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On nuclei and conuclei of S-quantales

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Abstract. S-quantales have been proved to be injectives in the category of S-posets with S-submultiplicative order-preserving mappings as morphisms. In this work, algebraic investigations on S-quantales are presented. A representation theorem of an S-quantale according to nuclei is obtained, quotients of an S-quantale with respect to nuclei and congruences are completely studied. Simultaneously, the relationship between S-subquantales and conuclei of an S-quantale is established.

1. Preliminary

Various quantale-like structures (quantales, locales, quantale modules, quantale algebras, unital quantales etc.) have been studied for decades and they have useful applications in algebra, logic and computer science ([3], [6], [11], [12]). In [11], algebraic properties and applications of quantales are well studied. The idea was then extended to groupoid quantales [7], involutive quantales [9], [5], sup-lattices [10], quantale mudules [4], [14], [13], and quantale algebras [15], [8], etc. Recently, Zhang and Laan in [16] introduced a new kind of quantale-like structure, named S-quantales. It has been shown that S-quantles play an important role in the theory of injectivity on the category of S-posets with S-submultiplicative order-preserving mappings as morphisms. In fact, injectives in this category are exactly S-quantales. The purpose of this paper is to make a contribution on algebraic investigations of S-quantales. Lets us first recall some basic definitions.

In this work, S is always a *pomonoid*, i.e., a monoid S equipped with a partial order $\leqslant$ such that $ss' \leqslant tt'$ whenever $s \leqslant t$, $s' \leqslant t'$ in S. A poset $(A, \leqslant)$ together with a mapping $A \times S \to A$ (under which a pair (a, s) maps to an element of A denoted by as) is called a *right S-poset*, denoted by A_S, if for any $a, b \in A,\ s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leqslant b,\ s \leqslant t$ imply that $as \leqslant bt$.

A left S-poset can be defined similarly. In this paper we only consider right S-posets, therefore we will omit the word "right".

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Let A_S and B_S be S-posets. A mapping $f : A_S \to B_S$ is said to be *S-submultiplicative* if $f(a)s \leqslant f(as)$ for any $a \in A_S,\ s \in S$. We call f an *S-poset homomorphism* if it preserves both S-actions and orders.

An S-poset A_S is said to be an *S-quantale* ([16]) if

(1) the poset A is a complete lattice;

(2) $(\bigvee M)s = \bigvee\{ms \mid m \in M\}$ for each subset M of A and each $s \in S$.

An *S-quantale homomorphism* is a mapping between S-quantales which preserves both S-actions and arbitrary joins. An *S-subquantale* of an S-quantale A_S is indeed the relative subposet of A_S which closed under S-actions and arbitrary joins.

We begin with properties of S-quantale homomorphisms and mappings between S-quantales with right adjoints. Then a representation theorem of quotients for S-quantales by nuclei is presented. The important topic of relations between the lattices of nuclei and congruences of an S-quantale is fully investigated. Dually, the connection on S-subquantales and conuclei is studied.

2. Mappings and homomorphisms

Let $f : P \to Q$ be a join-preserving mapping of posets. By the adjoint functor theorem ([1]), f has a unique right adjoint $f_* : Q \to P$, fulfilling

$$f(x) \leqslant y \Longleftrightarrow x \leqslant f_*(y), \tag{1}$$

for any $x \in P,\ y \in Q$, and hence

$$f(f_*(y)) \leqslant y, \quad x \leqslant f_*(f(x)). \tag{2}$$

Given an S-quantale Q_S, and any $s \in S$, the mapping $s_- : Q_S \to Q_S$ defined by $s(a) = as$ for each $a \in Q_S$, preserves all joins, and thus has a unique right adjoint, denoted by s_*, satisfying

$$s(a) \leqslant b \Longleftrightarrow a \leqslant s_*(b), \tag{3}$$

and

$$s(s_*(a)) \leqslant a,\ a \leqslant s_*(s(a)), \tag{4}$$

for each $a, b \in Q_S$. It holds evidently that $s_*(a)s \leqslant a,\ \forall a \in Q_S$.

