $\langle a \rangle \oplus x \neq \emptyset$. In fact, $x = 0 \oplus x \in \langle a \rangle \oplus x$. This implies that $G = \bigcup_{x\in G} \langle a \rangle \oplus x$. Suppose that $x, y \in G$ are such that $\langle a \rangle \oplus x \cap \langle a \rangle \oplus y$ is not empty, namely $b \in \langle a \rangle \oplus x \cap \langle a \rangle \oplus y$. Then $b = ma \oplus x = na \oplus y$ for some $m, n \in \mathbb{Z}$. To complete the proof, we show that $\langle a \rangle \oplus x = \langle a \rangle \oplus y$. Let $z \in \langle a \rangle \oplus x$. Then $z = ka \oplus x$. We compute

$$\begin{aligned} z &= ka \oplus x \\ &= ((k-m)a \oplus ma) \oplus x \\ &= (k-m)a \oplus (ma \oplus \mathrm{gyr}\,[ma, (k-m)a]x) \\ &= (k-m)a \oplus (ma \oplus x) \\ &= (k-m)a \oplus (na \oplus y) \\ &= ((k-m)a \oplus na) \oplus \mathrm{gyr}\,[(k-m)a, na]y \\ &= (k-m+n)a \oplus y, \end{aligned}$$

which implies $z \in \langle a \rangle \oplus y$. We have the second equation from Proposition 3.7 of [18]; the third equation from the right gyroassociative law; the forth equation from Proposition 3.10 of [18]; the sixth equation from the left gyroassociative law; the last equation from Propositions 3.7 and 3.10 of [18]. This proves $\langle a \rangle \oplus x \subseteq \langle a \rangle \oplus y$. Similarly, $z = k'a \oplus y$ for some $k' \in \mathbb{Z}$ implies $z = (k' - n + m)a \oplus x$, and we have the reverse inclusion $\langle a \rangle \oplus y \subseteq \langle a \rangle \oplus x$. □

**Lemma 4.2.** *Let $G$ be a gyrogroup and let $a$ be an element of $G$ of finite order. If $n \in \mathbb{Z}$ and $na = 0$, then $n$ is divisible by $|a|$.*

*Proof.* If $n = 0$, the statement is trivial. We may therefore assume that $n \neq 0$. Set $|a| = m$. Using the division algorithm, we write $n = mk + r$ for some $k, r \in \mathbb{Z}$ such that $0 \leq r < m$. From Proposition 3.7 of [18], we have

$$0 = na = (mk + r)a = k(ma) \oplus ra = 0 \oplus ra = ra.$$

By the minimality of $m$, $r = 0$. Hence, $n = mk$ and so $m \mid n$. □

**Lemma 4.3.** *Let $q$ be a prime and let $n$ be a positive integer such that $n < q$. Let $G$ be a gyrogroup of order $nq$. If $a$ is an element of $G$ of order $q$, then for all $x \in G$, $|x| = q$ implies $x \in \langle a \rangle$.*

*Proof.* Let $a \in G$ with $|a| = q$. Suppose that $x \in G$ and $|x| = q$. We prove that there must be distinct integers $i, j \in \{0, 1, \ldots, q-1\}$ such that $\langle a \rangle \oplus ix \cap \langle a \rangle \oplus jx \neq \emptyset$. Note that $|\langle a \rangle \oplus b| = |\langle a \rangle| = q$ for all $b \in G$ by the remark above. Suppose to the contrary that for all $i, j \in \{0, 1, \ldots, q-1\}$, if $i \neq j$, then $\langle a \rangle \oplus ix \cap \langle a \rangle \oplus jx = \emptyset$. Hence, $\left| \bigcup_{i=0}^{q-1} \langle a \rangle \oplus ix \right| = \sum_{i=0}^{q-1} |\langle a \rangle \oplus ix| = \sum_{i=0}^{q-1} q = q^2$, which is impossible, since $\bigcup_{i=0}^{q-1} \langle a \rangle \oplus ix \subseteq G$ and so $\left| \bigcup_{i=0}^{q-1} \langle a \rangle \oplus ix \right| \leq |G| = nq < q^2$. Hence, there are integers $i, j$ with $0 \leq i \neq j < q$ for which $\langle a \rangle \oplus ix \cap \langle a \rangle \oplus jx \neq \emptyset$. There is no loss in assuming that $i < j$. By Theorem 4.1, $\langle a \rangle \oplus ix = \langle a \rangle \oplus jx$. This implies $jx = c \oplus ix$ for some $c \in \langle a \rangle$. By the right cancellation law,

$$c = (c \oplus ix) \boxplus (\ominus ix) = jx \boxplus (\ominus ix) = jx \oplus \operatorname{gyr}[jx, ix](\ominus ix) = jx \ominus ix = (j - i)x.$$

By Corollary 3.15 (2) of [18], $|(j - i)x| = \dfrac{|x|}{\gcd(|x|, j - i)} = q$ for $0 < j - i < q$. Since $c \in \langle a \rangle$, $\langle c \rangle \leqslant \langle a \rangle$. Since $|c| = q = |a|$, $\langle c \rangle = \langle a \rangle$. Similarly, $\langle (j - i)x \rangle = \langle x \rangle$. Therefore, $\langle x \rangle = \langle a \rangle$ and hence $x \in \langle a \rangle$. □

**Theorem 4.4.** *Let $q$ be a prime and let $n$ be a positive integer such that $n < q$. Let $G$ be a gyrogroup of order $nq$. Define*

$$G_q = \{a \in G \colon qa = 0\}. \tag{4}$$

*Then $G_q$ is either the trivial subgyrogroup or the unique subgyrogroup of $G$ order $q$.*

*Proof.* If $G_q = \{0\}$, then we are done. We may therefore assume that $G_q \neq \{0\}$. Hence, $qa = 0$ for some $a \in G \setminus \{0\}$. By Lemma 4.2, $|a|$ divides $q$. Since $q$ is a prime and $a \neq 0$, $|a| = q$. It follows that $\langle a \rangle$ is a subgyrogroup of $G$ of order $q$. If $K$ is a subgyrogroup of $G$ of order $q$, then $K = \langle b \rangle$ for some $b \in K$ by Theorem 6.2 of [18]. Since $|b| = q$, Lemma 4.3 implies $b \in \langle a \rangle$. Hence, $K = \langle b \rangle = \langle a \rangle$. This proves existence and uniqueness of the subgyrogroup of $G$ of order $q$.

