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Covering semigroups of topological n-ary semigroups

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Abstract. We construct a topology on the covering (enveloping) semigroup of an n-ary topological semigroup, and study the properties of the constructed topology. Conditions under which this covering semigroup is a topological semigroup are obtained too.

1. Introduction

An n-ary semigroup $(G, [\,])$ with a topology τ is called a ***topological n-ary semigroup*** if (G,τ) is a topological space such that the n-ary operation $[\,]$ defined on G is continuous (in all variables together). Such n-ary semigroups and groups were studied by many authors in various directions. Čupona [4] proved that each topological n-ary group $(G,[\,])$ can be embedded into some topological (binary) group called the ***universal covering group*** of $(G,[\,])$. Moreover, on this universal covering group G^* of $(G,[\,])$ one can define a topology τ such that G^*, endowed with this topology, is a topological group (cf. [4]). The base of this topology is formed by sets of the form $U_1 \cdot U_2 \cdot \ldots \cdot U_k$, where U_i, $i = 1, 2, \ldots, k < n$ are open subsets of G. Crombez and Six [3] showed that each topological n-ary group is homeomorphic to some topological group. Stronger result was obtained by Endres [8]: a topological n-ary group and a normal subgroup of index $n-1$ of the corresponding covering group are homeomorphic. On the other hand, any topological n-ary group is uniquely determined by some topological group and some its homeomorphism (cf. [14]). Hence topological properties of topological groups may be moved to topological n-ary groups and conversely.

In the case of n-ary semigroups the situation is more complicated. Similarly as in case of n-ary groups, for any topological n-ary semigroup can be constructed the covering semigroup. Connections between the topology of this covering semigroup and the topology of its initial an n-ary semigroup are described in [10] (see also [7] and [11]). In some cases an n-ary semigroup with a locally compact topology can be topologically embedded into a locally compact binary group as an open set (for deteils see [10]). If additionaly, this n-ary semigroup is cancellative and commutative, and all its inner translations (shifts), i.e., mappings of the form $\varphi_i(x) = [a_1 ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n]$, where $a_1, \ldots, a_n$ are fixed elements,

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are both continuous and open, then this n-ary semigroupis can be topologically embedded into a locally compact n-ary group as an open n-ary subsemigroup [12].

In this paper, the construction of a free covering semigroup of a topological n-ary semigroup presented in [6] is generalized to an arbitrary covering semigroup. On this covering semigroup is constructed a topology with the following properties: the right and left shifts are continuous mappings (Theorem 2.2); if an n-ary operation is continuous in all variables, then this n-ary semigroup is an open subspace of the corresponding covering semigroup (Theorem 3.1). In Theorem 3.3 are given sufficient conditions under which a Hausdorff topology of an n-ary semigroup can be extended to a Hausdorff topology of its covering semigroup. An explicit description of a base of a topology of an n-ary topological semigroup with some open translations is presented in Theorem 3.7.

2. Topologies on covering semigroups

Let $(G,[\,])$ be an n-ary semigroup with $n>2$. The symbol $[x_1,\ldots,x_s]$ means that $s=k(n-1)+1$ and the operation $[\,]$ is applied k times to the sequence $x_1,\ldots,x_s$. Consequently, $[x]$ means x.

By G^k we denote the Cartesian product of G. If G is a subset of a semigroup $(S,\cdot)$, then by $G^{(k)}$ we denote the set $G{\cdot}G{\cdot}\ldots{\cdot}G$ (k times).

A binary semigroup $(S,\cdot)$ is a *covering* (*enveloping*) *semigroup* of an n-ary semigroup $(G,[\,])$ if S is generated by the set G and $[x_1,x_2,\ldots,x_n]=x_1\cdot x_2\cdot\ldots\cdot x_n$ for all $x_1,x_2,\ldots,x_n\in G$. If additionally, the sets G, $G^{(2)}$, $G^{(3)},\ldots,G^{(n-1)}$ are disjoint and their union gives S, then $(S,\cdot)$ is called the *universal covering semigroup*. For each n-ary semigroup there exists such universal covering semigroup [5].