Proposition 2.1. *Let Q_S be an S-quantale. Then for any $b \in Q_S,\ s, t \in S$, the following statements hold.*

1. $s_*(t_*(b)) = (st)_*(b)$,
2. $s_*(b)s = b \Longleftrightarrow (\exists a \in Q_S)\ as = b$,
3. $s_*(bs) = b \Longleftrightarrow (\exists a \in Q_S)\ b = s_*(a)$.

Proof. We note that for any $x, b \in Q_S,\ s, t \in S$,

$$x \leqslant s_*(t_*(b)) \Longleftrightarrow xs \leqslant t_*(b) \Longleftrightarrow xst \leqslant b \Longleftrightarrow x \leqslant (st)_*(b),$$

by (3), so we obtain 1. 2 and 3 can be proved similarly. □

Proposition 2.2. *Let $f : P_S \to Q_S$ be an S-quantale homomorphism. Then*

$$f_*(s_*(a)) = s_*(f_*(a))$$

for any $a \in Q_S,\ s \in S$.

Proof. By (1) and f preserving S-actions, we have

$$\begin{aligned} f_*(s_*(a)) \leqslant s_*(f_*(a)) &\Longleftrightarrow f_*(s_*(a))s \leqslant f_*(a) \Longleftrightarrow f(f_*(s_*(a))s) \leqslant a \\ &\Longleftrightarrow f(f_*(s_*(a))s \leqslant a \Longleftrightarrow f(f_*(s_*(a)) \leqslant s_*(a), \end{aligned}$$

for each $a \in Q_S$. But the final inequality natural follows by (2), we soon get that $f_*(s_*(a)) \leqslant s_*(f_*(a))$. One may dually gain that $s_*(f_*(a)) \leqslant f_*(s_*(a))$. □

Recall that for a poset P, a monotone mapping j on P is said to be a *closure operator* if it is both increasing and idempotent.

Definition 2.3. Let Q_S be an S-quantale, j a closure operator on Q_S. We call j a *nucleus* if it is S-submultiplicative, i.e.,

$$j(a)s \leqslant j(as)$$

for each $a \in Q_S,\ s \in S$.

Lemma 2.4. *Let Q_S be an S-quantale, j a nucleus on Q_S. Then*

$$j(s_*(a)) \leqslant s_*(j(a))$$

for all $a \in Q_S,\ s \in S$.

Proof. Keep in mind that $s_*(a)s \leqslant a,\ \forall a \in Q_S,\ s \in S$, we immediately get that $j(s_*(a))s \leqslant j(s_*(a)s) \leqslant j(a)$, and thus $j(s_*(a)) \leqslant s_*(j(a))$ by (3). □

Lemma 2.5. *Let $f : P_S \to Q_S$ be an S-quantale homomorphism. Then $f_* : Q_S \to P_S$ is S-submultiplicative.*

Proof. By (2), $f(f_*(a)) \leqslant a,\ \forall a \in Q_S$, it follows that $f(f_*(a)s) = f(f_*(a))s \leqslant as$, and hence $f_*(a)s \leqslant f_*(as)$ by (1). □

Lemma 2.6. *Let $f : P_S \to Q_S$ be an S-quantale homomorphism. Then f_*f is a nucleus on P_S.*

Proof. If $a \leqslant b$ for $a, b \in P_S$, then $f(a) \leqslant f(b)$, and thus $f_*f(a) \leqslant f_*f(b)$ by the fact that f_* preserves arbitrary meets.

Directly applying (2), we hence obtain that $a \leqslant f_*f(a)$ and

$$f_*f(a) \leqslant f_*f(f_*f(a)) = f_*(ff_*)(f(a)) \leqslant f_*(f(a)),$$

for any $a \in P_S$. So f_*f is a closure operator.

