Quasigroups and Related Systems **25** (2017), 109 − 120

Characterizations of ordered k-regular semirings by ordered quasi k-ideals

Pakorn Palakawong na Ayutthaya and *Bundit Pibaljommee*

Abstract. We introduce the notion of ordered quasi k-ideals of ordered semirings and use them to characterize ordered k-regular semirings.

1. Introduction

In 1936, von Neumann [7] defined a ring S to be regular if for any $a \in S$ there exists $x \in S$ such that $a = axa$. Later, Bourne [3] defined a semiring S to be regular if for any $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. In 1996, Adhikari, Sen and Weinert [1] renamed the Bourne regularity to be k-regular and investigated some of its properties. The notion of a quasi-ideal was defined by Steinfeld [11] for semigroups in 1956. Then, in 2004, Shabir, Ali and Batool [10] investigated some properties of quasi-ideals and used quasi-ideals to characterize regular semirings. In 2011, Bhuniya and Jana [2] defined k-bi-ideals on semirings and used them to characterize k-regular and intra-k-regular semirings. Later, Jana [5] introduced the notion of quasi k-ideals on semirings and characterized k-regular and intra-k-regular semirings by their quasi k-ideals which were a continuation of [2]. In 2011, Gan and Jiang [4] introduced the notion of ordered semirings, defined their ordered ideals and studied some of their properties. In 2014, Mandal [6] called an ordered semiring S to be regular if for any $a \in S$ there exists $x \in S$ such that $a \leqslant axa$ and to be k-regular if for any $a \in S$ there exist $x, y \in S$ such that $a + axa \leqslant aya$. Later, Patchakhieo and Pibaljommee [9] introduced the notion of ordered k-regular semirings as a generalization of k-regular ordered semirings, defined ordered k-ideals on ordered semirings and characterized ordered k-regular semirings using their ordered k-ideals.

In this paper, we introduce the notion of ordered quasi k-ideals of ordered semirings, investigate some of their properties, study connections between them and other ordered k-ideals and use them to characterize ordered k-regular semirings.

2010 Mathematics Subject Classification: 16Y60, 06F25.
Keywords: ordered semiring, ordered quasi k-ideal, ordered k-regular.
This work has been supported by Faculty of Science and The Research Fund for Supporting Lecturer to Admit High Potential Student to Study and Research on His Expert Program Year 2017, Graduate School, Khon Kaen University, Khon Kaen, Thailand.

2. Preliminaries

A *semiring* is an algebraic structure $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups which are connected by the distributive law. An *ordered semiring* is a system $(S, +, \cdot, \leqslant)$ such that $(S, +, \cdot)$ is a semiring, $(S, \leqslant)$ is a partially ordered set and the relation $\leqslant$ is compatible with the operations $+$ and $\cdot$ on S. An ordered semiring S is called *additively commutative* if $a + b = b + a$ for all $a, b \in S$.

In this paper, we assume that S is an additively commutative ordered semiring.

For any nonempty subsets A, B of S, we denote $AB = \{ab \in S \mid a \in A, b \in B\}$, $A + B = \{a + b \in S \mid a \in A, b \in B\}$ and $(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}$.

A nonempty subset A of S such that $A + A \subseteq A$ and $A = (A]$ is called a *left ordered ideal* (*right ordered ideal*) of S if $SA \subseteq A$ ($AS \subseteq A$). We call A an *ordered ideal* [4] if A is both a left ordered ideal and a right ordered ideal.

Let A, B be nonempty subsets of S. We denote some notations as follows.

$$\Sigma A = \left\{ \sum_{i=1}^{n} a_i \in S \mid a_i \in A, n \in \mathbb{N} \right\},$$

$$\Sigma AB = \left\{ \sum_{i=1}^{n} a_i b_i \in S \mid a_i \in A, b_i \in B, n \in \mathbb{N} \right\}.$$

In case of $A = \{a\}$ for some $a \in S$, we write Σa instead of $\Sigma\{a\}$.

Let $\emptyset \neq A \subseteq S$. Then A is called an *ordered quasi-ideal* [8] of S if $A + A \subseteq A$, $A = (A]$ and $(\Sigma SA] \cap (\Sigma AS] \subseteq A$. Obviously, every ordered quasi-ideal is a subsemiring. We call A an *ordered bi-ideal* (*ordered interior ideal*) of S if $A^2 \subseteq A$, $A = (A]$ and $ASA \subseteq A$ ($SAS \subseteq A$).

The *k-closure* [9] of a nonempty subset A of S is defined by

$$\overline{A} = \{x \in S \mid x + a \leqslant b \text{ for some } a, b \in A\}.$$

Now, we give some properties on an ordered semiring which will be used later as the following two lemmas such that their proofs are not difficult.