Below we describe connections between the the topology of an n-ary semigroup and the topology of its free covering semigroup. For this we use the construction of free covering semigroup proposed in [5] and the following proposition from [2] (Chapter 1, §3, Proposition 6).

Proposition 2.1. *Let ρ be an equivalence relation on a topological space X. Then a map f of X/ρ into a topological space Y is continuous if and only if $f\circ\varphi$, where φ is a cannonical map of X onto X/ρ, is continuous on X.* □

Let $(S,\cdot)$ be a covering semigroup of an n-ary semigroup $(G,[\,])$. Consider the free semigroup F over the set G. Then $F=\bigcup\limits_{k=1}^{\infty}G^k$ and the operation on F is defined by

$$(x_1,\ldots,x_p)\cdot(y_1,\ldots,y_m)=(x_1,\ldots,x_p,y_1,\ldots,y_m). \qquad (1)$$

For any elements $\alpha=(x_1,x_2,\ldots,x_p),\beta=(y_1,y_2,\ldots,y_m)$ from F we define the relation Ω by putting:

$$\alpha\Omega\beta \iff x_1\cdot x_2\cdot\ldots\cdot x_p=y_1\cdot y_2\cdot\ldots\cdot y_m. \qquad (2)$$

Such defined relation is a congruence on F. Thus, the set $\overline{F} = F/_{\Omega} = \{\overline{\alpha} : \alpha \in F\}$, where $\overline{\alpha} = \{\beta \in F : \alpha\Omega\beta\}$, with the operation $\overline{\alpha} * \overline{\beta} = \overline{\alpha\beta}$ is a semigroup. $\varphi : \alpha \mapsto \overline{\alpha}$ is a canonical mapping from F onto $\overline{F}$. Moreover, the mapping $\pi : \overline{\alpha} \mapsto x_1 \cdot x_2 \cdot \ldots \cdot x_p$ is an isomorphism of semigroups $(\overline{F}, *)$ and $(S, \cdot)$. Because $\pi(\varphi(G^i)) = G^{(i)}$ for $i = 1, 2, \ldots, n-1$ and the union of all $\varphi(G^i)$ covers $\overline{F}$, then, in the case when $(S, \cdot)$ is the universal covering semigroup of $(G, [\,])$, the sets $\varphi(G^i)$ are pairwise disjoint. So, the semigroups $(\overline{F}, *)$ and $(S, \cdot)$ can be identified. Also can be assumed that $\varphi(G^i) = G^{(i)}$ for $i = 1, 2, \ldots, n-1$.

Let τ be a topology on G, $\tau_k = \tau \times \cdots \times \tau$ (k times) – a topology on G^k. By τ_F we denote this topology on F which is the union of all topologies τ_k. Then, obviously, the operation (1) is continuous in the topology τ_F. The quotient topology (with respect to the relation Ω) of the topology τ_F is denoted by $\overline{\tau}$. It is the strongest topology on $\overline{F}$ for which the mapping φ is continuous.

Theorem 2.2. *Let $(G, [\,])$ be an n-ary semigroup with a free covering semigroup F and τ be a topology on G. Then each left and each right shift on $(\overline{F}, \overline{\tau})$ is a continuous mapping. Each set $\overline{F}_i = \varphi(G^i)$, $i = 1, 2, \ldots, n-1$, is open. If $(S, \cdot)$ is the universal covering semigroup of $(G, [\,])$, then each set $\overline{F}_i$ is open-closed.*

Proof. Let R_a and $r_{\overline{a}}$ be right shifts in F and $\overline{F}$, respectively. Then $\varphi \circ R_a = r_{\overline{a}} \circ \varphi$. Since φ and R_a are continuous, by Proposition 2.1, $r_{\overline{a}}$ is continuous too. Analogously we can prove the continuity of left shifts.

The second statement of the theorem follows from the fact that the sets $\varphi^{-1}(\overline{F_i}) = \bigcup_{k=0}^{\infty} G^{k(n-1)+i} \in \tau_F$ are saturated with respect to the relation Ω.