In addition, Lemma 2.5 provides that

$$(f_*f)(a)s = (f_*(f(a))s \leqslant f_*(f(a)s) = (f_*f)(as),$$

for any $a \in Q_S, s \in S$. Consequently, f_*f is a nucleus as desired. □

3. Nuclei and a representation theorem

For an S-quantale Q_S, we write $\mathsf{Nuc}(Q_S)$ for the set of all nuclei on Q_S. $\mathsf{Nuc}(Q_S)$ will therefore become a complete lattice if it is equipped with the pointwise order. The following properties of nuclei can be easily gained.

Lemma 3.1. (cf. [16]) *Let Q_S be an S-quantale, j a nucleus on Q_S. Then for any $a \in Q_S, s \in S$, $j(as) = j(j(a)s)$.*

Lemma 3.2. *Let Q_S be an S-quantale, j a nucleus on Q_S. Then*

$$j\Big(\bigvee_{i\in I} j(a_i)\Big) = j\Big(\bigvee_{i\in I} a_i\Big), \quad \forall a_i \in Q_S,\ i \in I.$$

Proof. Follows from the property of j being a closure operator. □

Lemma 3.3. *Let Q_S be an S-quantale, $j,\ \widetilde{j} \in \mathsf{Nuc}(Q_S)$. Then the following statements hold.*

1. $j \leqslant \widetilde{j} \Longleftrightarrow \widetilde{j}j = \widetilde{j}$;
2. $j \leqslant \widetilde{j} \Longleftrightarrow \forall x, y \in Q_S,\ j(x) = j(y) \Rightarrow \widetilde{j}(x) = \widetilde{j}(y)$.

Given a nucleus j on an S-quantale Q_S. Write

$$Q_j = \{a \in Q_S \mid j(a) = a\}.$$

Then Q_j becomes an S-quantale with the S-action defined by

$$a \circ s = j(as), \quad a \in A, \quad s \in S,$$

and the order inherited from Q_S, where the joins are given by

$$\bigvee\nolimits^j D = j\left(\bigvee D\right)$$

for any $D \subseteq Q_j$ (cf. [16]).

Proposition 3.4. *Let Q_S be an S-quantale, $P_S \subseteq Q_S$ an S-subquantale. Then $P_S = Q_j$ for some nucleus j iff P_S is closed under meets and $s_*(a) \in P_S$ whenever $a \in P_S$.*

Proof. Suppose that $P_S = Q_j$ for some nucleus j on Q_S. It is routine to check that $\bigwedge A \in P_S$ for any $A \subseteq P_S$. Note that for any $a \in P_S$, $j(s_*(a)) \leqslant s_*(j(a)) = s_*(a)$ by Lemma 2.4, one gets that $s_*(a) \in P_S$, as well.

On the contrary, define a mapping j on Q_S by

$$j(x) = \bigwedge \{a \in P_S \mid x \leqslant a\}, \ \forall x \in Q_S.$$

Straightforward verification shows that j is a closure operator.

For any $x \in Q_S, s \in S$, $a \in P_S$, since $xs \leqslant a \Leftrightarrow x \leqslant s_*(a)$ by (3), and $s_*(a) \in P_S$ by the assumption, it follows that

$$j(xs) \leqslant a \Rightarrow xs \leqslant j(xs) \leqslant a \Rightarrow j(x) \leqslant s_*(a) \Rightarrow j(x)s \leqslant a,$$

and results in $j(x)s \leqslant j(xs)$. Therefore, j is a nucleus on Q_S.

By the definition of j and the fact that P_S being closed under meets, we finally achieve that $P_S = Q_j$. □

Let Q_S be an S-quantale, $\mathcal{P}(Q)$ the power set of Q. Define an S-action on $\mathcal{P}(Q)$ by

$$I \cdot s = \{as \mid a \in I, s \in S\}, \ \forall I \subseteq Q.$$

Then $(\mathcal{P}(Q)_S, \cdot, \subseteq)$ becomes an S-quantale. The following theorem provides a representation of an S-quantale according to quotients w.r.t. nuclei.