Lemma 2.1. *Let A, B, C be nonempty subsets of S. Then*

(i) $A \subseteq \Sigma A$ *and* $\Sigma(\Sigma A) = \Sigma A$;

(ii) *if* $A \subseteq B$ *then* $\Sigma A \subseteq \Sigma B$;

(iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$, $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;

(iv) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$;

(v) $\Sigma(A \cup B) = \Sigma A \cup \Sigma B$;

(vi) $\Sigma(A \cap B) \subseteq \Sigma A \cap \Sigma B$;

(vii) $\Sigma(A] \subseteq (\Sigma A]$;

(viii) $A \subseteq (A]$ *and* $((A]] = (A]$;

(ix) *if* $A \subseteq B$ *then* $(A] \subseteq (B]$;

(x) $A(B] \subseteq (A](B] \subseteq (AB]$, $(A]B \subseteq (A](B] \subseteq (AB]$;

(xi) $A + (B] \subseteq (A] + (B] \subseteq (A + B]$;

(xii) $(A \cup B] = (A] \cup (B]$;

(xiii) $(A \cap B] \subseteq (A] \cap (B]$.

Lemma 2.2. *Let A, B be nonempty subsets of S. Then*

(*i*) $\Sigma\overline{A} \subseteq \overline{\Sigma A}$;

(*ii*) *if* $A + A \subseteq A$, *then* $A \subseteq \overline{A}$ *and* $\overline{\overline{A}} = \overline{(A]} = \overline{\overline{(A]}}$;

(*iii*) *if* $A \subseteq B$, *then* $\overline{A} \subseteq \overline{B}$;

(*iv*) $A\overline{B} \subseteq \overline{AB}$ *and* $\overline{A}B \subseteq \overline{AB}$;

(*v*) *if A and B are closed under addition, then* $\overline{A} + \overline{B} \subseteq \overline{A+B}$;

(*vi*) $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$;

(*vii*) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ (*the equality holds if A, B are closed under addition,* $\overline{A} = A$ *and* $\overline{B} = B$ *and also holds for arbitrary intersection*);

(*viii*) *if* $A + A \subseteq A$, *then* $A \subseteq (A] \subseteq (\overline{A}] = \overline{A} \subseteq \overline{(A]}$.

As a consequence of Lemma 2.1 and 2.2, we obtain the following lemma.

Lemma 2.3. *Let A, B be nonempty subsets of S such that A and B are closed under addition. Then:*

(*i*) $A\overline{(B]} \subseteq \overline{(AB]}$ *and* $\overline{(A]}B \subseteq \overline{(AB]}$;

(*ii*) $\overline{(A]}\,\overline{(B]} \subseteq \overline{(\Sigma AB]}$;

(*iii*) $\Sigma A\overline{(B]} \subseteq \overline{(\Sigma A\overline{(B]}]} \subseteq \overline{(\Sigma\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$,
$\Sigma\overline{(A]}B \subseteq \overline{(\Sigma\overline{(A]}B]} \subseteq \overline{(\Sigma\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$;

(*iv*) $\overline{(\overline{(A]} + \overline{(B]}]} \subseteq \overline{(A+B]}$.

It is not difficult to prove that if a nonempty subset A of S is closed under addition then $(A]$, $\overline{A}$ and $\overline{(A]}$ are also closed.

Now, we recall the notions of some types of ordered k-ideals which occur in [9] as follows. A *left ordered k-ideal* (resp. *right ordered k-ideal*, *ordered k-ideal*, *ordered k-bi-ideal*, *ordered k-interior ideal*) A of S is a left ordered ideal (resp. right ordered ideal, ordered ideal, ordered bi-ideal, ordered interior ideal) of S satisfying the condition if $x \in S$ such that $x + a \in A$ for some $a \in A$ then $x \in A$.

It is easy to prove that the following lemma is true on ordered semirings.

Lemma 2.4. *Let $\emptyset \neq A \subseteq S$. Then the following statements hold:*

(*i*) $\overline{(\Sigma SA]}$ *is a left ordered k-ideal of S;*

(*ii*) $\overline{(\Sigma AS]}$ *is a right ordered k-ideal of S;*

(*iii*) $\overline{(\Sigma SAS]}$ *is an ordered k-ideal of S.*

As a spacial case of Lemma 2.4, if $A = \{a\}$ then we obtain that $\overline{(Sa]}, \overline{(aS]}$ and $\overline{(\Sigma SaS]}$ is a left ordered k-ideal, right ordered k-ideal and ordered k-ideal of S, respectively.

By $L_k(A)$, $R_k(A)$, $J_k(A)$ and $B_k(A)$ we denote the smallest left ordered k-ideal, right ordered k-ideal, ordered k-ideal and ordered k-bi-ideal of S containing A, respectively.

Theorem 2.5. (cf. [9]) *For any* $\emptyset \neq A \subseteq S$ *we have:*

(i) $L_k(A) = \overline{(\Sigma A + \Sigma SA]}$;
(ii) $R_k(A) = \overline{(\Sigma A + \Sigma AS]}$;
(iii) $J_k(A) = \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]}$.

It is not difficult to prove that a subsemiring B of S is an ordered k-bi-ideal of S if and only if $BSB \subseteq B$ and $B = \overline{B}$.

Theorem 2.6. $B_k(A) = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}$ *for any* $\emptyset \neq A \subseteq S$.

