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On some algebraic properties of order of an element of a multigroup

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Abstract. The concept of multigroups is a generalization of groups whereby the underlying structure is a multiset over a group X. As a continuation of the study of various algebraic structures of multisets, the concept of order of an element with respect to multigroup is introduced and some of its related results outlined. Also, the Lagrange's theorem for regular multigroup is described, and the restriction to regular multigroup makes the theorem flexible showing an analogy to that of group theory.

1. Introduction

The conception of multiset was introduced by N.G. de Bruijn under the idea of classical set theory. According to George Cantor,

By a set we are to understand any collection M of definite and distinct objects m of our intuition or thought (which will be called the "element" of M) into a whole.

One unavoidable consequence of Cantor's definition is that no element can occur more than once in a classical set. Indeed, this aspect of Cantorian set theory does not go hand in hand with many situations arising in solving real world problems. For example, the repeated roots of $x^2 - 2x + 1 = 0$, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule, H_2O, etc. need to be considered significant. Once we admit the restriction of definiteness on the nature of objects forming a set, we have multisets. Details on fundamentals of multiset, multiset applications and various algebraic structures defined via multiset can be found in [3], [6], [7], [8], [9].

Very recently, [4] introduced multigroups as a natural generalization of the concept of groups which differs from the earlier definition given in [2], and established some of its fundamental properties. The recent definition of multigroup which follows [5] is adopted for the results presented in this paper. The aim of this paper is to present the notion of order of an element with respect to multigroup and outline some of its related results.

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2. Preliminaries

Definition 2.1. A *multiset* (*mset*) A drawn from a crisp (ordinary) set X is represented by a count function C_A defined as $C_A : X \to D = \{0, 1, 2, \ldots\}$.

For $x \in X$, $C_A(x)$ denotes the number of times the element x in the mset A occurs. The representation of the mset A drawn from $X = \{x_1, x_2, \ldots, x_n\}$ is

$$[x_1, x_2, ...x_n]_{m_1, m_2, ...m_n}$$

such that x_i appears m_i $(i = 1, 2, ..., n)$ times in A.

Definition 2.2. An mset is called *regular* or *constant* if all its elements occur with the same multiplicity.

Definition 2.3. Let X be a group. A multiset A over X is called a *multigroup* over X if the count function A or C_A satisfies the following conditions.

(i) $C_A(xy) \geqslant C_A(x) \wedge C_A(y), \ \forall x, y \in X,$

(ii) $C_A(x^{-1}) \geqslant C_A(x), \ \forall x \in X.$

The set of all multigroups over X is denoted by $MG(X)$.

If $A \in MG(X)$, it follows that $C_A(x^{-1}) = C_A(x)$ and $C_A(e) \geqslant C_A(x)$.

Definition 2.4. Let $H \in MG(X)$. For any $x \in X$, xH and Hx defined by

$$C_{xH}(y) = C_H(x^{-1}y)$$

and

$$C_{Hx}(y) = C_H(yx^{-1}), \ \forall y \in X,$$

are respectively called the *left* and *right mcosets* of H in X.

Definition 2.5. Let $A \in MG(X)$. Then A is called *regular* if the count function A occurs with the same multiplicity. The set of all regular multigroups over X is denoted by $RMG(X)$.

Proposition 2.6. (cf. [4]) *Let $A \in MG(X)$. Then the following assertions are equivalent.*

(*i*) $C_A(xy) = C_A(yx), \ \forall x, y \in X.$
(*ii*) $C_A(xyx^{-1}) = C_A(y), \ \forall x, y \in X.$
(*iii*) $C_A(xyx^{-1}) \geqslant C_A(y), \ \forall x, y \in X.$
(*iv*) $C_A(xyx^{-1}) \leqslant C_A(y), \ \forall x, y \in X.$

Other definitions and facts one can find in [1].

3. Order of an element of a multigroup

Definition 3.7. Let $A \in MG(X)$ and $x \in X$. If there exists a positive integer n such that $C_A(x^n) = C_A(e)$, then the least such positive integer is called the *order of an element x with respect to A*. If no such n exists, x is said to be of *infinite order with respect to A*. The order of an element x with respect to A is denoted by $O_A(x)$.