Theorem 3.5. (Representation Theorem) *Let Q_S be an S-quantale. Then there exists a nucleus j on $\mathcal{P}(Q)_S$ such that $Q_S \cong \mathcal{P}(Q)_j$.*

Proof. Define a mapping j on $\mathcal{P}(Q)_S$ by

$$j(I) = \big(\bigvee I\big)\big\downarrow, \ \forall I \in \mathcal{P}(Q)_S.$$

Clearly, j is a closure operator. Suppose that $I \subseteq Q_S$ and $x \in j(I)$. Then $xs \leqslant (\bigvee I)s = \bigvee(Is)$ for all $s \in S$, gives that $xs \in j(Is)$. Thus $j(I) \cdot s \subseteq j(Is)$.

We note that for any $I \subseteq Q_S$, $j(I) = I$ iff $I = a\downarrow$ for some $a \in Q_S$. Therefore,

$$\mathcal{P}(Q)_j = \{I \in \mathcal{P}(Q)_S \mid I = j(I)\} = \{I \subseteq Q_S \mid I = a\downarrow \text{ for some } a \in Q_S\}.$$

Now define a mapping $\sigma : Q_S \to \mathcal{P}(Q)_j$ by

$$\sigma(a) = a\downarrow, \ \forall a \in Q_S.$$

Then σ is certainly bijective. We remain to show that σ is a homomorphism. By virtue of

$$\sigma\Big(\bigvee_{i\in I} a_i\Big) = \Big(\bigvee_{i\in I} a_i\Big)\Big\downarrow = \Big(\bigvee\Big(\bigcup_{i\in I} a_i\downarrow\Big)\Big)\Big\downarrow = j\Big(\bigcup_{i\in I} a_i\downarrow\Big) = \bigvee_{i\in I}^{j} \sigma(a_i),$$

for any $a_i \in Q_S, \ i \in I$, and

$$\begin{aligned}\sigma(a) \circ s &= j\left(\sigma(a) \cdot s\right) = j(a\downarrow \cdot s) = \big(\bigvee(a\downarrow \cdot s)\big)\big\downarrow \\ &= \big(\bigvee\{xs \mid x \leqslant a\}\big)\big\downarrow = (as)\downarrow = \sigma(as),\end{aligned}$$

for each $a \in Q_S$, $s \in S$, we finally achieve that σ is an isomorphism between S-quantales Q_S and $\mathcal{P}(Q)_j$. □

4. Quotients of S-quantales

Let Q_S be an S-quantale. A congruence ρ on Q_S is an equivalence relation on Q_S which is compatible both with S-actions and joins, and has the further property that Q/ρ equipped with a partial order becomes an S-quantale, and the canonical mapping $\pi : Q_S \to (Q/\rho)_S$ is an S-quantale homomorphism. Similar to the case of S-posets ([2]), a simple way for Q/ρ being an S-quantale is that Q/ρ accompanies an order "$\sqsubseteq$" defined by a ρ-chain, that is,

$$[a]_\rho \sqsubseteq [b]_\rho \iff a \underset{\rho}{\leqslant} b,\ \forall a, b \in Q_S,$$

where $a \underset{\rho}{\leqslant} b$ is given by a sequence

$$a \leqslant a_1 \ \rho\ a_1^{'} \leqslant a_2 \ \rho\ a_2^{'} \leqslant .... \leqslant a_n \ \rho\ a_n^{'} \leqslant b,$$

for $a_i, a_i^{'} \in Q_S,\ i = 1, 2.., n$. We see at once that in the S-quantale $(Q/\rho, \sqsubseteq)$,

$$\bigvee_{i\in I}[a_i]_\rho = \Big[\bigvee_{i\in I} a_i\Big]_\rho,\ \forall a_i \in Q_S.$$

Let us denote by $\mathsf{Con}(Q_S)$ the set of all congruences on Q_S. Then $\mathsf{Con}(Q_S)$ is a complete lattice with the inclusion as order.

This section is devoted to presenting the intrinsic relationship between the posets $\mathsf{Nuc}(Q_S)$ and $\mathsf{Con}(Q_S)$, respectively. We begin with the following results.