Proof. Let $\emptyset \neq A \subseteq S$ and $B = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}$. Firstly, we show that B is an ordered k-bi-ideal of S. Since $\Sigma A + \Sigma A^2 + \Sigma ASA$ is closed under addition, B is also closed. By Lemma 2.3(ii) and Lemma 2.1(i), we obtain

$$\begin{aligned}
B^2 &= \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}\,\overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]} \\
&\subseteq \overline{(\Sigma(\Sigma A + \Sigma A^2 + \Sigma AS)(\Sigma A + \Sigma A^2 + \Sigma SA)]} \\
&\subseteq \overline{(\Sigma(\Sigma A^2 + \Sigma A^3 + \Sigma ASA + \Sigma A^4 + \Sigma A^2SA + \Sigma ASA + \Sigma ASA^2 + \Sigma ASSA)]} \\
&\subseteq \overline{(\Sigma A^2 + \Sigma ASA]} \subseteq B.
\end{aligned}$$

Using Lemma 2.3(i, ii), we have

$$\begin{aligned}
BSB &= \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}S\overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]} \\
&\subseteq \overline{(\Sigma A + \Sigma AS + \Sigma ASA]}\,\overline{(\Sigma SA + \Sigma SA^2 + \Sigma SASA]} \\
&\subseteq \overline{(\Sigma A + \Sigma AS]}\,\overline{(\Sigma SA]} \subseteq \overline{(\Sigma(\Sigma A + \Sigma AS)(\Sigma SA)]} \subseteq \overline{(\Sigma ASA]}.
\end{aligned}$$

Let $x \in \overline{(\Sigma ASA]}$. Then $x + (z + x) \leqslant z + x + x$ for every $z \in \Sigma A + \Sigma A^2$ and so $x \in \overline{\Sigma A + \Sigma A^2 + \overline{(\Sigma ASA]}}$, since $z + x, z + x + x \in \Sigma A + \Sigma A^2 + \overline{(\Sigma ASA]}$. Thus $\overline{(\Sigma ASA]} \subseteq \overline{\Sigma A + \Sigma A^2 + \overline{(\Sigma ASA]}}$. Using Lemma 2.2$(viii)$ and Lemma 2.3(iv), we have $BSB \subseteq \overline{(\Sigma ASA]} \subseteq \overline{\Sigma A + \Sigma A^2 + \overline{(\Sigma ASA]}} \subseteq \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]} = B$. By Lemma 2.2$(ii)$, we get $\overline{B} = B$. This means that B is an ordered k-bi-ideal of S.

Secondly, we show that $A \subseteq B$. Let $x \in \Sigma A$. Then $x + (x + w) \leqslant x + x + w$ for every $w \in \Sigma A^2 + \Sigma ASA$ and so $x \in \overline{\Sigma A + \Sigma A^2 + \Sigma ASA}$, since $x + w, x + x + w \in \Sigma A + \Sigma A^2 + \Sigma ASA$. It follows that $A \subseteq \Sigma A \subseteq \overline{\Sigma A + \Sigma A^2 + \Sigma ASA} \subseteq \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]} = B$.

Finally, let C be an ordered k-bi-ideal of S containing A. Then

$$B = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]} \subseteq \overline{(\Sigma C + \Sigma C^2 + \Sigma CSC]} \subseteq \overline{(\Sigma C]} = C.$$

Therefore, B is the smallest ordered k-bi-ideal of S containing A. □

3. Ordered quasi k-ideals

Here, we give the notion of ordered quasi k-ideals of ordered semirings, study their properties and investigate connections between them and other ordered k-ideals.

Definition 3.1. Let $\emptyset \neq Q \subseteq S$ such that $Q+Q \subseteq Q$. Then Q is called an *ordered quasi k-ideal* of S if

(i) $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq Q$;

(ii) if $x \leqslant y$ for some $y \in Q$ then $x \in Q$ (i.e., $Q = (Q]$);

(iii) if $x + a \in Q$ for some $a \in Q$ then $x \in Q$.

It is easy to see that every ordered quasi k-ideal Q of S is a subsemiring because $Q^2 \subseteq SQ \cap QS \subseteq Q$.

Theorem 3.2. *Let $\emptyset \neq Q \subseteq S$ and $Q + Q \subseteq Q$. Then Q is an ordered quasi k-ideal of S if and only if $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq Q$ and $Q = \overline{Q}$.*

Proof. Let Q be an ordered quasi k-ideal of S. Clearly, $Q \subseteq \overline{Q}$. Let $x \in \overline{Q}$. Then $x + y \leqslant z$ for some $y, z \in Q$ and so $x + y \in (Q] = Q$. Thus, $x \in Q$. Hence, $Q = \overline{Q}$.

Conversely, we consider $Q \subseteq (Q] \subseteq \overline{Q} = Q$. Thus, $Q = (Q]$. Let $x \in S$ such that $x + y \in Q = (Q]$ for some $y \in Q$. So, $x + y \leqslant q$ for some $q \in Q$. Hence, $x \in \overline{Q} = Q$. □

Note that every left ordered k-ideal (right ordered k-ideal, ordered k-ideal) of S is an ordered quasi k-ideal. The converse is not true as the following example shows.