Example 3.8. Let $X = (\mathbb{R}-\{0\}, \cdot)$ and $A = [1, -1]_{3,2}$. Then $C_A((-1)^2) = C_A(1)$. Therefore $O_A(-1) = 2$. But for any $x \in \mathbb{R} - \{1, 0, -1\}$, $\nexists\ n \in \mathbb{Z}^+$ such that $C_A(x^n) = C_A(1)$. Therefore $O_A(x) = \infty$, $\forall x \in \mathbb{R} - \{1, 0, -1\}$.

Equality of $O(x) = O(y)$ does not imply $O_A(x) = O_A(y)$, as shown in the below.

Example 3.9. Let $\{e, a, b, c\}$ be the Klein's 4-group and $A = [e, a, b, c]_{3,2,3,2}$. Clearly, $O(a) = O(b)$ but $O_A(a) = 2$ and $O_A(b) = 1$, since $C_A(b) = C_A(e)$.

Remark 3.10. If $H = \{x \in X \mid C_A(x) = C_A(e)\} \leqslant X$, then $O_A(x) = \hbar$, the order of x relative to H (i.e., the smallest positive integer n such that $x^n \in H$, if $\exists$ such a positive integer). In particular, if H is trivial subgroup $\{e\}$ of X, then $O_A(x) = O(x)$, the (classical) order of x in X.

Definition 3.11. Let $A \in MG(X)$. The *order of A* denoted by $O(A)$ is defined as $O(A) = \Sigma_{x\in X} C_A(x)$, i.e., the total number of all multiplicities of its element.

Proposition 3.12. *If $H \leqslant X$ and $A \in MG(X)$, then $O(A|H) \leqslant O(A)$, where $A|H$ means A restricted to H.*

Proof. Straightforward. □

Proposition 3.13. (Lagrange's theorem for RMG)
Let $H \leqslant X$, $A|H \in RMG(H)$ and $A \in RMG(X)$. Then $O(A|H)|O(A)$.

Proof. Let $O(A) = n$. By Proposition 2.6, we have $O(A|H) \leqslant n$. If $O(A|H) = n$, then the result is trivial. Now, we assume that $O(A|H) < n$. Let $O(A|H) = m$, $\forall x \in H$. Then if k is the count function of each left mcoset A in X, then $O(A) = C_{xA}(y) \cdot O(A|H)\ \forall y \in X$. By Lagrange's theorem for regular multigroup, $n|m$. Hence the proof. □

Example 3.14. Consider the subgroup $H = \{1, -1\}$ of $X = \{1, -1, i, -i\}$ such that $A = [1, -1, i, -i]_{2,2,2,2}$ and $A|H = [1, -1]_{2,2}$. Then $O(A) = 8$, $O(A|H) = 4$ and $C_{iA}(-i) = C_A(1) = 2$. Hence, $O(A|H)|O(A)$.

Corollary 3.15. *If $H \leqslant X$, $x \in H$ and $A|H \in RMG(H)$, then $O_{A|H}(x)|O(A|H)$.*

Proof. Since $A|H \in RMG(H)$, for some positive integer m we have $C_{A|H}(x^m) = C_{A|H}(e)$. Hence, $O_{A|H}(x) = m$. Now, H is a subgroup of X and $A|H \in RMG(H)$ such that $O(A|H) = n$. If for any $x \in H$, $r = C_{x(A|H)}(y) = C_{A|H}(x^{-1}y)\ \forall y \in H$, then $n = rm$. Hence $n|m$. □

Proposition 3.16. *Let $A \in MG(X)$. Then $O_A(x) = O_A(x^{-1})$.*

Proof. By definition, $O_A(x) = n$. So, $C_A(x^n) = C_A(e)$. Thus, $C_A((x^n)^{-1}) = C_A(e^{-1})$, i.e., $C_A((x^{-1})^n) = C_A(e)$, which implies $O_A(x^{-1}) \geqslant n$. Hence $m \geqslant n$.