Lemma 4.1. *Let Q_S be an S-quantale, $\rho \in \mathsf{Con}(Q_S)$, $\pi : Q_S \to (Q/\rho)_S$ be the canonical mapping. Then $\pi = \pi\pi_*\pi$.*

Proof. By Lemma 2.6, $\pi_*\pi$ is a nucleus on Q_S. So for any $a \in Q_S$, one has that $a \leqslant \pi_*\pi(a)$, and hence $\pi(a) \leqslant \pi\pi_*\pi(a)$. However, (1) indicates that $\pi\pi_*\pi(a) \leqslant \pi(a)$. Consequently, we get that $\pi(a) = \pi\pi_*\pi(a)$. □

Let us write $\pi_*\pi$ in Lemma 4.1 as j_ρ. As usual, π is a homomorphism on Q_S such that $\rho = \mathsf{ker}\pi$.

Lemma 4.2. *Let Q_S be an S-quantale, $\rho \in \mathsf{Con}(Q_S)$, $\pi : Q_S \to (Q/\rho)_S$ be the canonical mapping. Then $\mathsf{ker}j_\rho = \mathsf{ker}\pi$.*

Proof. Follows by Lemma 4.1. □

Lemma 4.3. *Let Q_S be an S-quantale, j a nucleus on Q_S. Then $\mathsf{ker}j$ is a congruence on Q_S.*

Proof. From Lemma 3.1, we have $j(as) = j(j(a)s),\ \forall a \in Q_S, s \in S$. Thus for any $(a, b) \in \mathsf{ker}j, s \in S$,

$$j(as) = j(j(a)s) = j(j(b)s) = j(bs),$$

that is, $(as, bs) \in \mathsf{ker}j$. Moreover, derived from Lemma 3.2, we obtain that

$$j\Big(\bigvee_{i\in I} a_i\Big) = j\Big(\bigvee_{i\in I} j(a_i)\Big) = j\Big(\bigvee_{i\in I} j(b_i)\Big) = j\Big(\bigvee_{i\in I} b_i\Big),$$

for any $(a_i, b_i) \in \mathsf{ker}j,\ i \in I$. Therefore, $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i) \in \mathsf{ker}j$ as needed. □

Now we are ready to characterize the concrete relationship between nuclei and congruences of an S-quantale.

Theorem 4.4. *Let Q_S be an S-quantale. Then there exists an isomorphism φ : $\mathsf{Nuc}(Q_S) \to \mathsf{Con}(Q_S)$ as posets. Moreover, for each $j \in \mathsf{Nuc}(Q_S)$, $Q_j \cong (Q/\varphi(j))_S$ as S-quantales.*

Proof. Define a mapping $\varphi: \mathsf{Nuc}(Q_S) \to \mathsf{Con}(Q_S)$ by

$$\varphi(j) = \mathsf{ker}j,$$

for each $j \in \mathsf{Nuc}(Q_S)$. Then by Lemma 4.3, $\mathsf{ker}j$ is a congruence on Q_S. From Lemma 3.3(2), we obtain that φ is an order embedding.

Suppose that $\rho \in \mathsf{Con}(Q_S)$, and $\pi : Q_S \to (Q/\rho)_S$ is the canonical mapping. Then by Lemma 4.2, we have

$$\varphi(j_\rho) = \mathsf{ker}j_\rho = \mathsf{ker}\pi = \rho.$$

We hence conclude that $\mathsf{Nuc}(Q_S)$ is isomorphic to $\mathsf{Con}(Q_S)$ as posets.

For each $j \in \mathsf{Nuc}(Q_S)$, define $f : (Q/\mathsf{ker}j)_S \to Q_j$ and $g : Q_j \to (Q/\mathsf{ker}j)_S$ as

$$f([a]_{\mathsf{ker}j}) = j(a),$$

for each $[a]_{\mathsf{ker}j} \in (Q/\mathsf{ker}j)_S$, and

$$g(a) = [a]_{\mathsf{ker}j},$$

for any $a \in Q_j$. We need to show that f and g are invertible S-quantale homomorphisms.

