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Prime ordered k-bi-ideals in ordered semirings

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Abstract. Various types of ordered k-bi-ideals of ordered semirings are investigated. Several characterizations of ordered k-bi-idempotent semirings are presented.

1. Introduction

The notion of a semiring was introduced by Vandiver [8] as a generalization of a ring. Gan and Jiang [2] investigated an ordered semiring with zero and introduced several notions, for example, ordered ideals, minimal ideals and maximal ideals of an ordered semiring. Han, Kim and Neggers [3] investigated properties orders in a semiring. Henriksen [4] defined more restrict class of ideals in semiring known as k-ideals. Several characterizations of k-ideals of a semiring were obtained by Sen and Adhikari in [6, 7]. In [1], Akram and Dudek studied properties of intuinistic fuzzy left k-ideals of semirings. An ordered k-ideal in an ordered semiring was characterized by Patchakhieo and Pibaljommee [5].

In this paper, we introduce the notion of an ordered k-bi-ideal, a prime ordered k-bi-ideal, a strongly prime ordered k-bi-ideal, an irreducible and a strongly irreducible ordered k-bi-ideals of an ordered semiring. We introduce the concept of an ordered k-bi-idempotent semiring and characterize it using prime, strongly prime, irreducible and strongly irreducible ordered k-bi-ideals.

2. Preliminaries

A *semiring* is a triplet $(S, +, \cdot)$ consisting of a nonempty set S and two operations $+$ (addition) and $\cdot$ (multiplication) such that $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in S$.

A semiring $(S, +, \cdot)$ is called a *commutative* if $(S, \cdot)$ is a commutative semigroup. An element $0 \in S$ is called a *zero element* if $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 = 0 \cdot a$.

A nonempty subset A of a semiring $(S, +, \cdot)$ is called a *left* (*right*) *ideal* of S if $x + y \in A$ for all $x, y \in A$ and $SA \subseteq A$ ($AS \subseteq A$). We call A an *ideal* of S if it is both a left and a right ideal of S. A subsemiring B of a semiring S is called a *bi-ideal* of S if $BSB \subseteq B$.

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Let $(S, \leqslant)$ be a partially ordered set. Then $(S, +, \cdot, \leqslant)$ is called an *ordered semiring* if $(S, +, \cdot)$ is a semiring and the relation $\leqslant$ is compatible with the operations $+$ and $\cdot$, i.e., if $a \leqslant b$, then $a + x \leqslant b + x$, $x + a \leqslant x + b$, $ax \leqslant bx$ and $xa \leqslant xb$ for all $a, b, x \in S$.

Let $(S, +, \cdot, \leqslant)$ be an ordered semiring. For nonempty subsets A, B of S and $a \in S$, we denote

$$
\begin{aligned}
(A] &= \{x \in S \mid x \leqslant a \text{ for some } a \in A\}, \\
AB &= \{xy \in S \mid x \in A, y \in B\}, \\
\Sigma A &= \{\sum_{i \in I} a_i \in S \mid a_i \in A \text{ and } I \text{ is a finite subset of } \mathbb{N}\}, \\
\Sigma AB &= \{\sum_{i \in I} a_i b_i \in S \mid a_i \in A, b_i \in B \text{ and } I \text{ is a finite subset of } \mathbb{N}\} \text{ and} \\
\mathbb{N}a &= \{na \in S \mid n \in \mathbb{N}\}.
\end{aligned}
$$

Instead of writing an ordered semiring $(S, +, \cdot, \leqslant)$, we simply denote S as an ordered semiring.

A left (right) ideal A of an ordered semiring S is called a *left* (*right*) *ordered ideal* of S if for any $x \leqslant a$ for some $a \in A$ implies $x \in A$. We call A an *ordered ideal* if it is both a left and a right ordered ideal of S.

A left (right) ordered ideal of A of a semiring S is called a *left (right) ordered k-ideal* of S if $x + a = b$ for some $a, b \in A$ implies $x \in A$. We call A an *ordered k-ideal* of S if it is both a left and a right ordered k-ideal of S.

The k-closure of a nonempty subset A of an ordered semiring S is defined by

$$\overline{A} = \{x \in S \mid \exists\, a,\ b \in A,\ x + a \leqslant b\}.$$

Now, we recall the results concerning to the k-closure given in [5].

Lemma 2.1. *Let S be an ordered semiring and A, B be nonempty subsets of S.*

(i) $(\overline{A}] \subseteq \overline{(A]}$.

