On bi-bases of a semigroup

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Abstract. Based on the results of bi-ideals generated by a non-empty subset of a semigroup S, we introduce the concept which is called bi-bases of the semigroup S. Using the quasi-order defined by the principal bi-ideals of S, we give a characterization when a non-empty subset of S is a bi-base of S.

1. Preliminaries

Let S be a semigroup. A subset A of the semigroup S is called a *two-sided base* (or simply *base*) of S if it satisfies the following two conditions:

(i) $S = A \cup SA \cup AS \cup SAS$;

(ii) if B is a subset of A such that $S = B \cup SB \cup BS \cup SBS$, then $B = A$.

This notion was first introduced and studied by I. Fabrici [3]. In fact, using the quasi-order defined by principal two-sided ideals of S, the author gave a characterization when a non-empty subset of S is a base of S. Moreover, the structure of semigroups containing two-sided bases was described. Indeed, using the concepts of left ideals and right ideals generated by a non-empty set, the concepts of left bases and right bases of a semigroup were introduced by T. Tamura before the concept of two-sided bases (see [7]). In [2], I. Fabrici studied the structure of a semigroup containing one-sided bases. In [4], I. Fabrici and T. Kepka showed that there is a relation between bases and maximal ideals of a semigroup. The results obtaind by I. Fabrici [3] have been extended to ordered semigroups by T. Changphas and P. Summaprab (see [1]). As in the line of I. Fabrici ([3], [2]) mentioned before, the main purpose of this paper is to introduce the concept which is called bi-bases of a semigroup. We also define the quasi-order using principal bi-ideals of S, and give a characterization when a non-empty subset of S is a bi-base of S.

Let S be a semigroup, and A, B non-empty subsets of S. The set product AB of A and B is defined to be the set of all elements ab with a in A and b in B. That is

$$AB = \{ab \mid a \in A, b \in B\}.$$

For $a \in S$, we write Ba for $B\{a\}$, and similarly for aB.

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A subsemigroup B of a semigroup S is called a *bi-ideal* ([5], [6]) of S if

$$BSB \subseteq B.$$

This notion generalizes the notion of one-sided ideals and two-sided ideals of a semigroup.

Let S be a semigroup, and B_i a bi-ideal of S for each i in an indexed set I. It is known that if $\bigcap_{i \in I} B_i \neq \emptyset$, then $\bigcap_{i \in I} B_i$ is a bi-ideal of S. Moreover, for a non-empty subset A of S, the intersection of all bi-ideals of S containing A, denoted by $(A)_b$, is the smallest bi-ideal of S containing A. And it is of the form

$$(A)_b = A \cup AA \cup ASA.$$

In particular, for $A = \{a\}$, we write $(\{a\})_b$ by $(a)_b$ (see [6]).

2. Main Results

We begin this section with the following definition of bi-bases of a semigroup.

Definition 2.1. Let S be a semigroup. A subset B of S is called a *bi-base* of S if it satisfies the following two conditions:

(i) $S = (B)_b$ (i.e. $S = B \cup BB \cup BSB$);

(ii) if A is a subset of B such that $S = (A)_b$, then $A = B$.

Example 2.2. Let $S = \{r, s, t, u\}$ be a semigroup with the binary operation defined by:

$\cdot$	r	s	t	u
r	r	s	r	r
s	s	r	s	s
t	r	s	t	u
u	r	s	u	t

We have that the bi-bases of S are: $B_1 = \{t\}$ and $B_2 = \{u\}$.

Example 2.3. Let $S = \{p, q, r, s\}$ be a semigroup with the binary operation defined by:

$\cdot$	p	q	r	s
p	p	p	p	p
q	p	p	p	p
r	p	p	q	q
s	p	p	q	q

It is a routine matter to check that S has only one bi-base: $B = \{r, s\}$.

Example 2.4. Let $S = \{a, b, c, d, x, y\}$ be a semigroup with the binary operation defined by:

$\cdot$	a	b	c	d	x	y
a	a	b	c	d	x	y
b	b	c	a	y	d	x
c	c	a	d	x	y	d
d	d	x	y	a	b	c
x	x	y	d	c	a	b
y	y	d	x	b	c	a

We have that the singleton sets consisting of an element of S are bi-bases of S.