In the case when $(S, \cdot)$ is a universal covering semigroup of $(G, [\,])$ the open sets $\overline{F_i}$, $i = 1, \ldots, n-1$, form a partition of $\overline{F}$ and, therefore, are open-closed. □

We will need also the following result proved in [9].

Proposition 2.3. *Let S be a locally compact, σ-compact Hausdorff topological semigroup and θ be a closed congruence on S. Then S/θ is a topological semigroup.* □

3. Topologies on universal covering semigroups

An n-ary semigroup $(G, [\,])$ with a topology τ is called a *topological n-ary semigroup* if (G, τ) is a topological space such that the n-ary operation $[\,]$ is continuous (in all variables together).

Theorem 3.1. *If $(G, [\,], \tau)$ is a topological n-ary semigroup, then topologies $\overline{\tau}$ and τ coincide on G.*

Proof. Let $U \in \overline{\tau}$, $U \subset G$. Then $\varphi^{-1}(U) \in \tau_F$. Thus $U = \varphi^{-1}(U) \cap G \in \tau$.

Let now $U \in \tau$ and $\alpha = (a_1, \ldots, a_p) \in \varphi^{-1}(U)$. Then, $\overline{a}_1 * \ldots * \overline{a}_p \in U$, where $p = k(n-1) + 1$, and consequently, $[a_1, \ldots, a_p] = \overline{a}_1 * \ldots * \overline{a}_p \in U$. Since the operation [ ] is continuous in all variables, in the topology τ there are the neighborhoods $V_1,\dots,V_p$ of points $\overline{a}_1,\dots,\overline{a}_p$ such that $[x_1,\dots,x_p]\in U$ for all $x_i\in V_i$, $i=1,\dots,p$. Therefore, $\varphi(x_1,\dots,x_p)=\overline{x}_1*\dots*\overline{x}_p=[x_1,\dots,x_p]\in U$. Consequently, $\varphi^{-1}(U)\supset V_1\times\dots\times V_p\in\tau_F$. So $\varphi^{-1}(U)\in\tau_F$. This together with saturation of $\varphi^{-1}(U)$ gives $U\in\overline{\tau}$. □

Example 3.2. Consider on the real interval $G=(1,+\infty)$ the ternary operation $[x_1,x_2,x_3]=x_1+x_2+x_3$ and the topology τ which is a union on the topology τ_1 on $(1,2]$, the discrete topology on $(2,3]$ and the usual topology on $(3,+\infty)$, where the sets $(a,b]$ with $1\leqslant a\leqslant b\leqslant 2$ form the basis of the topology τ_1. Such defined ternary operation is continuous in all variables together and the semigroup $(G,+)$ is the covering semigroup for $(G,[\ ])$. The shift $x\mapsto x+1.5$ is not a continuous map, since the preimage of the open set $\{3\}$ is not an open set. So, on the set G the topologies $\overline{\tau}$ and τ are different.

Note that the topology $\overline{\tau}$ is the union of the usual topology on $(3,+\infty)$ and the topology on $(1,3]$ with the base of the form $(a,b]$, where $1\leqslant a\leqslant b\leqslant 3$.

Consider the set $S=G\cup G_1$, where $G_1=(2,+\infty)\times\{0\}$, with the commutative binary operation $*$ defined for $x,y\in G$ in the following way:

$$\begin{aligned} x*y&=(x+y,0),\\ x*(y,0)&=(y,0)*x=x+y,\\ (x,0)*(y,0)&=(x+y,0). \end{aligned}$$

It is easy to verify that $(S,*)$ a commutative universal covering semigroup of an n-ary semigroup $(G,[\ ])$. On G the topology $\overline{\tau}$ coincides with the topology τ, but the restriction of $\overline{\tau}$ to G_1 gives the topology with the base formed by sets $(a,b]\times\{0\}$ and $(c,d)\times\{0\}$, where $2\leqslant a\leqslant b\leqslant 4\leqslant c\leqslant d$.