(ii) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(iii) $\overline{(A]}B \subseteq \overline{(AB]}$ *and* $A\overline{(B]} \subseteq \overline{(AB]}$. □

Lemma 2.2. *Let A be a nonempty subset of an ordered semiring S. If A is closed under addition, then $(A]$ and $\overline{(A]}$ are also closed.* □

Lemma 2.3. *Let S be an ordered semiring and A, B be nonempty subsets of S with $A + A \subseteq A$ and $B + B \subseteq B$. Then*

(i) $A \subseteq (A] \subseteq \overline{A} \subseteq \overline{(A]}$;

(ii) $\overline{(A]} = \overline{\overline{(A]}}$;

(*iii*) $A+B \subseteq \overline{A}+\overline{B} \subseteq \overline{A+B}$;

(*iv*) $\overline{(A]}+\overline{(B]} \subseteq \overline{\overline{(A]}+\overline{(B]}} \subseteq \overline{(A+B]}$;

(*v*) $\overline{A}\ \overline{B} \subseteq \overline{(A]}\ \overline{(B]} \subseteq \overline{(\Sigma AB]}$;

(*vi*) $A(\Sigma B) \subseteq \Sigma AB$ *and* $(\Sigma A)B \subseteq \Sigma AB$. □

Lemma 2.4. *Let S be an ordered semiring and A be a nonempty subset of S with $A+A \subseteq A$. Then $\overline{(\overline{(A]}]} = \overline{(A]}$.* □

Theorem 2.5. *Let S be an ordered semiring and A be a left ideal (resp. right ideal, ideal). Then the following conditions are equivalent:*

(*i*) *A is a left ordered k-ideal (resp. right ordered k- ideal, ordered k-ideal) of S;*

(*ii*) *if $x \in S, x+a \leqslant b$ for some $a, b \in A$, then $x \in A$;*

(*iii*) $\overline{A} = A$. □

Theorem 2.6. *Let S be an ordered semiring and A be a nonempty subset of S. If A is a left ideal (resp. right ideal, ideal), then $\overline{(A]}$ is the smallest left ordered k-ideal (resp. right ordered k-ideal, ordered k-ideal) containing A.* □

From Theorem 2.6, we have A is an ordered k-ideal if and only if $\overline{(A]} = A$.

Theorem 2.7. *Let S be an ordered semiring. If the intersection of a family of left ordered k-ideals (resp. right ordered k-ideal, ordered k-ideal) is not empty, then it is a left ordered k-ideal (resp. right ordered k-ideal, ordered k-ideal).*

For a nonempty subset A of an ordered semiring S, we denote by $L_k(A)$, $R_k(A)$ and $M_k(A)$ the smallest left ordered k-ideal, the smallest right ordered k-ideal and the smallest ordered k-ideal of S containing A, respectively. For any $a \in S$, we denote $L_k(a) = L_k(\{a\})$, $R_k(a) = R_k(\{a\})$ and $M_k(a) = M_k(\{a\})$.

Theorem 2.8. *Let S be an ordered semiring and $a \in S$. Then*

(*i*) $L_k(A) = \overline{(\Sigma A + \Sigma SA]}$;

(*ii*) $R_k(A) = \overline{(\Sigma A + \Sigma AS]}$;

(*iii*) $M_k(a) = \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]}$. □

Corollary 2.9. *Let S be an ordered semiring and $a \in S$. Then*

(*i*) $L_k(a) = \overline{(\mathbb{N}a + Sa]}$;

(*ii*) $R_k(a) = \overline{(\mathbb{N}a + aS]}$;

(*iii*) $M_k(a) = \overline{(\mathbb{N}a + Sa + Sa + \Sigma SaS]}$. □

3. Prime ordered k-bi-ideals

First, we begin with the definition of an ordered k-bi-ideal of an ordered semiring and give some concepts in ordered semirings that we need in this section.

Definition 3.1. An ordered subsemiring B of an ordered semiring S is said to be an *ordered k-bi-ideal* of S if

(i) $BSB \subseteq B$;

(ii) if $x \in S$, $a + x = b$ for some $a, b \in B$, then $x \in B$;

(iii) if $x \in S$, $x \leqslant b$ for some $b \in B$, then $x \in B$.

We note that every right ordered k-ideal or left ordered k-ideal is an ordered k-bi-ideal of S.

Example 3.2. Let $S = \{A, B, C, D, E, F\}$, where $A = \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$, $B = \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$, $C = \begin{bmatrix} \{1\} & \{1\} \\ \emptyset & \emptyset \end{bmatrix}$, $D = \begin{bmatrix} \{1\} & \emptyset \\ \{1\} & \emptyset \end{bmatrix}$, $E = \begin{bmatrix} \{1\} & \{1\} \\ \{1\} & \emptyset \end{bmatrix}$, $F = \begin{bmatrix} \{1\} & \{1\} \\ \{1\} & \{1\} \end{bmatrix}$.
We defined operations $+$ and $\cdot$ on S by letting $U = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $V = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$U + V = \begin{bmatrix} a_1 \cup b_1 & a_2 \cup b_2 \\ a_3 \cup b_3 & a_4 \cup b_4 \end{bmatrix} \quad \text{and}$$
$$U \cdot V = \begin{bmatrix} (a_1 \cap b_1) \cup (a_2 \cap b_3) & (a_1 \cap b_2) \cup (a_2 \cap b_4) \\ (a_3 \cap b_1) \cup (a_4 \cap b_3) & (a_3 \cap b_2) \cup (a_4 \cap b_4) \end{bmatrix}.$$

The tables of both operations are shown as follows.