First, we have the following useful lemma:

Lemma 2.5. *Let B be a bi-base of a semigroup S, and $a, b \in B$. If $a \in bb \cup bSb$, then $a = b$.*

Proof. Assume that $a \in bb \cup bSb$, and suppose that $a \neq b$. We consider

$$A = B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$, $b \in A$. We will show that $(A)_b = S$. Clearly, $(A)_b \subseteq S$. Let $x \in S$. Then, by $(B)_b = S$, we have $x \in B \cup BB \cup BSB$. There are three cases to consider:

CASE 1: $x \in B$.

SUBCASE 1.1: $x \neq a$. Then $x \in B \setminus \{a\} = A \subseteq (A)_b$.

SUBCASE 1.2: $x = a$. By assumption,

$$x = a \in bb \cup bSb \subseteq AA \cup ASA \subseteq (A)_b.$$

CASE 2: $x \in BB$. Then $x = b_1 b_2$ for some $b_1, b_2 \in B$.

SUBCASE 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned} x = b_1 b_2 &\in (bb \cup bSb)(bb \cup bSb) = bbbb \cup bbbSb \cup bSbbb \cup bSbbSb \\ &\subseteq AAAA \cup AAASA \cup ASAAA \cup ASAASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1 b_2 &\in (B \setminus \{a\})(bb \cup bSb) = (B \setminus \{a\})bb \cup (B \setminus \{a\})bSb \\ &\subseteq AAA \cup AASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1 b_2 &\in (bb \cup bSb)(B \setminus \{a\}) = bb(B \setminus \{a\}) \cup bSb(B \setminus \{a\}) \\ &\subseteq AAA \cup ASAA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$,

$$x = b_1 b_2 \in (B \setminus \{a\})(B \setminus \{a\}) = AA \subseteq (A)_b.$$

CASE 3: $x \in BSB$. Then $x = b_3 s b_4$ for some $b_3, b_4 \in B$ and $s \in S$.
SUBCASE 3.1: $b_3 = a$ and $b_4 = a$. By assumption,

$$\begin{aligned} x = b_3 s b_4 &\in (bb \cup bSb)S(bb \cup bSb) = bbSbb \cup bbSbSb \cup bSbSbb \cup bSbSbSb \\ &\subseteq AASAA \cup AASASA \cup ASASAA \cup ASASASA \\ &\subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3 s b_3 &\in (B \setminus \{a\})S(bb \cup bSb) = (B \setminus \{a\})Sbb \cup (B \setminus \{a\})SbSb \\ &\subseteq ASAA \cup ASASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3 s b_4 &\in (bb \cup bSb)S(B \setminus \{a\}) = bbS(B \setminus \{a\}) \cup bSbS(B \setminus \{a\}) \\ &\subseteq AASA \cup ASASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_3 s b_4 \in (B \setminus \{a\})S(B \setminus \{a\}) = ASA \subseteq (A)_b.$$

Hence, $(A)_b = S$. And this is a contradiction. Thus $a = b$. □

Lemma 2.6. *Let B be a bi-base of a semigroup S. Let $a, b, c \in B$. If $a \in cb \cup cSb$, then $a = b$ or $a = c$.*

Proof. Assume that $a \in cb \cup cSb$, and suppose that $a \neq b$ and $a \neq c$. We set

$$A = B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(A)_b = S$. Clearly, $(A)_b \subseteq S$. Let $x \in S$. By $(B)_b = S$, $x \in B \cup BB \cup BSB$.
We consider three cases:
CASE 1: $x \in B$.
SUBCASE 1.1: $x \neq a$. Then $x \in B \setminus \{a\} = A \subseteq (A)_b$.
SUBCASE 1.2: $x = a$. By assumption, $x = a \in cb \cup cSb \subseteq AA \cup ASA \subseteq (A)_b$.
CASE 2: $x \in BB$. Then $x = b_1 b_2$ for some $b_1, b_2 \in B$.
SUBCASE 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned} x = b_1 b_2 &\in (cb \cup cSb)(cb \cup cSb) = cbcb \cup cbcSb \cup cSbcb \cup cSbcSb \\ &\subseteq AAAA \cup AAASA \cup ASAAA \cup ASAASA \\ &\subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1b_2 &\in (B \setminus \{a\})(cb \cup cSb) = (B \setminus \{a\})cb \cup (B \setminus \{a\})cSb \\ &\subseteq AAA \cup AASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1b_2 &\in (cb \cup cSb)(B \setminus \{a\}) = cb(B \setminus \{a\}) \cup cSb(B \setminus \{a\}) \\ &\subseteq AAA \cup ASAA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_1b_2 \in (B \setminus \{a\})(B \setminus \{a\}) = AA \subseteq (A)_b.$$