Next, we prove that $G_q = \langle a \rangle$. Let $x \in \langle a \rangle$. Then either $x = 0$ or $|x| = q$. In either case, $qx = 0$. Hence, $x \in G_q$. This proves $\langle a \rangle \subseteq G_q$. Let $x \in G_q$. Then $qx = 0$. If $x = 0$, then $x \in \langle a \rangle$. We may therefore assume that $x \neq 0$. By Lemma 4.2, $|x|$ divides $q$ and so $|x| = q$. By Lemma 4.3, $x \in \langle a \rangle$ and we have the reverse inclusion $G_q \subseteq \langle a \rangle$. □

A subgyrogroup $H$ of a gyrogroup $G$ is called an *L-subgyrogroup* of $G$ if

$$\mathrm{gyr}\,[a, h](H) = H$$

for all $a \in G$, $h \in H$. One of the main aspects of L-subgyrogroups is that they partition $G$ into left cosets of equal size [19, Theorem 20].

**Theorem 4.5.** *If $G_q$ given in Theorem 4.4 is nontrivial, then it is an L-subgyrogroup of $G$ of index $n$.*

*Proof.* Assume that $G_q \neq \{0\}$. Let $a, b \in G$. Since $\mathrm{gyr}\,[a, b]$ is a gyrogroup automorphism of $G$, $\mathrm{gyr}\,[a, b](G_q)$ forms a subgyrogroup of $G$ of order $q$. By the uniqueness of $G_q$, $\mathrm{gyr}\,[a, b](G_q) = G_q$. By definition, $G_q \leqslant_L G$. As $G_q \leqslant_L G$, the index formula holds and hence $[G\colon G_q] = |G|/|G_q| = n$. □

**Theorem 4.6.** *If $G$ is a gyrogroup of order $pq$, where $p$ and $q$ are primes with $p < q$, then $G$ contains the unique subgyrogroup of order $q$.*

*Proof.* By Cauchy's theorem for gyrogroups of order $pq$ [18, Theorem 6.9], $G$ has an element of order $q$. So, $G_q \neq \{0\}$ and the theorem follows directly from Theorem 4.4. □

**Theorem 4.7.** *Let $G$ be a gyrogroup of order $pq$, where $p$ and $q$ are primes with $p < q$. If the unique subgyrogroup of $G$ of order $q$ is normal in $G$, then $G$ is solvable.*

*Proof.* Let $N$ be the unique subgyrogroup of $G$ of order $q$ and assume that $N \trianglelefteq G$. By Theorem 6.2 of [18], $N$ is a cyclic group of order $q$ and hence is an abelian group. Since $N \trianglelefteq G$, $G/N$ has the quotient gyrogroup structure and $|G/N| = [G\colon N] = p$. Hence, $G/N$ is an abelian group as well. Therefore, the series $\{0\} \leqslant N \leqslant G$ fulfills the condition of a solvable gyrogroup. □

Let $\Gamma$ be a group. A subgroup $\Xi$ of $\Gamma$ is said to be *characteristic* in $\Gamma$ if $\Xi$ is invariant under the automorphisms of $\Gamma$, that is, if $\tau(\Xi) = \Xi$ for all $\tau$ in $\mathrm{Aut}\,(\Gamma)$. Since group-theoretic conjugation $\kappa_g\colon x \mapsto gxg^{-1}$, $x \in \Gamma$, defines a group automorphism of $\Gamma$ for all $g \in \Gamma$, every characteristic subgroup of $\Gamma$ is normal in $\Gamma$. From this point of view, characteristic subgroups are sometimes called *strongly normal* subgroups.

In light of Theorem 4.4, not only the structure of a finite gyrogroup, but also the structure of a finite group, is revealed, as shown in the following theorem.

**Theorem 4.8.** *Let $q$ be a prime and let $n$ be a positive integer such that $n < q$. Every group of order $nq$ contains the unique characteristic subgroup of order $q$.*

*Proof.* Let $\Gamma$ be a group of order $nq$. By Cauchy's theorem for groups, $\Gamma$ has an element of order $q$. By Theorem 4.4, $\Gamma$ has the unique subgroup of order $q$, say $\Xi$. If $\tau$ is a group automorphism of $\Gamma$, then $\tau(\Xi)$ is indeed a subgroup of $\Gamma$ of order $q$. Hence, $\tau(\Xi) = \Xi$. This proves that $\Xi$ is characteristic in $\Gamma$. $\square$

Note that if the integer $n$ in Theorem 4.8 becomes a prime, we recover the well-known result in abstract algebra that any group of order $pq$, where $p$ and $q$ are primes with $p < q$, contains the unique normal subgroup of order $q$. This result arises as an application of the Sylow theorems, see for instance [4, p. 143]. Further, it is not difficult to see that Theorem 4.8 can be obtained as a consequence of the Sylow theorems as well.

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