Also, $O_A(x^{-1}) = m$ implies $C_A((x^{-1})^m) = C_A(e)$. So, $C_A((x^m)^{-1}) = C_A(e)$, i.e., $C_A(x^m) = C_A(e)$. Thus, $O_A(x) \geqslant m$. Hence $n \geqslant m$. Therefore, $n = m$. □

Proposition 3.17. *If $x \in X$ and $A \in MG(X)$ such that $O(A)$ is even, then $C_A(x^{O(A)}) = C_A(e)$.*

Proof. Let $O_A(x) = n$. Then $O(A) = m \cdot O_A(x)$, where

$$x^{O(A)} = x^m \cdot O_A(x) = (x^n)^m.$$

Then

$$C_A(x^{O(A)}) = C_A((X^n)^m) \geqslant C_A(x^n) = C_A(e).$$

Therefore, $C_A(x^{O(A)}) \geqslant C_A(e)$.

Since $A \in MG(X)$, then $C_A(e) \geqslant C_A(y)\ \forall y \in X$. So, $C_A(x^{O(A)}) \leqslant C_A(e)$. Hence, $C_A(x^{O(A)}) = C_A(e)$. □

Proposition 3.18. *Let $A \in MG(X)$ and $x \in X$. If there exists $m \in \mathbb{Z}^+$, such that $C_A(x^m) = C_A(e)$, then $O_A(x)|m$.*

Proof. Let $O_A(x) = n$. By division algorithm, there exists integers s and t such that $m = ns + t$, $0 \leqslant t < n$. Then

$$\begin{aligned} C_A(x^t) = C_A(x^{m-ns}) &= C_A(x^m(x^n)^{-s}) \geqslant C_A(x^m) \wedge C_A((x^n)^{-s}) \\ &= C_A(e) \wedge C_A((x^{ns})^{-1}) = C_A((x^{ns})^{-1}) \\ &= C_A(x^{ns}) = C_A((x^n)^s) \geqslant C_A(x^n) = C_A(e). \end{aligned}$$

Thus, $C_A(x^t) = C_A(e)$. Hence, $t = 0$ by minimality of n, i.e., $m = ns$. □

Proposition 3.19. *Let $A \in MG(X)$ and let $x, y \in X$ be such that $(O_A(x), O_A(y)) = 1$ and $xy = yx$. If $C_A(xy) = C_A(e)$, then $C_A(x) = C_A(y) = C_A(e)$.*

Proof. Let $O_A(x) = n$ and $O_A(y) = m$. Then

$$C_A(e) = C_A(xy) \leqslant C_A((xy)^m) = C_A(x^m y^m).$$

Hence, $C_A(x^m y^m) = C_A(e)$. Now,

$$C_A(x^m) = C_A(x^m y^m y^{-m}) \geqslant C_A(x^m y^m) \wedge C_A(y^{-m}) = C_A(e) \wedge C_A(e) = C_A(e).$$

Thus, $C_A(x^m) = C_A(y^m) = C_A(e)$. Therefore, $n|m$ by Proposition 3.18. But $(n, m) = 1$. Thus, $n = 1$ i.e., $C_A(x) = C_A(x^n) = C_A(e)$. Similarly, $C_A(y) = C_A(e)$. □

Proposition 3.20. *Let $A \in MG(X)$. Then $O_A(x^m) \leqslant O_A(x)$.*

Proof. By definition, $O_A(x) = n$ means $C_A(x^n) = C_A(e)$. Then $C_A((x^n)^m) = C_A(e^m)$, hence $C_A(x^{nm}) = C_A(e)$. So, $C_A((x^m)^n) = C_A(e)$, i.e, $O_A(x^m) \leqslant n$. Consequently, $O_A(x^m) \leqslant O_A(x)$. □

Proposition 3.21. *Let $A \in MG(X)$. Then $O_A(xyx^{-1}) \leqslant O_A(y)$.*

Proof. Let $O_A(xyx^{-1}) = m$ and $O_A(y) = n$. Then

$$\begin{aligned} C_A((xyx^{-1})^2) &= C_A((xyx^{-1})(xyx^{-1})) = C_A(xy(x^{-1}x)yx^{-1}) \\ &= C_A(x(ye)yx^{-1}) = C_A(xy^2x^{-1}). \end{aligned}$$

In general, $C_A((xyx^{-1})^n) = C_A(xy^nx^{-1}) \leqslant C_A(y^n) = C_A(e) = O_A(y)$. □

Remark 3.22. If $A \in MG(X)$, then $O_A(xyx^{-1}) = O_A(y)$.