Example 3.3. Let $S = \{a, b, c\}$. Define a binary operation $+$ on S by $b + b = b$ and $a + x = x + a = x$, $c + x = x + c = c$ for all $x \in S$. Define a binary operation $\cdot$ on S by for any $y \in S$, $xy = a$ if $x = a$ and $xy = b$, otherwise. Define a binary relation $\leq$ on S by $\leqslant := \{(a, a), (b, b), (c, c), (a, b)\}$. Now, $(S, +, \cdot, \leqslant)$ is an additively commutative ordered semiring. Let $Q = \{a\}$. Clearly, $Q + Q \subseteq Q$ and $Q = (Q]$. We have $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} = \overline{(\Sigma\{a, b\}]} \cap \overline{(\Sigma\{a\}]} = \{a, b\} \cap \{a\} = \{a\} = Q$ and $\overline{Q} = Q$. This shows that Q is an ordered quasi k-ideal. Since $SQ = \{a, b\} \nsubseteq Q$, this follows that Q is not a left ordered k-ideal of S.

Also every ordered quasi k-ideal of S is an ordered k-bi-ideal, but not conversely.

Example 3.4. Let $S = \{a, b, c, d, e\}$. Define a binary operation $+$ on S by $a + x = x + a = x$ for all $x \in S$, $b + b = b$, $e + e = e$ and $x + y = d$ otherwise. Define a binary operation $\cdot$ on S by for any $y \in S$, $xy = yx = a$ if $x \in \{a, b\}$ and $xy = b$ otherwise. Define a binary relation $\leqslant$ on S by

$$\leqslant \ := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, e), (a, d), (b, d), (c, d), (e, d)\}.$$

Now, $(S, +, \cdot, \leqslant)$ is an additively commutative ordered semiring. Let $B = \{a, e\}$. It is easy to show that B is an ordered k-bi-ideal of S, but not an ordered quasi k-ideal, since $\overline{(\Sigma SB]} \cap \overline{(\Sigma BS]} = \{a, b\} \nsubseteq B$.

Theorem 3.5. *The intersection of a right ordered k-ideal and a left ordered k-ideal of S is an ordered quasi k-ideal.*

Proof. Let R and L be a right and a left ordered k-ideal of S, respectively. Then

$$\overline{(\Sigma(R \cap L)S]} \cap \overline{(\Sigma S(R \cap L)]} \subseteq \overline{(\Sigma RS]} \cap \overline{(\Sigma SL]} \subseteq \overline{(\Sigma R]} \cap \overline{(\Sigma L]} = R \cap L.$$

We consider $\overline{R \cap L} = \overline{R} \cap \overline{L} = R \cap L$. By Theorem 3.2, we obtain $R \cap L$ is an ordered quasi k-ideal of S. □

The converse of Theorem 3.5 is not true as the following example shows.

Example 3.6. Let $S = \{a, b, c, d, e, f, g, h\}$. Define binary operations $+$ and $\cdot$ by the following tables:

$+$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	e	f	c	d	h	g
c	c	e	a	g	b	h	d	f
d	d	f	g	a	h	b	c	e
e	e	c	b	h	a	g	f	d
f	f	d	h	b	g	a	e	c
g	g	h	d	c	f	e	a	b
h	h	g	f	e	d	c	b	a

$\cdot$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	g	a	h	b	g	h
c	a	d	a	a	d	d	a	d
d	a	d	a	a	d	d	a	d
e	a	f	g	a	e	f	g	e
f	a	f	g	a	e	f	g	e
g	a	a	a	a	a	a	a	a
h	a	b	g	a	h	b	g	h

Define a binary relation $\leqslant$ on S by $\leqslant := \{(x, x) \mid x \in S\}$.

Then $(S, +, \cdot, \leqslant)$ is an additively commutative ordered semiring. Let $Q = \{a, c\}$. Clearly, $Q + Q \subseteq Q$ and $Q = (Q]$. We consider

$$\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} = \overline{\{a, g\}} \cap \overline{\{a, d\}} = \{a, g\} \cap \{a, d\} = \{a\} \subseteq Q.$$

It is easy to see that $\overline{Q} = Q$. By Theorem 3.2, Q is an ordered quasi k-ideal of S. If $Q = R \cap L$ for some a right ordered k-ideal R and a left ordered k-ideal L of S, then $c \in R \cap L$. We have $g = c + cb \in R$ and $g = bc \in L$. Then $g \in R \cap L = Q$, but $g \notin Q$. This give a contradiction.

As a consequence of Lemma 2.4 and Theorem 3.5, we have that $\overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$ is an ordered quasi k-ideals of S for any $\emptyset \neq A \subseteq S$.

For $\emptyset \neq A \subseteq S$, we denote $Q_k(A)$ as the smallest ordered quasi k-ideal of S containing A. Now, we give the construction of $Q_k(A)$ as follows.