CASE 3: $x \in BSB$. Then $x = b_3sb_4$ for some $b_3, b_4 \in B$ and $s \in S$.
SUBCASE 3.1: $b_3 = a$ and $b_4 = a$. By assumption we have

$$\begin{aligned} x = b_3sb_4 &\in (cb \cup cSb)S(cb \cup cSb) = cbScb \cup cbScSb \cup cSbScb \cup cSbScSb \\ &\subseteq AASAA \cup AASASA \cup ASASAA \cup ASASASA \\ &\subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3sb_3 &\in (B \setminus \{a\})S(cb \cup cSb) = (B \setminus \{a\})Scb \cup (B \setminus \{a\})ScSb \\ &\subseteq ASAA \cup ASASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3sb_4 &\in (cb \cup cSb)S(B \setminus \{a\}) = cbS(B \setminus \{a\}) \cup cSbS(B \setminus \{a\}) \\ &\subseteq AASA \cup ASASA \subseteq ASA \subseteq (A)_b. \end{aligned}$$

SUBCASE 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_3sb_4 \in (B \setminus \{a\})S(B \setminus \{a\}) = ASA \subseteq (A)_b.$$

Hence $(A)_b = S$. This is a contradiction, and thus $a = b$. □

To give a characterization when a non-empty subset of a semigroup is a bi-base of the semigroup we need the quasi-order defined as follows:

Definition 2.7. *Let S be a semigroup. Define a quasi-order on S by, for any $a, b \in S$,*

$$a \leqslant_b b :\Leftrightarrow (a)_b \subseteq (b)_b.$$

The following example shows that the relation $\leqslant_b$ defined above is not, in general, a partial order.

Example 2.8. From Example 2.4, we have that $(a)_b \subseteq (b)_b$ (i.e., $a \leqslant_b b$) and $(b)_b \subseteq (a)_b$ (i.e., $b \leqslant_b a$), but $a \neq b$. Thus, $\leqslant_b$ is not a partial order on S.

Lemma 2.9. *Let B be a bi-base of a semigroup S. If $a, b \in B$ such that $a \neq b$, then neither $a \leqslant_b b$, nor $b \leqslant_b a$.*

Proof. Assume that $a, b \in B$ such that $a \neq b$. Suppose that $a \leqslant_b b$; then

$$a \in (a)_b \subseteq (b)_b.$$

By assumption we have $a \neq b$, so $a \in bb \cup bSb$. By Lamma 2.5, $a = b$. This is a contradiction. The case $b \leqslant_b a$ can be proved similarly. □

Lemma 2.10. *Let B be a bi-base of a semigroup S. Let $a, b, c \in B$ and $s \in S$:*

(1) *If $a \in bc \cup bcbc \cup bcSbc$, then $a = b$ or $a = c$.*

(2) *If $a \in bsc \cup bscbsc \cup bscSbsc$, then $a = b$ or $a = c$.*

Proof. (1). Assume that $a \in bc \cup bcbc \cup bcSbc$, and suppose that $a \neq b$ and $a \neq c$. Let

$$A = B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, if suffices to show that $B \subseteq (A)_b$. Let $x \in B$. If $x \neq a$, then $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$x = a \in bc \cup bcbc \cup bcSbc \subseteq AA \cup AAAA \cup AASAA \subseteq ASA \subseteq (A)_b.$$

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of S,

$$S = (B)_b \subseteq (A)_b \subseteq S.$$

Therefore $S = (A)_b$. This is a contradiction.

(2). Assume that $a \in bsc \cup bscbsc \cup bscSbsc$, and suppose that $a \neq b$ and $a \neq c$. Let

$$A = B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, if suffices to show that $B \subseteq (A)_b$. Let $x \in B$. If $x \neq a$, then $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$\begin{aligned} x = a \in bsc \cup bscbsc \cup bscSbsc &\subseteq ASA \cup ASAASA \cup ASASASA \\ &\subseteq ASA \subseteq (A)_b. \end{aligned}$$

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of S,

$$S = (B)_b \subseteq (A)_b \subseteq S.$$

Therefore, $S = (A)_b$. This is a contradiction. □

Lemma 2.11. *Let B be a bi-base of a semigroup S.*

(1) *For any $a, b, c \in B$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b bc$.*

(2) *For any $a, b, c \in B$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b bsc$.*

Proof. (1). For any $a, b, c \in B$, let $a \neq b$ and $a \neq c$. Suppose that $a \leqslant_b bc$, we have

$$a \in (a)_b \subseteq (bc)_b = bc \cup bcbc \cup bcSbc.$$

By Lamma 2.10 (1), it follows that $a = b$ or $a = c$. This contradicts to assumption.