$+$	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	B	C	D	E	F
C	C	C	C	E	E	F
D	D	D	E	D	E	F
E	E	E	E	E	E	F
F	F	F	F	F	F	F

and

$\cdot$	A	B	C	D	E	F
A	A	A	A	A	A	A
B	A	B	B	D	D	D
C	A	C	C	F	F	F
D	A	B	B	D	D	D
E	A	C	C	F	F	F
F	A	C	C	F	F	F

We defined a partially ordered relation $\leqslant$ on L by

$$U \leqslant V \text{ if and only if } a_1 \subseteq b_1,\ a_2 \subseteq b_2,\ a_3 \subseteq b_3 \text{ and } a_4 \subseteq b_4.$$

Then $A \leqslant B \leqslant C \leqslant E \leqslant F$ and $A \leqslant B \leqslant D \leqslant E \leqslant F$.

We can see that $(S, +, \cdot, \leqslant)$ is an ordered semiring and $T = \{A\}$ is its ordered k-ideal, $Y = \{A, B, C\}$ is a left ordered k-ideal but not a right ordered k-ideal, $Z = \{A, B, D\}$ is a right ordered k-ideal but not a left ordered k-ideal and $X = \{A,\ B\}$ is an ordered k-bi-ideal but not a left or a right ordered k-ideal.

Theorem 3.3. *Let B be a bi-ideal of an ordered semiring S. Then the following statements are equivalent.*

(i) *B is an ordered k-bi-ideal of S.*

(ii) *If $a + x \leqslant b$ for some $a, b \in B$, then $x \in B$.*

(iii) *$\overline{B} = B$.*

Proof. (i) $\Rightarrow$ (ii): Let B be an ordered k-bi-ideal of S. If $x + a \leqslant b$ for some $a, b \in B$ and $x \in S$. Then $x + a \in B$. It follows that there exists $p \in B$ such that $x + a = p$. By assumption, $x \in B$.

(ii) $\Rightarrow$ (iii): Let $x \in \overline{B}$. Then there exist $a, b \in B$ such that $x + a \leqslant b$. By assumption, we have $x \in B$. Thus, $\overline{B} = B$.

(iii) $\Rightarrow$ (i): Assume that $\overline{B} = B$. Let $x \in S$ such that $x + a = b$ for some $a, b \in B$. Then $x \in \overline{B}$. By assumption, we have $x \in B$. By Lemma 2.3(i), $(B] \subseteq \overline{B} = B$. Altogether, B is an ordered k-bi-ideal of S. □

Theorem 3.4. *Let B be a bi-ideal of an ordered semiring S. Then $\overline{(B]}$ is the smallest ordered k-bi-ideal of S containing B.*

Proof. It is clear that $B \subseteq \overline{(B]}$. By Lemma 2.2, $\overline{(B]}$ is closed under addition. By Lemma 2.1(iii) and Lemma 2.4, we have $\overline{(B]}\ \overline{(B]} \subseteq \overline{(\overline{(B]}B]} \subseteq \overline{(\overline{(BB]}]} \subseteq \overline{(\overline{(B]}]} = \overline{(B]}$. By Lemma 2.1($iii$), $\overline{(B]}S\overline{(B]} \subseteq \overline{(\Sigma BSB]} \subseteq \overline{(B]}$. Thus, $\overline{(B]}$ is a bi-ideal of S. By Lemma 2.3(ii), we have $\overline{\overline{(B]}} = \overline{(B]}$. By Theorem 3.3, $\overline{(B]}$ is an ordered k-bi-ideal of S. Let K be an ordered k-bi-ideal of S containing B. Then $(B] \subseteq (K] = K$ and $\overline{(B]} \subseteq \overline{K} = K$. Then $\overline{(B]}$ is the smallest ordered k-bi-ideal of S containing B. □

Corollary 3.5. *A bi-ideal B of an ordered semiring S is an ordered k-bi-ideal if and only if $\overline{(B]} = B$.* □

Theorem 3.6. *If intersection of a family of ordered k-bi-ideals of an ordered semiring S is not empty, then it is an ordered k-bi-ideal of S.* □

Definition 3.7. An ordered k-bi-ideal B of S is called a *semiprime ordered k-bi-ideal* if $\overline{(\Sigma A^2]} \subseteq B$ implies $A \subseteq B$ for any ordered k-bi-ideal A of S.