Theorem 3.3. *If in the universal covering semigroup $(S,\cdot)$ of an n-ary semigroup $(G,[\ ])$ with the Hausdorff topology τ for any $x_1,\dots,x_i,y_1,\dots,y_i\in G$ such that $x_1\cdot\ldots\cdot x_i\neq y_1\cdot\ldots\cdot y_i$, where $1\leqslant i<n$, there are $z_{i+1},\dots,z_n\in G$ such that*

$$x_1\cdot\ldots\cdot x_i\cdot z_{i+1}\cdot\ldots\cdot z_n\neq y_1\cdot\ldots\cdot y_i\cdot z_{i+1}\cdot\ldots\cdot z_n \quad \text{or}$$
$$z_{i+1}\cdot\ldots\cdot z_n\cdot x_1\cdot\ldots\cdot x_i\neq z_{i+1}\cdot\ldots\cdot z_n\cdot y_1\cdot\ldots\cdot y_i,$$

then the topology $\overline{\tau}$ on $\overline{F}$ is the Hausdorff topology, too.

Proof. Consider the first case when for some $x_1,\dots,x_i,y_1,\dots,y_i,z_{i+1},\dots z_n\in G$ we have $\widetilde{x}=x_1\cdot\ldots\cdot x_i\neq y_1\cdot\ldots\cdot y_i=\widetilde{y}$ and $x=x_1\cdot\ldots\cdot x_i\cdot z_{i+1}\cdot\ldots\cdot z_n\neq y_1\cdot\ldots\cdot y_i\cdot z_{i+1}\cdot\ldots\cdot z_n=y$. τ is the Hausdorff topology, so there are neighborhoods U_x and U_y of x and y such that $U_x\cap U_y=\emptyset$. Since shifts in $(\overline{F},\overline{\tau})$ are continuous and $x=\widetilde{x}\cdot\widetilde{z}$, $y=\widetilde{y}\cdot\widetilde{z}$ for $\widetilde{z}=z_{i+1}\cdot\ldots\cdot z_n$, there are neighborhoods W_x and W_y of points $\widetilde{x}$ and $\widetilde{y}$ such that $W_x\cdot\widetilde{z}\subset U_x$ and $W_y\cdot\widetilde{z}\subset U_y$. So, $W_x\cap W_y=\emptyset$. Thus $\overline{\tau}$ is the Hausdorff topology.

The second case can be proved analogously. □

Corollary 3.4. *If the universal covering semigroup of an n-ary semigroup $(G,[\,])$ with the Hausdorff topology τ is left or right cancellative, then the topology $\overline{\tau}$ on $\overline{F}$ is the Hausdorff topology.* □

Theorem 3.5. *If the universal covering semigroup $(S,\cdot)$ of a topological n-ary semigroup $(G,[\,])$ with the Hausdorff topology τ has at least one left or right cancellable element, then the congruence Ω is a closed subset in a topological space $(F\times F,\tau_F\times\tau_F)$.*

Proof. Suppose that in $(F\times F,\tau_F\times\tau_F)$ the sequence $(\alpha_\xi,\beta_\xi)_{\xi\in A}\in\Omega$ converges to (α,β). This means that in the topological space (F,τ_F) the sequences $(\alpha_\xi)_{\xi\in A}$ and $(\beta_\xi)_{\xi\in A}$ converge to α and β, respectively.

Let $\alpha=(x_1,\ldots,x_p)\in G^p$, $\beta=(y_1,\ldots,y_q)\in G^q$. Since G^p, G^q are disjoint open-closed subsets in (F,τ_F), there is an index $\xi_0\in A$ such that $\alpha_\xi=(x_1^\xi,\ldots,x_p^\xi)\in G^p$ and $\beta_\xi=(y_1^\xi,\ldots,y_q^\xi)\in G^q$ for all $\xi>\xi_0$. Therefore, for $\xi>\xi_0$ we have $x_1^\xi\cdot\ldots\cdot x_p^\xi=y_1^\xi\cdot\ldots\cdot y_q^\xi$. Consequently,

$$a^f\cdot x_1^\xi\cdot\ldots\cdot x_p^\xi=a^f\cdot y_1^\xi\cdot\ldots\cdot y_q^\xi \tag{3}$$

for any left cancellable element $a\in S$ and all natural f.