Theorem 3.7. *Let $\emptyset \neq A \subseteq S$. Then* $Q_k(A) = \overline{(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}]}$.

Proof. Let $\emptyset \neq A \subseteq S$ and $Q = \overline{(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}]}$. Firstly, we show that Q is an ordered quasi k-ideal. It is easy to show that Q is closed under addition. Using Lemma 2.3(i) and (iv), we obtain

$$\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq \overline{(\Sigma SQ]} = \overline{(\Sigma S\overline{(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}]}]} \subseteq \overline{(\Sigma S\overline{(\Sigma A + \overline{(\Sigma SA]}]}]}$$
$$\subseteq \overline{(\Sigma S\overline{(\Sigma A + \Sigma SA]}} \subseteq \overline{(\Sigma\overline{(\Sigma SA + \Sigma SSA]}]} \subseteq \overline{(\overline{(\Sigma(\Sigma SA)]}]} \subseteq \overline{(\Sigma SA]}.$$

Similarly, we have $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq \overline{(\Sigma AS]}$. So, $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$. If $q \in \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$ then $q + a' + q \leqslant a' + q + q \in \Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$ for every $a' \in \Sigma A$. So, $\overline{(\Sigma SA]} \cap \overline{(\Sigma AS]} \subseteq \overline{\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}}$. By Lemma 2.2($ii$), we get

$$\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]} \subseteq \overline{\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}} \subseteq Q.$$

Using Lemma 2.2($viii$), $\overline{Q} = Q$. By Theorem 3.2, Q is an ordered quasi k-ideal.

Secondly, we show that $A \subseteq Q$. If $a \in \Sigma A$ then $a + a + w \leqslant a + a + w$ and $a + a + w \in \Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$, for every $w \in \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$. This implies

$$A \subseteq \Sigma A \subseteq \overline{\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}} \subseteq \overline{(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}]} = Q.$$

Finally, let K be an ordered quasi k-ideal of S such that $A \subseteq K$. Then

$$Q = \overline{(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}]} \subseteq \overline{(\Sigma K + \overline{(\Sigma SK]} \cap \overline{(\Sigma KS]}]} \subseteq \overline{(K + K]} \subseteq \overline{(K]} = K.$$

Therefore, Q is the smallest ordered quasi k-ideal of S containing A. □

As a spacial case of Theorem 3.7, if $A = \{a\}$ for some $a \in S$ then we obtain $Q_k(a) = \overline{(\Sigma a + \overline{(Sa]} \cap \overline{(aS]}]}$.

Note that a nonempty intersection of a family of ordered quasi k-ideals of S is an ordered quasi k-ideal of S.

An element e of S is called an *identity* of S if $ea = a = ae$ for all $a \in S$.

Corollary 3.8. *Let $\emptyset \neq A \subseteq S$. If S has an identity then*

(i) $L_k(A) = \overline{(\Sigma SA]}$;

(ii) $R_k(A) = \overline{(\Sigma AS]}$;

(iii) $J_k(A) = \overline{(\Sigma SAS]}$;

(iv) $B_k(A) = \overline{(\Sigma ASA]}$;

(v) $Q_k(A) = \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}$.

As a spacial case of Corollary 3.8, if $A = \{a\}$ then we have $L_k(a) = \overline{(Sa]}$, $R_k(a) = \overline{(aS]}$, $J_k(a) = \overline{(\Sigma SaS]}$, $Q_k(a) = \overline{(Sa]} \cap \overline{(aS]}$ and $B_k(a) = \overline{(aSa]}$.

If S has an identity element, then the converse of Theorem 3.5 is true.

Theorem 3.9. *If S has an identity, then ordered quasi k-ideals and ordered k-bi-ideals coincide.*

Proof. Assume that S has an identity. Let B be an ordered k-bi-ideal of S and let $x \in \overline{(\Sigma SB]} \cap \overline{(\Sigma BS]}$. Using Lemma 2.3$(i), (iii)$, we obtain

$$x \in B_k(x) = \overline{(xSx]} \subseteq \overline{(\overline{(\Sigma BS]}S\overline{(\Sigma SB]}]} \subseteq \overline{(\Sigma BSSSB]} \subseteq \overline{(\Sigma BSB]} \subseteq \overline{(\Sigma B]} = B.$$

This shows that B is an ordered quasi k-ideal of S. □

Theorem 3.10. *If S has an identity, then every ordered quasi k-ideal of S can be written in the form $Q = R \cap L$ for some a right ordered k-ideal R and a left ordered k-ideal L of S.*

Proof. Let Q be an ordered quasi k-ideal of S. Clearly, $Q \subseteq R_k(Q) \cap L_k(Q)$. By Corollary 3.8, we have $R_k(Q) = \overline{(\Sigma QS]}$ and $L_k(Q) = \overline{(\Sigma QS]}$. Hence, $R_k(Q) \cap L_k(Q) = \overline{(\Sigma QS]} \cap \overline{(\Sigma QS]} \subseteq Q$. Therefore, $Q = R_k(Q) \cap L_k(Q)$. □

4. Ordered k-regular semirings

First, we review the notion of a k-regular ordered semiring given by Mandal [6] and the notion of an ordered k-regular semiring defined by Patchakhieo and Pibaljommee [9] which is a generalization of Mandal k-regularity as follows.