Definition 3.8. An ordered k-bi-ideal B of S is called a *prime ordered k-bi-ideal* if $\overline{(\Sigma AC]} \subseteq B$ implies $A \subseteq B$ or $C \subseteq B$ for any ordered k-bi-ideal A, C of S.

Definition 3.9. An ordered k-bi-ideal B of S is called a *strongly prime ordered k-bi-ideal* if $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq B$ implies $A \subseteq B$ or $C \subseteq B$ for any ordered k-bi-ideal A, C of S.

Obviously, every strongly prime ordered k-bi-ideal of S is a prime ordered k-bi-ideal and every prime ordered k-bi-ideal of S is a semiprime ordered k-bi-ideal.

The following example shows that every prime ordered k-bi-ideal need not to be a strongly prime ordered k-bi-ideal.

Example 3.10. Let $S = \{a, b, c\}$. We define operations $+$ and $\cdot$ on S as the following tables.

$+$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

and

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

We defined a partially ordered relation $\leqslant$ on S by $\leqslant := \{(a, a), (b, b), (c, c), (a, b)\}$.

We can show that $(S, +, \cdot, \leqslant)$ is an ordered semiring and $\{a\}, \{a, b\}, \{a, c\}$ and S are all ordered k-bi-ideals of S. Now, we have $\{a\}$ is prime but not strongly prime, since $\overline{(\Sigma\{a, b\}\{a, c\}]} \cap \overline{(\Sigma\{a, c\}\{a, b\}]} = \{a\}$ but $\{a, b\} \nsubseteq \{a\}$ and $\{a, c\} \nsubseteq \{a\}$.

Example 3.11. Let $S = \{a, b, c, d, e, f\}$. We define operations $+$ and $\cdot$ on S as the following tables.

$+$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	c	d	e	f
c	c	c	c	e	e	f
d	d	d	e	d	e	f
e	e	e	e	e	e	f
f	f	f	f	f	f	f

and

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	b	b	c
c	a	b	c	b	c	c
d	a	a	a	d	d	f
e	a	b	c	d	e	f
f	a	d	f	d	f	f

We defined a partially ordered relation $\leqslant$ on S by

$$\begin{aligned}\leqslant := \{&(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, b), (a, c), (a, d), (a, e),\\ &(b, c), (b, d), (b, e), (c, e), (d, e)\}.\end{aligned}$$

The sets $T = \{a\}, X = \{a, b\}, Y = \{a, b, c\}, Z = \{a, b, d\}$ and S are all ordered k-bi-ideals of S. We find that Y, Z and S are strongly prime ordered k-bi-ideals, X is a semiprime ordered k-bi-ideal but not prime and T is not a semiprime ordered k-bi-ideal.

Definition 3.12. An ordered k-bi-ideal B of S is called an *irreducible ordered k-bi-ideal* if for any ordered k-bi-ideal A and C of S, $A \cap C = B$ implies $A = B$ or $C = B$.

Definition 3.13. An ordered k-bi-ideal B of S is called a *strongly irreducible ordered k-bi-ideal* if for any ordered k-bi-ideal A and C of S, $A \cap C \subseteq B$ implies $A \subseteq B$ or $C \subseteq B$.

It is clear that every strongly irreducible ordered k-bi-ideal of S is an irreducible ordered k-bi-ideal of S.

Theorem 3.14. *If intersection of any family of prime ordered k-bi-ideals* (*or semiprime ordered k-bi-ideals*) *of S is not empty, then it is a semiprime ordered k-bi-ideal.*

Proof. Let $\{K_i \mid i \in I\}$ be a family of prime ordered k-bi-ideals of S. Assume that $\bigcap_{i\in I} K_i \neq \emptyset$. For any ordered k-bi-ideal B of S, $\overline{(\Sigma B^2]} \subseteq \bigcap_{i\in I} K_i$ implies $\overline{(\Sigma B^2]} \subseteq K_i$ for all $i \in I$. Since K_i are prime ordered k-bi-ideals, $B \subseteq K_i$ for all $i \in I$. Hence, $B \subseteq \bigcap_{i\in I} K_i$. Thus, $\bigcap_{i\in I} K_i$ is semiprime. □

Theorem 3.15. *If B is a strongly irreducible and semiprime ordered k-bi-ideal of an ordered semiring S, then B is a strongly prime ordered k-bi-ideal of S.*