Proposition 3.23. *Let $A \in MG(X)$ and $O_A(x) = n$, where $x \in X$. If $m \in \mathbb{Z}$ with $(m, n) = d$, then $O_A(x^m) = \frac{n}{d}$.*

Proof. Let $O_A(x^m) = t$. Now, for $\frac{m}{d} = k \in \mathbb{Z}^+$,

$$C_A((x^m)^{\frac{n}{d}}) = C_A(x^{nk}) \geqslant C_A(x^n) = C_A(e).$$

By Proposition 3.18, $t|(\frac{n}{d})$. Since $(m, n) = d$, then $\exists\, i, j \in \mathbb{Z}$ such that $ni+mj = d$. Therefore,

$$\begin{aligned} C_A(x^{td}) = C_A(x^{t(ni+mj)}) &\geqslant C_A((x^n)^{ti}) \wedge C_A(((x^m)^t)^j) \\ &\geqslant C_A(x^n) \wedge C_A((x^m)^t) \\ &\geqslant C_A(e) \wedge C_A(e) = C_A(e). \end{aligned}$$

Thus, $n|(\frac{t}{d})$ by Proposition 3.18, this implies $(\frac{n}{d})|t$, consequently $t = \frac{n}{d}$. □

Putting in the above Proposition $d = 1$ we obtain

Corollary 3.24. *Let $A \in MG(X)$ and $O_A(x) = n$, where $x \in X$. If $m \in \mathbb{Z}$ with $(m, n) = 1$, then $O_A(x^m) = O_A(x)$.*

Proposition 3.25. *Let $A \in MG(X)$ and $O_A(x) = n$, where $x \in X$. Then for all $i \equiv j (mod\ n)$, $i, j \in \mathbb{Z}$, we have $O_A(x^i) = O_A(x^j)$.*

Proof. Let $O_A(x^i) = t$ and $O_A(x^j) = s$. Assume $i = j + nk$ and $k \in \mathbb{Z}$. Then

$$\begin{aligned} C_A((x^i)^s) = C_A((x^{j+nk})^s) &\geqslant C_A((x^j)^s) \wedge C_A((x^n)^{ks}) \\ &\geqslant C_A(e) \wedge C_A(e) = C_A(e) \end{aligned}$$

implies $C_A((x^i)^s) = C_A(e)$. Therefore, $t|s$. Similarly, by $C_A((x^j)^t) = C_A(e)$ we obtain $s|t$. Thus, $t = s$. □

References

[1] **J.A. Awolola and A.M. Ibrahimm** *Some results on multigroups*, Quasigroups and Related Systems **24** (2016), 169 – 177.

[2] **M. Dresher and O. Ore**, *Theory of multigroups*, American J. Math. **60** (1938), 705 – 733.

[3] **K.P. Girish and J.J. Sunil**, *On multiset topolpgies*, Theory Appl. Math. Computer Sci. **2** (2013), 37 – 52.

[4] **Sk. Nazmul, P. Majumdar and S.K. Samanta**, *On multisets and multigroups*, Annals Fuzzy Math. Inform. **6** (2013), 643 – 656.

[5] **A. Rosenfeld**, *Fuzzy groups*, J. Math. Analysis and Appl. **35** (1971), 512 – 517.

[6] **D. Singh, A.M. Ibrahim, T. Yohanna and J.N. Singh**, *An overview of the application of multisets*, Novi Sad J. Math. **33** (2007), no. 2, 73 – 92.

[7] **D. Singh, A.M. Ibrahim, T. Yohanna and J.N. Singh**, *A systematization of fundamentals of multisets*, Lecturas Matematicas **29** (2008), 33 – 48.

[8] **A. Syropoulous**, *Mathematics of multisets*, Springer-Verlag Berlin Heidelberge, (2001), 347 – 358.

[9] **Y. Tella and S. Daniel**, *Symmetry groups under multiset perspective*, IOSR J. Math. **7** (2013) , no. 5, 47 – 52.

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