Definition 4.1. An element a of S is called *regular* (resp. *k-regular*, *ordered k-regular*) if $a \leqslant axa$ (resp. $a + axa \leqslant aya$, $a \in \overline{(aSa]}$) for some $x, y \in S$. We call S *regular* (resp. *k-regular, ordered k-regular*) if every element of S is regular (resp. k-regular, ordered k-regular).

Obviously, S is ordered k-regular if and only if $A \subseteq \overline{(\Sigma ASA]}$ for each $A \subseteq S$.

Theorem 4.2. (cf. [9]) *An ordered semiring S is ordered k-regular if and only if $R \cap L = \overline{(RL]}$ for every right ordered k-ideal R and left ordered k-ideal L of S.*

Corollary 4.3. *An ordered semiring S is ordered k-regular if and only if $A \subseteq \overline{(R_k(A)L_k(A)]}$ for each $A \subseteq S$.*

Remark 4.4. *If S is ordered k-regular then ordered k-ideals and ordered k-interior ideals coincide.*

Proof. Let J be an ordered k-ideal of S. Then $SJS \subseteq SJ \subseteq JS \subseteq J$ and so J is an ordered k-interior ideal. Conversely, let I be an ordered k-interior ideal of S. If $x \in IS$, then $x \in \overline{(xSx]} \subseteq \overline{(ISSIS]} \subseteq \overline{(ISIS]} \subseteq \overline{(II]} \subseteq \overline{(I]} = I$. So, $IS \subseteq I$ Similarly, we obtain $SI \subseteq I$. Therefore, I is an ordered k-ideal of S. □

Now, we show that if S is ordered k-regular, then the converse of Theorem 3.5 is true.

Theorem 4.5. *If S is ordered k-regular, then their ordered quasi k-ideals coincide with their ordered k-bi-ideals.*

Proof. Assume that S is ordered k-regular. Let B be an ordered k-bi-ideal of S and let $x \in \overline{(\Sigma SB]} \cap \overline{(\Sigma BS]}$. Using Lemma 2.3$(i), (iii)$ and by assumption, we get

$$x \in \overline{(xSx]} \subseteq \overline{(\overline{(\Sigma BS]}S\overline{(\Sigma SB]}]} \subseteq \overline{(\Sigma BSSSB]} \subseteq \overline{(\Sigma BSB]} \subseteq \overline{(\Sigma B]} = B.$$

This shows that B is an ordered quasi k-ideal of S. □

Theorem 4.6. *If S is ordered k-regular, then every ordered quasi k-ideal of S can be written in the form $Q = R \cap L$ for some a right ordered k-ideal R and a left ordered k-ideal L of S.*

Proof. Let Q be an ordered quasi k-ideal. Clearly, $Q \subseteq R_k(Q) \cap L_k(Q)$. If $x \in \Sigma Q$ then $x \in \overline{(xSx]} \subseteq \overline{(xS]} \subseteq \overline{((\Sigma Q)S]} \subseteq \overline{(\Sigma QS]}$. Thus $\Sigma Q \subseteq \overline{(\Sigma QS]}$. We consider $\overline{(\Sigma QS]} \subseteq \overline{(\Sigma Q + \Sigma QS]} \subseteq \overline{(\overline{(\Sigma QS]} + \Sigma QS]} \subseteq \overline{(\Sigma QS]}$. This means that $R_k(Q) = \overline{(\Sigma Q + \Sigma QS]} = \overline{(\Sigma QS]}$. Similarly, we can show that $L_k(Q) = \overline{(\Sigma SQ]}$. It follows that $R_k(Q) \cap L_k(Q) = \overline{(\Sigma QS]} \cap \overline{(\Sigma SQ]} \subseteq Q$. Therefore, $Q = R_k(Q) \cap L_k(Q)$. □

Here, we use ordered quasi k-ideals to characterize ordered k-regular semirings.

Theorem 4.7. *The following statements are equivalent:*

(i) *S is ordered k-regular;*

(ii) *$B = \overline{(BSB]}$ for every ordered k-bi-ideal of S;*

(iii) *$Q = \overline{(QSQ]}$ for every ordered quasi k-ideal of S.*

Proof. $(i) \Rightarrow (ii)$: Let S be ordered k-regular and B be an ordered k-bi-ideal of S. Clearly, $\overline{(BSB]} \subseteq \overline{(B]} = B$. If $x \in B$ then $x \in \overline{(xSx]} \subseteq \overline{(BSB]}$. So, $B = \overline{(BSB]}$.

$(ii) \Rightarrow (iii)$: It is clear, since every ordered quasi k-ideal is an ordered k-bi-ideal.