Proof. Let B be a strongly irreducible and semiprime ordered k-bi-ideal of S. Let $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq B$ for any ordered k-bi-ideals A and C of S. Since $\overline{(\Sigma (A\cap C)^2]} \subseteq \overline{(\Sigma AC]}$ and $\overline{(\Sigma (A\cap C)^2]} \subseteq \overline{(\Sigma CA]}$. We have $\overline{(\Sigma (A\cap C)^2]} \subseteq \overline{(\Sigma AC]} \cap \overline{(\Sigma CA]}$. Since $A\cap C$ is an ordered k-bi-ideal and B is a semiprime ordered k-bi-ideal, $A\cap C \subseteq B$. Since B is a strongly irreducible ordered k-bi-ideal, $A \subseteq B$ or $C \subseteq B$. Thus, B is a strongly prime ordered k-bi-ideal of S. □

Theorem 3.16. *If B is an ordered k-bi-ideal of an ordered semiring S and $a \in S$ such that $a \notin B$, then there exists an irreducible ordered k-bi-ideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proof. Let $\mathcal{K}$ be the set of all ordered k-bi-ideals of S containing B but not containing a. Then $\mathcal{K}$ is a nonempty set, since $B \in \mathcal{K}$. Clearly, $\mathcal{K}$ is a partially ordered set under the inclusion of sets. Let $\mathcal{H}$ be a chain subset of $\mathcal{K}$. Then $\cup\mathcal{H} \in \mathcal{K}$. By Zorn's Lemma, there exists a maximal element in $\mathcal{K}$. Let I be a maximal element in $\mathcal{K}$. Let A and C be any two ordered k-bi-ideals of S such that $A \cap C = I$. Suppose that $I \subset A$ and $I \subset C$. Since I is a maximal element in $\mathcal{K}$, we have $a \in A$ and $a \in C$. Then $a \in A \cap C = I$ which is a contradiction. Thus, $C = I$ or $A = I$. Therefore, I is an irreducible ordered k-bi-ideal. □

Theorem 3.17. *A prime ordered k-bi-ideal B of an ordered semiring S is a prime one sided ordered k-ideal of S.*

Proof. Let B be a prime ordered k-bi-ideal of S. Suppose B is not a one sided ordered k-ideal of S. It follows $\overline{(BS]} \nsubseteq B$ and $\overline{(BS]} \nsubseteq B$. Then $\overline{(\Sigma BS]} \nsubseteq B$ and $\overline{(\Sigma SB]} \nsubseteq B$. Since B is a prime ordered k-bi-ideal, $\overline{(\Sigma \overline{(\Sigma BS]}\ \overline{(\Sigma SB]}]} \nsubseteq B$. By Lemma 2.3($v$),

$$\begin{aligned}\overline{(\Sigma \overline{(\Sigma BS]}\ \overline{(\Sigma SB]}]} &\subseteq \overline{\Sigma\overline{\Sigma(\Sigma BS)(\Sigma SB)]}]} \subseteq \overline{\Sigma\overline{\Sigma(\Sigma BSSB)]}]} \\ &\subseteq \overline{\Sigma\overline{(\Sigma BSSB)]}]} \subseteq \overline{\Sigma\overline{(\Sigma B)]}]} \subseteq \overline{(\Sigma B]} = B.\end{aligned}$$

This is a contradiction. Therefore, $\overline{(\Sigma BS]} \subseteq B$ or $\overline{(\Sigma SB]} \subseteq B$. Thus, B is a prime one sided ordered k-ideal of S. □

Theorem 3.18. *Let B be an ordered k-bi-ideal of an ordered semiring S. Then B is prime if and only if for a right ordered k-ideal R and a left ordered k-ideal L of S, $\overline{(\Sigma RL]} \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Assume that B is a prime ordered k-bi-ideal of S. Let R be a right ordered k-ideal and L be a left ordered k-ideal of S such that $\overline{(\Sigma RL]} \subseteq B$. Since R and L are ordered k-bi-ideals of S, $R \subseteq B$ or $L \subseteq B$. Conversely, let A and C be any two ordered k-bi-ideals of S such that $\overline{(\Sigma AC]} \subseteq B$. Suppose that $C \nsubseteq B$. Let $a \in A$ and $c \in C \backslash B$. Then $\overline{(\mathbb{N}a + aS]} \subseteq A$ and $\overline{(\mathbb{N}c + Sc]} \subseteq C$. We have $\overline{(\Sigma \overline{(\mathbb{N}a + aS]}\ \overline{(\mathbb{N}c + Sc]}]} \subseteq \overline{(\Sigma AC]} \subseteq B$. By assumption, $\overline{(\mathbb{N}a + aS]} \subseteq B$ or $\overline{(\mathbb{N}c + Sc]} \subseteq B$. But $\overline{(\mathbb{N}c + Sc]} \nsubseteq B$ implies that $\overline{(\mathbb{N}a + aS]} \subseteq B$. Then $a \in B$. Thus, $A \subseteq B$ and B is a prime ordered k-bi-ideal of S. □

4. Fully ordered k-bi-idempotent semirings

In this section, we assume that S is an ordered semiring with zero.