(2). For any $a, b, c \in B$ and $s \in S$, let $a \neq b$ and $a \neq c$. Suppose that $a \leqslant_b bsc$, we have $a \in (a)_b \subseteq (bsc)_b = bsc \cup bscbsc \cup bscSbsc$. By Lamma 2.10 (2), it follows that $a = b$ or $a = c$. This contradicts to assumption. □

We now prove the main result of this paper.

Theorem 2.12. *A non-empty subset B of a semigroup S is a bi-base of S if and only if B satisfies the following conditions:*

(1) *For any $x \in S$,*

  (1.a) *there exists $b \in B$ such that $x \leqslant_b b$; or*

  (1.b) *there exist $b_1, b_2 \in B$ such that $x \leqslant_b b_1b_2$; or*

  (1.c) *there exist $b_3, b_4 \in B, s \in S$ such that $x \leqslant_b b_3sb_4$.*

(2) *For any $a, b, c \in B$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b bc$.*

(3) *For any $a, b, c \in B$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b bsc$.*

Proof. Assume first that B is a bi-base of S; then $S = (B)_b$. To show that (1) holds, let $x \in S$. Then $x \in B \cup BB \cup BSB$.

We consider three cases:

Case 1: $x \in B$. Then $x = b$ for some $b \in B$. This implies $(x)_b \subseteq (b)_b$. Hence $x \leqslant_b b$.

Case 2: $x \in BB$. Then $x = b_1b_2$ for some $b_1, b_2 \in B$. This implies $(x)_b \subseteq (b_1b_2)_b$. Hence $x \leqslant_b b_1b_2$.

Case 3: $x \in BSB$. Then $x = b_3sb_4$ for some $b_3, b_4 \in B, s \in S$. This implies $(x)_b \subseteq (b_3sb_4)_b$. Hence $x \leqslant_b b_3sb_4$.

The validity of (2) and (3) follow, respectively, from Lemma 2.11 (1), and Lemma 2.11 (2).

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that B is a bi-base of S. Clearly, $(B)_b \subseteq S$. By (1), $S \subseteq (B)_b$, and $S = (B)_b$. It remains to show that B is a minimal subset of S with the property $S = (B)_b$.

Suppose that $S = (A)_b$ for some $A \subset B$. Since $A \subset B$, there exists $b \in B \setminus A$. Since $b \in B \subseteq S = (A)_b$ and $b \notin A$, it follows that $b \in AA \cup ASA$.

There are two cases to consider:

CASE 1: $b \in AA$. Then $b = a_1a_2$ for some $a_1, a_2 \in A$. We have $a_1, a_2 \in B$. Since $b \notin A$, so $b \neq a_1$ and $b \neq a_2$. Since $b = a_1a_2$, so $(b)_b \subseteq (a_1a_2)_b$. Hence $b \leqslant_b a_1a_2$. This contradicts to (2).

CASE 2: $b \in ASA$. Then $b = a_3sa_4$ for some $a_3, a_4 \in A$ and $s \in S$. Since $b \notin A$, we have $b \neq a_3$ and $b \neq a_4$. Since $A \subset B$, $a_3, a_4 \in B$. Since $b = a_3sa_4$, so $(b)_b \subseteq (a_3sa_4)_b$. Hence, $b \leqslant_b a_3sa_4$. This contradicts to (3).

Therefore, B is a bi-base of S as required, and the proof is completed. □

In Example 2.2, we have that $\{u\}$ is a bi-base of S where as it is not a subsemigroup of S. So, we find a condition in order that a bi-base is a subsemigroup.

Theorem 2.13. *Let B be a bi-base of a semigroup S. Then B is a subsemigroup of S if and only if B satisfies the following conditions: For any $b, c \in B$, $bc = b$ or $bc = c$.*

Proof. By Lemma 2.6, and B is a subsemigroup of S implies for any $b, c \in B$, $bc = b$ or $bc = c$. The opposit direction is clear. □

Question. It was proved in [3] (Theorem 3) that for any two two-sided bases of a semigroup have the same cardinality. This is hold true for an ordered semigroup (see [1], Theorem 2.10). Here, we ask for bi-bases of a semigroup. Indeed, *is it true that for any two bi-bases of a semigroup have the same cardinality?*

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