$(iii) \Rightarrow (i)$: Assume that (iii) holds. Let $A \subseteq S$. Then

$$A \subseteq Q_k(A) = \overline{(Q_k(A)SQ_k(A)]} \subseteq \overline{(R_k(A)SL_k(A)]} \subseteq \overline{(R_k(A)L_k(A)]}.$$

By Corollary 4.3, we obtain that S is ordered k-regular. □

Theorem 4.8. *An ordered semiring S is ordered k-regular if and only if for every ordered k-bi-ideal B, ordered k-ideal J and left ordered k-ideal L of S we have $B \cap J \cap L \subseteq \overline{(BJL]}$.*

Proof. Assume that S is ordered k-regular. Let B, J and L be an ordered k-bi-ideal, an ordered k-ideal and a left ordered k-ideal of S, respectively. Let $x \in B \cap J \cap L$. By assumption, we get $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(BSJSL]} \subseteq \overline{(BSL]}$. Hence, $B \cap J \cap L \subseteq \overline{(BJL]}$.

Conversely, let R and L be a right ordered k-ideal and a left ordered k-ideal of S, respectively. We obtain $R \cap L = R \cap S \cap L = \overline{(RSL]} \subseteq \overline{(RL]}$. On the other hand, we know that $\overline{(RL]} \subseteq R \cap L$. So, $\overline{(RL]} = R \cap L$. By Theorem 4.2, S is ordered k-regular. □

Theorem 4.9. *The following statements are equivalent:*

- (i) *S is ordered k-regular;*
- (ii) *$Q \cap I = \overline{(QIQ]}$ for every ordered quasi k-ideal Q and ordered k-interior ideal I of S;*
- (iii) *$Q \cap J = \overline{(QJQ]}$ for every ordered quasi k-ideal Q and ordered k-ideal J of S;*
- (iv) *$Q \cap L \subseteq \overline{(QL]}$ for every ordered quasi k-ideal Q and left ordered k-ideal L of S;*
- (v) *$R \cap Q \subseteq \overline{(RQ]}$ for every right ordered k-ideal R and ordered quasi k-ideal Q of S;*
- (vi) *$R \cap Q \cap L \subseteq \overline{(RQL]}$ for every right ordered k-ideal R, ordered quasi k-ideal Q and left ordered k-ideal L of S.*

Proof. Let Q, I, J, R and L be an ordered quasi k-ideal, an ordered k-interior ideal, an ordered k-ideal, a right ordered k-ideal and a left ordered k-ideal of S, respectively.

$(i) \Rightarrow (ii)$: Assume that S is ordered k-regular and let $x \in Q \cap I$. By assumption, we obtain $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(QSISQ]} \subseteq \overline{(QIQ]}$. For the opposite inclusion, we consider $\overline{(QIQ]} \subseteq \overline{(QSQ]} \subseteq \overline{(Q]} = Q$ and $\overline{(QIQ]} \subseteq \overline{(SIS]} \subseteq \overline{(I]} = I$. Therefore, $Q \cap I = \overline{(QIQ]}$.

$(ii) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (i)$: Assume that (iii) holds. By assumption, we get $Q = Q \cap S = \overline{(QSQ]}$. By Theorem 4.7, S is ordered k-regular.

$(i) \Rightarrow (iv)$: If $x \in Q \cap L$, then $x \in \overline{(xSx]} \subseteq \overline{(QSL]} \subseteq \overline{(QL]}$.

$(iv) \Rightarrow (i)$: Assume that (iv) holds. Then we obtain $R \cap L \subseteq \overline{(RL]}$, since every right ordered k-ideal is an ordered quasi k-ideal. Clearly, $\overline{(RL]} \subseteq R \cap L$. So, $R \cap L = \overline{(RL]}$. By Theorem 4.2, S is ordered k-regular.

$(i) \Rightarrow (v)$: If $x \in R \cap Q$, then $x \in \overline{(xSx]} \subseteq \overline{(RSQ]} \subseteq \overline{(RQ]}$.

$(v) \Rightarrow (i)$: It can be proved in a similar way of $(iv) \Rightarrow (i)$.

$(i) \Rightarrow (vi)$: Assume that S is ordered k-regular and let $x \in R \cap Q \cap L$. Then $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(RSQSL]} \subseteq \overline{(RQL]}$.

$(vi) \Rightarrow (i)$: Assume that (vi) holds. We get $R \cap L = R \cap S \cap L \subseteq \overline{(RSL]} \subseteq \overline{(RL]}$. Clearly, $\overline{(RL]} \subseteq R \cap L$. So, $R \cap L = \overline{(RL]}$. By Theorem 4.2, S is ordered k-regular. □

Definition 4.10. An ordered semiring S is said to be an *ordered k-duo-semiring* if every one-sided (left or right) ordered k-ideal of S is an ordered k-ideal of S.

It is clear that every multiplicatively commutative ordered semiring is an ordered k-duo-semiring, but the converse is not true as the following example shows.