Definition 4.1. An ordered semiring S is said to be *fully ordered k-bi-idempotent* if $\overline{(\Sigma B^2]} = B$ for any ordered k-bi-ideal B of S.

Example 4.2. The ordered semiring S defined in Example 3.2 is fully ordered k-bi-idempotent. The ordered semiring S defined in Example 3.11 is not fully ordered k-bi-idempotent, since $\overline{(\Sigma X^2]} = T \neq X$.

Theorem 4.3. *Let S be an ordered semiring. Then the following statements are equivalent.*

(i) *S is fully ordered k-bi-idempotent.*

(ii) *$A \cap C = \overline{(\Sigma AC]} \cap \overline{(\Sigma CA]}$ for any ordered k-bi-ideal A and C of S.*

(iii) *Each ordered k-bi-ideal of S is semiprime.*

Proof. $(i) \Rightarrow (ii)$: Assume that $\overline{(\Sigma B^2]} = B$ for any ordered k-bi-ideal B of S. Let A and C be any two ordered k-bi-ideals of S. By Theorem 3.6, $A \cap C$ is an ordered k-bi-ideal of S. By assumption, $A \cap C = \overline{(\Sigma (A \cap C)^2]} = \overline{(\Sigma (A \cap C)(A \cap C)]} \subseteq \overline{(\Sigma AC]}$. Similarly, we get $A \cap C \subseteq \overline{(\Sigma CA]}$. Therefore, $A \cap C \subseteq \overline{(\Sigma AC]} \cap \overline{(\Sigma CA]}$. Since ΣAC is closed under addition, by Lemma 2.2, $\overline{(\Sigma AC]}$ is also closed under addition. By Lemma 2.3(vi),

$$(\Sigma AC)(\Sigma AC) \subseteq \Sigma ACAC \subseteq \Sigma ASAC \subseteq \Sigma AC.$$

Then ΣAC is an ordered subsemiring of S. By Lemma 2.3(vi),

$$(\Sigma AC)S(\Sigma AC) \subseteq (\Sigma ACS)(\Sigma AC) \subseteq \Sigma ACSAC \subseteq \Sigma ASSAC \subseteq \Sigma ASAC \subseteq \Sigma AC.$$

Thus, ΣAC is a bi-ideal of S. By Theorem 3.4, $\overline{(\Sigma AC]}$ is an ordered k-bi-ideal of S. Similarly, $\overline{(\Sigma CA]}$ is an ordered k-bi-ideal. By Theorem 3.6, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]}$ is an ordered k-bi-ideal of S. By assumption, Lemma 2.3$(v), (vi)$ and Lemma 2.4, we have

$$\begin{aligned}\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} &= \overline{(\Sigma(\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]})(\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]}))]}\\ &\subseteq \overline{(\Sigma\overline{(\Sigma AC]}\ \overline{(\Sigma CA]}]} \subseteq \overline{(\Sigma\overline{(\Sigma ACCA]}]} \subseteq \overline{(\Sigma\overline{(\Sigma ASA]}]} \subseteq A.\end{aligned}$$

Similarly, we can show that $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq C$. Thus, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq A \cap C$. Hence, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} = A \cap C$.

$(ii) \Rightarrow (iii)$: Let B be an ordered k-bi-ideal of S. Suppose that $\overline{(\Sigma A^2]} \subseteq B$ for any ordered k-bi-ideal A of S. By assumption, we have $A = A \cap A = \overline{(\Sigma AA]} \cap \overline{(\Sigma AA]} = \overline{(\Sigma AA]} \subseteq B$. Hence, B is semiprime.

$(iii) \Rightarrow (i)$: Let B be an ordered k-bi-ideal of S. Since $\overline{(\Sigma B^2]}$ is an ordered k-bi-ideal, by assumption, $\overline{(\Sigma B^2]}$ is semiprime. Since $\overline{(\Sigma B^2]} \subseteq \overline{(\Sigma B^2]}$, $B \subseteq \overline{(\Sigma B^2]}$. Clearly, $\overline{(\Sigma B^2]} \subseteq B$. This shows that S is ordered k-bi-idempotent. □

Theorem 4.4. *Let S be a fully ordered k-bi-idempotent semiring and B be an ordered k-bi-ideal of S. Then B is strongly irreducible if and only if B is strongly prime.*