Obviously, f is well-defined. For any $a \in Q_S, s \in S$, since $j(j(a)s) = j(as)$ by Lemma 3.1, we obtain that

$$f([a]_{\mathsf{ker}j}s) = f([as]_{\mathsf{ker}j}) = j(as) = j(j(a)s) = j(a) \circ s = f([a]_{\mathsf{ker}j}) \circ s.$$

Moreover, Lemma 3.2 yields that

$$f\Big(\bigvee_{i\in I}[a_i]_{\mathsf{ker}j}\Big) = f\Big(\Big[\bigvee_{i\in I} a_i\Big]_{\mathsf{ker}j}\Big) = j\Big(\bigvee_{i\in I} a_i\Big) = j\Big(\bigvee_{i\in I} j(a_i)\Big) = \bigvee_{i\in I} j(a_i) = \bigvee_{i\in I}(f[a_i]_{\mathsf{ker}j}),$$

for each $[a_i]_{\mathsf{ker}j} \in (Q/\mathsf{ker}j)_S$, $i \in I$. Therefore, f is an S-quantale homomorphism.

It is clear that g is an S-poset homomorphism. Furthermore, the equalities

$$g\Big(\bigvee_{i\in I} a_i\Big) = g\Big(j\Big(\bigvee_{i\in I} a_i\Big)\Big) = \Big[j\Big(\bigvee_{i\in I} a_i\Big)\Big]_{\mathsf{ker}j} = \Big[\bigvee_{i\in I} a_i\Big]_{\mathsf{ker}j} = \bigvee_{i\in I}[a_i]_{\mathsf{ker}j} = \bigvee_{i\in I} g(a_i),$$

for any $a_i \in Q_j$, $i \in I$ indicate that g is an S-quantale homomorphism. We then achieve our aim by the final step, that is, for all $a \in Q_j$,

$$f(g(a)) = f([a]_{\mathsf{ker}j}) = j(a) = a,$$

and

$$g(f([a]_{\mathsf{ker}j})) = g(j(a)) = [j(a)]_{\mathsf{ker}j} = [a]_{\mathsf{ker}j},$$

for any $a \in Q_S$. $\square$

5. Conuclei and S-subquantales

In this section, we introduce the notion of conuclei on an S-quantale Q_S, and discuss the relationship between conuclei and S-subquantales of Q_S.

Definition 5.1. Let Q_S be an S-quantale. We call a coclosure operator g on Q_S a conucleus if it is S-submultiplicative.

Dually to Theorem 3.5, which represented quotients of an S-quantale by nuclei, the following theorem establishes the relation between conuclei and S-subquantales of an S-quantale.

Theorem 5.2. *Let Q_S be an S-quantale, g a conucleus on Q_S. Then*

$$Q_g = \{a \in Q_S \mid g(a) = a\}$$

is an S-subsquantale of Q_S. Moreover, for any S-subquantale P_S of Q_S, there is a conucleus g on Q_S, such that $P_S = Q_g$.

Proof. Firstly, we have

$$\bigvee A = \bigvee\{g(a) \mid a \in A\} \leqslant g\left(\bigvee\{a \mid a \in A\}\right) = g\left(\bigvee A\right),$$

for any $A \subseteq Q_g$, and

$$as = g(a)s \leqslant g(as) \leqslant as,$$

for each $a \in Q_g, s \in S$. It turns out that Q_g is an S-subquantale of Q_S.

Next, suppose that P_S is an S-subquantale of Q_S. Define a mapping g on Q_S as

$$g(b) = \bigvee\{a \in P_S \mid a \leqslant b\},\ \forall b \in Q_S.$$

Straightforward proving shows that g is order-preserving and $g(b) \leqslant b,\ \forall b \in Q_S$. Recall that P_S is join closed, $g(b) \in P_S$, and hence

$$g(b) \leqslant \bigvee\{a \in P_S \mid a \leqslant g(b)\} = g(g(b)).$$

So g is a coclosure operator. Together with the inequalities

$$\begin{aligned} g(b)s = \bigvee\{a \in P_S \mid a \leqslant b\} \cdot s &= \bigvee\{as \in P_S \mid a \leqslant b\} \\ &\leqslant \bigvee\{a \in P_S \mid a \leqslant bs\} = g(bs), \end{aligned}$$

for any $b \in Q_S,\ s \in S$, we consequently obtain that g is a conucleus on Q_S.