Obviously, $a=a_1\cdot\ldots\cdot a_k$ for some $a_1,\ldots,a_k\in G$ and $k<n$. Moreover, for each natural f such that $fk\geqslant n$ there is a natural r satisfying the condition $r(n-1)+1\leqslant fk+p<(r+1)(n-1)+1$. Thus $fk+p-s=r(n-1)+1$ for some $0\leqslant s<k$. Consequently,

$$a_1\cdot\ldots a_s\cdot[a_{s+1},\ldots,a_k,\underbrace{\underbrace{a_1,\ldots,a_k},\ldots,\underbrace{a_1,\ldots,a_k}}_{f-1\ times},x_1^\xi,\ldots,x_p^\xi]=$$

$$a_1\cdot\ldots a_s\cdot[a_{s+1},\ldots,a_k,\underbrace{\underbrace{a_1,\ldots,a_k},\ldots,\underbrace{a_1,\ldots,a_k}}_{f-1\ times},y_1^\xi,\ldots,y_p^\xi].$$

By previous results, $\overline{\tau}$ is the Hausdorff topology which on G coincides with τ and each left shift in $(\overline{F},\overline{\tau})$ is a continuous mapping. So, if in (G,τ) the sequence $(x_i^\xi)_{\xi\in A}$ converge to x_i and $(y_i^\xi)_{\xi\in A}$ converge to y_i, then (3) implies $a\cdot x_1\cdot\ldots\cdot x_p=a\cdot y_1\cdot\ldots\cdot y_q$, which, by the cancellativity of a, gives $x_1\cdot\ldots\cdot x_p=y_1\cdot\ldots\cdot y_q$. Thus $(\alpha,\beta)\in\Omega$ and Ω is a closed subset of $(F\times F,\tau_F\times\tau_F)$.

For a right cancellable element the proof is similar. □

Theorem 3.6. *If the universal covering semigroup $(S,\cdot)$ of a topological n-ary semigroup $(G,[\,])$ with the locally compact and σ-compact Hausdorff topology τ has at least one left or right cancellable element, then $(\overline{F},*)$ is a topological semigroup with respect to the topology $\overline{\tau}$.*

Proof. Note that the topology τ_F on F is a locally compact, σ-compact, and the congruence Ω is a closed subset of F. Then, by Proposition 2.3, $(\overline{F},*,\overline{\tau})$ is a topological semigroup. □

Theorem 3.7. *Let in a topological n-ary semigroup $(G,[\,],\tau)$ for certain $1\leqslant p<n$ all translations $x\mapsto[c_1,\ldots,c_p,x,c_{p+1}\ldots,c_{n-1}]$ be continuous. If the universal covering semigroup $(S,\cdot)$ of $(G,[\,])$ is cancellative, then $(\overline{F},*,\overline{\tau})$ is a topological semigroup, G is an open-closed subset in $\overline{F}$ and the family*

$$\mathcal{B}=\{A_1\cdot\ldots\cdot A_k : A_1,\ldots,A_k\in\tau,\ k=1,\ldots,n-1\}$$

forms the base of the topology $\overline{\tau}$.

Proof. Let $A_1,\ldots,A_k$ be open sets in τ. We will show that the set $A_1\cdot\ldots\cdot A_k$ is open in $\overline{\tau}$.