Proof. Assume that B is strongly irreducible. Let A and C be any two ordered k-bi-ideals of S such that $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq B$. By Theorem 4.3, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} = A \cap C$. Hence, $A \cap C \subseteq B$. By assumption, we have $A \subseteq B$ or $C \subseteq B$. Thus, B is a strongly prime ordered k-bi-ideal of S. Conversely, assume that B is strongly prime. Let A and C be any two ordered k-bi-ideals of S such that $A \cap C \subseteq B$. By Theorem 4.3, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} = A \cap C \subseteq B$. By assumption, we have $A \subseteq B$ or $C \subseteq B$. Thus, B is a strongly irreducible ordered k-bi-ideal of S. □

Theorem 4.5. *Every ordered k-bi-ideal of an ordered semiring S is a strongly prime ordered k-bi-ideal if and only if S is a fully ordered k-bi-idempotent semiring and the set of all ordered k-bi-ideals of S is totally ordered.*

Proof. Assume that every ordered k-bi-ideal of S is strongly prime. Then every ordered k-bi-ideal of S is semiprime. By Theorem 4.3, S is a fully ordered k-bi-idempotent semiring. Let A and C be any two ordered k-bi-ideals of S. By Theorem 3.6, $A \cap C$ is an ordered k-bi-ideal of S. By assumption, $A \cap C$ is a strongly prime ordered k-bi-ideal of S. By Theorem 4.3, $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} = A \cap C$. Then $A \subseteq A \cap C$ or $C \subseteq A \cap C$. Therefore, $A = A \cap C$ or $C = A \cap C$. Thus, $A \subseteq C$ or $C \subseteq A$. Conversely, assume that S is a fully ordered k-bi-idempotent semiring and the set of all ordered k-bi-ideals of S is a totally ordered set. Let B be any ordered k-bi-ideal of S. Let A and C be any two ordered k-bi-ideals of S such that $\overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq B$. By Theorem 4.3, $A \cap C = \overline{(\Sigma AC]} \cap \overline{(\Sigma CA]} \subseteq B$. By assumption, $A \subseteq C$ or $C \subseteq A$. Hence, $A \cap C = A$ or $A \cap C = C$. Thus, $A \subseteq B$ or $C \subseteq B$. Therefore, B is a strongly prime ordered k-bi-ideal of S. □

Since every strongly prime ordered k-bi-ideal is a prime ordered k-bi-ideal and by Theorem 4.3 and 4.5, we have the following corollary.

Corollary 4.6. *Let the set of all ordered k-bi-ideals of S be a totally ordered set under inclusion of sets. Then every ordered k-bi-ideal of S is strongly prime if and only if every ordered k-bi-ideal of S is prime.* □

Theorem 4.7. *If the set of all ordered k-bi-ideals of an ordered semiring S is a totally ordered set under inclusion of sets, then S is a fully ordered k-bi-idempotent if and only if each ordered k-bi-ideal of S is prime.*

Proof. Assume that S is a fully ordered k-bi-idempotent semiring. Let B be any ordered k-bi-ideal of S and A, C be any two ordered k-bi-ideals of S such that $\overline{(\Sigma AC]} \subseteq B$. By assumption, we have $A \subseteq C$ or $C \subseteq A$. Without loss of generality, suppose that $A \subseteq C$. Then $A = \overline{(\Sigma AA]} \subseteq \overline{(\Sigma AC]} \subseteq B$. Hence, B is a prime ordered k-bi-ideal of S. Conversely, assume that every ordered k-bi-ideal of S is prime. Then every ordered k-bi-ideal of S is semiprime. By Theorem 4.3, S is a fully ordered k-bi-idempotent semiring. □

Theorem 4.8. *If S is a fully ordered k-bi-idempotent semiring and B is a strongly irreducible ordered k-bi-ideal of S, then B is a prime ordered k-bi-ideal.*

Proof. Let B be a strongly irreducible ordered k-bi-ideal of a fully ordered k-bi-idempotent semiring S. Let A and C be any two ordered k-bi-ideals of S such that $\overline{(\Sigma AC]} \subseteq B$. Since $A \cap C$ is also an ordered k-bi-ideal of S. By assumption, $\overline{(\Sigma(A \cap C)^2]} = A \cap C$. Consider $A \cap C = \overline{(\Sigma(A \cap C)^2]} = \overline{(\Sigma(A \cap C)(A \cap C)]} \subseteq \overline{(\Sigma(AC]} \subseteq B$. Since B is a strongly irreducible ordered k-bi-ideal of S, $A \subseteq B$ or $C \subseteq B$. Hence, B is a prime ordered k-bi-ideal of S. □

5. Right ordered k-weakly regular semirings

First, we recall the definition of a right ordered k-weakly regular semiring and some of its properties given by Patchakhieo and Pibaljommee [5] which we need to use in this section. Then we give characterizations of right ordered k-weakly regular semirings using ordered k-bi-ideals.