Example 4.11. Let $S = \{a, b, c, d, e\}$. Define a binary operation $+$ on S by $a+x = x+a = x$ for all $x \in S$ and $x+y = c$ otherwise. Define a binary operation $\cdot$ on S by $ax = xa = a$ for all $x \in S$, $bb = bd = dd = e$ and $xy = c$ otherwise. Define a binary relation $\leqslant$ on S by

$$\leqslant \ := \{(a,a),(b,b),(c,c),(d,d),(e,e),(e,c)\}.$$

Then $(S, +, \cdot, \leqslant)$ is an ordered semiring which is not multiplicatively commutative, since $bd \neq db$. We have $\{a\}$ and S are only ordered one-sided k-ideals of S. Obviously, all of them are ordered ideals of S. This shows that S is an ordered k-duo-semiring.

Theorem 4.12. *The following statements are equivalent:*

(i) *S is an ordered k-duo-semiring;*

(ii) *$R_k(A) = L_k(A)$ for each $A \subseteq S$;*

(iii) *$R_k(a) = L_k(a)$ for each $a \in S$.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$: Assume that (iii) holds and let R be a right ordered k-ideal of S. Let $x \in R$, $s \in S$. By assumption, we obtain $sx \in SL_k(x) \subseteq L_k(x) = R_k(x) \subseteq R_k(R) = R$. This shows that R is a left ordered k-ideal of S. Similarly, we can show that if L is a left ordered k-ideal of S then L is a right ordered k-ideal of S. Therefore, S is an ordered k-duo-semiring. □

As a consequence of Theorem 4.5, 4.6 and Definition 4.10, we obtain the following corollary.

Corollary 4.13. *If an ordered k-duo-semiring S is ordered k-regular, then its ordered k-ideals, ordered k-interior ideals, ordered quasi k-ideals and its ordered k-bi-ideals coincide.*

Theorem 4.14. *Let S be an ordered k-duo-semiring. Then the following statements are equivalent:*

(i) *S is ordered k-regular;*

(ii) *$B_1 \cap B_2 = \overline{(B_1B_2]}$ for every ordered k-bi-ideals B_1 and B_2 of S;*

(iii) *$Q_1 \cap Q_2 = \overline{(Q_1Q_2]}$ for every ordered quasi k-ideals Q_1 and Q_2 of S;*

(iv) *$J_1 \cap J_2 = \overline{(J_1J_2]}$ for every ordered k-ideal J_1 and J_2 of S.*

Proof. $(i) \Rightarrow (ii)$: Let B_1, B_2 be ordered k-bi-ideals of S. By Corollary 4.13, B_1 and B_2 are ordered k-ideals of S. By Theorem 4.2, we obtain $B_1 \cap B_2 = \overline{(B_1B_2]}$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are obvious.

$(vi) \Rightarrow (i)$: Assume that (iv) holds. Let $A \subseteq S$. Since S is an ordered k-duo-semiring, $J_k(A) = L_k(A) = R_k(A)$. By assumption, we obtain

$$A \subseteq J_k(A) = J_k(A) \cap J_k(A) = \overline{(J_k(A)J_k(A)]} = \overline{(R_k(A)L_k(A)]}.$$

By Corollary 4.3, we get S is ordered k-regular. □

References

[1] **M. R. Adhikari, M. K. Sen and H. J. Weinert**, *On k-regular semirings*, Bull. Calcutta Math. Soc. **88** (1996), 141 – 144.

[2] **A. K. Bhuniya and K. Jana**, *Bi-ideals in k-regular and intra k-regular semirings*, Discuss. Math. Gen. Algebra Appl. **31** (2011), 5 – 23.

[3] **S. Bourne**, *The Jacobson radical of a semiring*, Proc. Nati. Acad. Sci. USA **31** (1951), 163 – 170.

[4] **A. P. Gan and Y. L. Jiang**, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition **31** (2011), 989 – 996.

[5] **K. Jana**, *Quasi k-ideals in k-regular and intra k-regular semirings*, Pure. Math. Appl. **22** (2011), 65 – 74.

[6] **D. Mandal**, *Fuzzy ideals and fuzzy interior ideals in ordered semirings*, Fuzzy Inf. Eng. **6** (2014), 101 – 114.

[7] **J. von Neumann**, *On regular rings*, Proc. Natl. Acad. Sci. USA **22** (1936), 707 – 713.

[8] **P. Palakawong na Ayuthaya and B. Pibaljommee**, *Characterizations of regular ordered semirings by ordered quasi-ideals*, Int. J. Math. Math. Sci. (2016), Article ID 4272451.

[9] **S. Patchakhieo and B. Pibaljommee**, *Characterizations of ordered k-regular semirings by ordered k-ideals*, Asian-Eur. J. Math. **10** (2017), Article ID 1750020.

[10] **M. Shabir, A. Ali and S. Batool**, *A note on quasi-ideals in semirings*, Southeast. Asian. Bull. Math. **27** (2004), 923 – 928.

[11] **O. Steinfeld**, *Über die quasiideale von halbgruppen*. Publ. Math. Debrecan. **4** (1956), 262 – 275.

[12] **H. S. Vandiver**, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934), 914 – 920.

Received October 4, 2016

Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen, Thailand, 40002
E-mails: pakorn1702@gmail.com, banpib@kku.ac.th