Let $a\in G$, $a_1\in A_1,\ldots,a_k\in A_k$. Then

$$[\overset{(l)}{a},a_1,\ldots,a_{i-1},A_i,a_{i+1},\ldots,a_k,\overset{(n-k-l)}{a}]\subset[\overset{(l)}{a},A_1,\ldots,A_k,\overset{(n-k-l)}{a}]$$

for all $k+l\leqslant n$, $i\leqslant k$ and $l+i=p+1$, where $\overset{(s)}{a}$ means the sequence $a,\ldots,a$ (s times). By hypothesis, the set $[\overset{(l)}{a},a_1,\ldots,a_{i-1},A_i,a_{i+1},\ldots,a_k,\overset{(n-k-l)}{a}]$ is open in G. Since

$$[\overset{(l)}{a},A_1,\ldots,A_k,\overset{(n-k-l)}{a}]=\bigcup_{\substack{i=1\\ a_j\in A_j}}^{k}[\overset{(l)}{a},a_1,\ldots,a_{i-1},A_i,a_{i+1},\ldots,a_k,\overset{(n-k-l)}{a}],$$

the set $[\overset{(l)}{a},A_1,\ldots,A_k,\overset{(n-k-l)}{a}]$ also is open in G.

As was noted earlier, $(\overline{F},*)$ as a semigroup isomorphic to $(S,\cdot)$, can be identified with $(S,\cdot)$ and treated as a cancellative semigroup.

Consider the translation $\lambda:\overline{F}\to\overline{F}$ defined by $\lambda(x)=a^pxa^{n-p-1}$. We have

$$\lambda^{-1}([\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}])=A_1\cdot\ldots\cdot A_k. \tag{4}$$

Indeed, if $x\in\lambda^{-1}([\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}])$, then

$$\lambda(x)=a^pxa^{n-p-1}\in[\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}]=a^p\cdot A_1\cdot\ldots\cdot A_k\cdot a^{n-p-1}=a^pya^{n-p-1}$$

for some $y\in A_1\cdot\ldots\cdot A_k$, which, by cancellativity, implies $x=y$. So, $x\in A_1\cdot\ldots\cdot A_k$.

On the other hand, if $x\in A_1\cdot\ldots\cdot A_k$, then

$$a^pxa^{n-p-1}\in a^p\cdot A_1\cdot\ldots\cdot A_k\cdot a^{n-p-1}=[\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}].$$

Thus $x\in\lambda^{-1}([\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}])$. This completes the proof of (4).

The set $[\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}]$ is open in G, hence, by Theorem 3.1, it is open in $(\overline{F},\overline{\tau})$. By Theorem 2.2, the mapping λ is continuous and therefore $A_1\cdot\ldots\cdot A_k=\lambda^{-1}([\overset{(p)}{a},A_1\cdot\ldots\cdot A_k,\overset{(n-p-1)}{a}])\in\overline{\tau}$.

If $U\subset G^{(k)}$, $U\in\overline{\tau}$ and $a_1,\ldots,a_k\in G$ such that $a_1\cdot\ldots\cdot a_k\in U$, then $W=\varphi^{-1}(U)\in\overline{\tau}$, where $\varphi(x)=a_1\cdot\ldots\cdot a_k\cdot x$ is a left shift in $\overline{F}$. Consequently $W\in\tau$, because $W\subset G$. So, for any $a\in G$, the set

$$[\overset{(n-k+p)}{a}, a_1, \ldots, a_{k-1}, W, \overset{(n-p-1)}{a}] = [\overset{(p-1)}{a}, [\overset{(n-k+1)}{a}, a_1, \ldots, a_{k-1}], W, \overset{(n-p-1)}{a}]$$

is an open subset of G.

Since in (G, τ) the n-ary operation $[\,]$ is continuous in all variables, there exist the family of open neighborhoods $U_1, \ldots, U_k$ of the points $a_1, \ldots, a_k$, respectively, such that

$$[\overset{(n-k+p)}{a}, U_1, \ldots, U_k, \overset{(n-p-1)}{a}] \subset [\overset{(n-k+p)}{a}, a_1, \ldots, a_{k-1}, W, \overset{(n-p-1)}{a}].$$

Thus, in $\overline{F}$, we have

$$a^{n-k+p} \cdot U_1 \cdot \ldots \cdot U_k \cdot a^{n-p-1} \subset a^{n-k+p} \cdot a_1 \cdot \ldots \cdot a_{k-1} \cdot W \cdot a^{n-p-1}.$$

Because $a_1 \cdot \ldots \cdot a_{k-1} \cdot W \subset U$, the last implies

$$a^{n-k+p} \cdot U_1 \cdot \ldots \cdot U_k \cdot a^{n-p-1} \subset a^{n-k+p} \cdot U \cdot a^{n-p-1}.$$

This, in view of the cancellativity, gives $U_1 \cdot \ldots \cdot U_k \subset U$.