An ordered semiring S is said to be a *right ordered k-weakly regular semiring* if $a \in \overline{(\Sigma(aS)^2]}$ for all $a \in S$.

Theorem 5.1. *Let S be an ordered semiring. Then the following statements are equivalent.*

(*i*) *S is a right ordered k-weakly regular.*

(*ii*) *$\overline{(\Sigma A^2]} = A$ for every right ordered k-ideal A of S.*

(*iii*) *$A \cap I = \overline{(\Sigma AI]}$ for every right ordered k-ideal A of S and every ordered k-ideal I of S.* □

Theorem 5.2. *An ordered semiring S is right ordered k-weakly regular if and only if $B \cap I \subseteq \overline{(\Sigma BI]}$ for any ordered k-bi-ideal B and ordered k-ideal I of S.*

Proof. Let S be a right ordered k-weakly regular semiring, B be an ordered k-bi-ideal and I be an ordered k-ideal of S. Let $a \in B \cap I$. By Lemma 2.1(iii), Lemma 2.3(v) and Lemma 2.4, we have

$$\begin{aligned} a \in \overline{(\Sigma(aS)^2]} = \overline{(\Sigma(aS)(aS)]} &\subseteq \overline{(\Sigma(aS)\overline{(\Sigma(aS)(aS)]}S]} \subseteq \overline{(\Sigma\overline{((aS)(\Sigma(aS)(aS))S]}]} \\ &\subseteq \overline{(\Sigma\overline{(\Sigma(aSa)(SaSS)]}]} \subseteq \overline{(\overline{(\Sigma(BSB)(SIS)]}]} \subseteq \overline{(\overline{(\Sigma BI]}]} = \overline{(\Sigma BI]}. \end{aligned}$$

Therefore, $B \cap I \subseteq \overline{(\Sigma BI]}$.

Conversely, assume that $B \cap I \subseteq \overline{(\Sigma BI]}$ for any ordered k-bi-ideal B and ordered k-ideal I of S. Let R be a right ordered k-ideal of S. Then R is an ordered k-bi-ideal of S. By assumption, Lemma 2.8, Lemma 2.1(iii), Lemma 2.3(vi) and Lemma 2.4, we have

$$\begin{aligned} R &= R \cap M_k(R) \\ &\subseteq \overline{(\Sigma RM_k(R)]} = \overline{(\Sigma R\overline{(\Sigma R + \Sigma RS + \Sigma SR + \Sigma SRS]}]} \\ &\subseteq \overline{(\Sigma\overline{(\Sigma R^2 + \Sigma R^2S + \Sigma RSR + \Sigma RSRS]}]} \\ &\subseteq \overline{(\Sigma R^2 + \Sigma R^2 + \Sigma R^2 + \Sigma R^2]} \\ &= \overline{(\Sigma R^2]}. \end{aligned}$$

Then $R = \overline{(\Sigma R^2]}$. Thus, by Theorem 5.1, S is a right ordered k-weakly regular semiring. □

Theorem 5.3. *An ordered semiring S is right ordered k-weakly regular if and only if $B \cap I \cap R \subseteq \overline{(\Sigma BIR]}$ for any ordered k-bi-ideal B, ordered k-ideal I and right ordered k-ideal R of S.*

Proof. Let S be a right ordered k-weakly regular semiring, B be an ordered k-bi-ideal, I be an ordered k-ideal and R be a right ordered k-ideal of S. Let $a \in B \cap I \cap R$. By assumption, Lemma 2.1(iii), Lemma 2.3(vi) and Lemma 2.4, we have

$$\begin{aligned} a \in \overline{(\Sigma(aS)^2]} = \overline{(\Sigma(aS)(aS)]} &\subseteq \overline{(\Sigma(aS)\overline{(\Sigma(aS)(aS)]}S]} \\ &\subseteq \overline{(\Sigma\overline{(\Sigma a(SaS)(aS)]}]} \subseteq \overline{(\overline{(\Sigma BIR]}]} = \overline{(\Sigma BIR]}. \end{aligned}$$

Therefore, $B \cap I \cap R \subseteq \overline{(\Sigma BIR]}$.

Conversely, assume that $B \cap I \cap R \subseteq \overline{(\Sigma BIR]}$ for any ordered k-bi-ideal B, ordered k-ideal I and right ordered k-ideal R of S. Since R is an ordered k-bi-ideal of S and S is also an ordered k-ideal of S. By assumption, $R = R \cap S \cap R \subseteq \overline{(\Sigma RSR]} \subseteq \overline{(\Sigma R^2]}$. Therefore, $R = \overline{(\Sigma R^2]}$. By Theorem 5.1, S is right ordered k-weakly regular. □

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