By the definition of g, we immediately get that $b \leqslant g(b)$, $\forall b \in P_S$. So $P_S \subseteq Q_g$. Another inclusion is clear. Therefore, $P_S = Q_g$ as required. □

Given an S-quantale Q_S, write $\mathsf{CoNuc}(Q_S)$ as the poset of all conuclei on Q_S equipped with pointwise order, and $\mathsf{Sub}(Q_S)$ the poset of all S-subquantales of Q_S with inclusion as order, respectively. Theorem 5.3 describes the potential connection between the posets $\mathsf{Sub}(Q_S)$ and $\mathsf{CoNuc}(Q_S)$.

Theorem 5.3. *Let Q_S be a fixed S-quantale. Then there is an isomorphism $k : \mathsf{Sub}(Q_S) \to \mathsf{CoNuc}(Q_S)$ as posets, such that for any $M_S \in \mathsf{Sub}(Q_S)$ we have $M_S = Q_{k(M_S)}$.*

Proof. Define mappings $h : \mathsf{CoNuc}(Q_S) \to \mathsf{Sub}(Q_S)$ and $k : \mathsf{Sub}(Q_S) \to \mathsf{CoNuc}(Q_S)$ as

$$h(g) = Q_g, \ \forall g \in \mathsf{CoNuc}(Q_S),$$

and

$$k(M_S) = g_{M_S}, \ \forall M_S \in \mathsf{Sub}(Q_S),$$

respectively, where g_{M_S} is given by

$$g_{M_S}(a) = \bigvee\{m \in M_S \mid m \leqslant a\} = \bigvee\{M_S \cap a\downarrow\}, \ \forall a \in Q_S.$$

It is routine to check that g_{M_S} is a coclosure operator. In addition, for any $s \in S$, $a \in Q_S$, the inequalities

$$\begin{aligned} g_{M_S}(a)s &= \Big(\bigvee\{m \in M_S \mid m \leqslant a\}\Big)s = \bigvee\{ms \in M_S \mid m \leqslant a\} \\ &\leqslant \bigvee\{m \in M_S \mid m \leqslant as\} = g_{M_S}(as) \end{aligned}$$

show that g_{M_S} is S-submultiplicative, and hence a conucleus. k being order-preserving is clear.

By Theorem 5.2, h is well-defined. Assume that $m, n \in \mathsf{CoNuc}(Q_S)$ with $m \leqslant n$. Then $a = m(a) \leqslant n(a) \leqslant a$, for any $a \in Q_m$, indicate that $a \in Q_n$. Thus h is order-preserving.

We next show that $hk = id_{\mathsf{Sub}(Q_S)}$, i.e., $Q_{g_{M_S}} = M_S$, $\forall M_S \in \mathsf{Sub}(Q_S)$. This follows by the fact that

$$g_{M_S}(x) = \bigvee(M_S \cap x\downarrow) = x,$$

for any $x \in M_S$, and conversely, $Q_{g_{M_S}} \subseteq M_S$ by the reason that M_S is closed under joins.

It remains to prove that $kh = id_{\mathsf{CoNuc}(Q_S)}$, i.e., $g_{Q_f} = f$, $\forall f \in \mathsf{CoNuc}(Q_S)$. Suppose that $a \in Q_S$. Then

$$f(a) \leqslant a \leqslant \bigvee(Q_f \cap a\downarrow) = g_{Q_f}(a).$$

Conversely, for any $x \in Q_f \cap a\downarrow$, $x = f(x) \leqslant f(a)$ give rise to that $f(a)$ is an upper bound of $Q_f \cap a\downarrow$. Therefore, we achieve that $g_{Q_f}(a) = f(a)$ and finally, $\mathsf{Sub}(Q_S) \cong \mathsf{CoNuc}(Q_S)$ as needed. □

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