By virtue of the arbitrariness of the point $a_1 \cdot \ldots \cdot a_k \in U$, we conclude that the family $\mathcal{B}$ is a base of the topology $\overline{\tau}$ on $\overline{F}$.

Now we will show that the binary operation defined in $\overline{F}$ is continuous in the topology $\overline{\tau}$. Let $g = s \cdot t$ for some $s = a_1 \cdot \ldots \cdot a_i$, $t = b_1 \cdot \ldots \cdot b_j$, where $a_1, \ldots, a_i, b_1, \ldots, b_j \in G$ and $1 \leqslant i, j < n$. If $C \in \mathcal{B}$ and $g \in C$, then $C = C_1 \cdot \ldots \cdot C_k$ for some $k < n$ and $\emptyset \neq C_i \in \tau$. Let $g = c_1 \cdot \ldots \cdot c_k$ for some $c_i \in C_i$. If $i + j < n$, then $s \cdot t = a_1 \cdot \ldots \cdot a_i \cdot b_1 \cdot \ldots \cdot b_j = c_1 \cdot \ldots \cdot c_k$. Thus $i + j = k$.

From the cancellativity of the binary operation in $\overline{F}$ and the continuity of the n-ary operation $[\,]$, we conclude that there exist open neighborhoods $A_1, \ldots, A_i$ of the points $a_1, \ldots, a_i$, respectively, and open neighborhoods $B_1, \ldots, B_j$ of the points $b_1, \ldots, b_j$ such that $A_1 \cdot \ldots \cdot A_i \cdot B_1 \cdot \ldots \cdot B_j \subset C_1 \cdot \ldots \cdot C_k = C$. Since $A = A_1 \cdot \ldots \cdot A_i$ and $B = B_1 \cdot \ldots \cdot B_j$ are open neighborhoods of the points s, t, respectively, the last inclusion implies $A \cdot B \subset C$.

In the case $i + j \geqslant n$ we have $c_1 \cdot \ldots \cdot c_k = a_1 \cdot \ldots \cdot a_i \cdot b_1 \cdot \ldots \cdot b_j = a \cdot b_{n-i+1} \cdot \ldots \cdot b_j$ for $a = [a_1, \ldots, a_i, b_1, \ldots, b_{n-i}]$. So, as above, we conclude that $k = i + j - n$ and there are open neighborhoods $D, B_{n-i+1}, \ldots, B_j$ of the points $a, b_{n-i+1}, \ldots, b_j$, respectively, such that $D \cdot B_{n-i+1} \cdot \ldots \cdot B_j \subset C_1 \cdot \ldots \cdot C_k = C$. Since the n-ary operation $[\,]$ is continuous, then there are open neighborhoods $A_1, \ldots, A_i$ of the points $a_1, \ldots, a_i$ and open neighborhoods $B_1, \ldots, B_{n-i}$ of the points $b_1, \ldots, b_{n-i}$ such that $[A_1, \ldots, A_i, B_1, \ldots, B_{n-i}] \subset D$. Thus, for $A = A_1 \cdot \ldots \cdot A_i$, $B = B_1 \cdot \ldots \cdot B_j$ we have $A, B, \in \mathcal{B}$, $A \cdot B \subset C$ and $s \in A$, $t \in B$. This proves that the binary multiplication defined in $\overline{F}$ is continuous in the topology $\overline{\tau}$. □

Corollary 3.8. *If $(G, [\,], \tau)$ is a topological n-ary group, then its universal covering group $(\overline{F}, *)$ is a topological group with the topology $\overline{\tau}$.*

The proof follows immediately from the preceding theorem and the results of [4], where is proved that the operation of taking inverse element is continous if the family $\mathcal{B}$ is a base of the corresponding topology.

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