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On topological semi-hoops

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Abstract. We investigate topological structuers on a semi-hoop A and under conditions show that there exists a topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. We prove that for each cardinal number α, there exists a topological semi-hoop of order α. Finally, the separation axioms on topological semi-hoops are study and show that for any infinite cardinal number α there exists a Hausdorff topological semi-hoop of order α with non-trivial topology.

1. Introduction

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields and topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to topological algebra. In this paper, we introduce the notion of topological semi-hoop and derive here conditions that imply a semi-hoop to be a topological semi-hoop. We prove that for each cardinal number α, there exists at least a topological semi-hoop of order α. Also, we study separation axioms on topological semi-hoop and show that for any infinite cardinal number α there exists a Hausdorff topological semi-hoop of order α with non-trivial topology. We prove that a Hausdorff topological semi-hoop algebra exists and we try to study some properties of it. Also, we investigate that under what conditions a topological semi-hoop can be a Hausdorff, connected, T_0 and T_1-spaces.

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2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1. An algebra $(A, \odot, \to, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a *semi-hoop* if it satisfies the following conditions:
$(SH1)$ $(A, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1,
$(SH2)$ $(A, \odot, 1)$ is a commutative monoid,
$(SH3)$ $(x \odot y) \to z = x \to (y \to z)$, for all $x, y, z \in A$.

On a semi-hoop A we define $x \leqslant y$ if and only if $x \to y = 1$. It is easy to see that " $\leqslant$ " is a partial order relation on A and for any $x \in A$, $x \leqslant 1$. A semi-hoop A is *bounded* if there exists an element $0 \in A$ such that $0 \leqslant x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for all $n \in \mathbb{N}$. In a bounded semi-hoop A, we define the negation $'$ on A by, $x' = x \to 0$, for all $x \in A$. If $(x')' = x$, for all $x \in A$, then the bounded semi-hoop A is said to have the *Double Negation Property*, or (DNP) for short. A semi-hoop A is called a *hoop* if $x \odot (x \to y) = y \odot (y \to x)$, for all $x, y \in A$. A semi-hoop A is called a $\sqcup$-*semi-hoop*, if $x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ and $\sqcup$ is a join operation on A.

The following proposition provides some properties of semi-hoops.

Proposition 2.2. (cf. [7]) *Let A be a semi-hoop. Then the following hold, for all $x, y, z \in A$:*
(i) $x \odot y \leqslant z$ *if and only if* $x \leqslant y \to z$,
(ii) $x \odot y \leqslant x, y$,
(iii) $x \leqslant y \to x$,
(iv) $x \odot (x \to y) \leqslant y$,
(v) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,
(vi) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$,
(vii) $(x \to y) \leqslant (y \to z) \to (x \to z)$,
$(viii)$ $(x \to y) \odot (y \to z) \leqslant (x \to z)$.

Remark 2.3. (cf. [7]) *$\sqcup$-semi-hoop $(A, \sqcup, \wedge)$ is a distributive lattice.*

Definition 2.4. Let A be a semi-hoop. A non-empty subset F of A is called a *filter* of A if,
$(F1)$ $x, y \in F$ implies $x \odot y \in F$,
$(F2)$ $x \leqslant y$ and $x \in F$ imply $y \in F$, for any $x, y \in A$.

We use $\mathcal{F}(A)$ to denote the set of all filters of A. Clearly, $1 \in F$, for all $F \in \mathcal{F}(A)$. $F \in \mathcal{F}(A)$ is called a *proper filter* if $F \neq A$. It can be easily seen that, if A is a bounded semi-hoop, then a filter is proper if and only if it is not containing 0.

Let $(A, \odot, \to, \wedge, 1)$ be a semi-hoop and $F \in \mathcal{F}(A)$. We define a binary relation $\sim_F$ on A by $x \sim_F y$ if and only if $x \to y$, $y \to x \in F$, for any $x, y \in A$. Then $\sim_F$

is a congruence on A. Let $A/F = \{x/F \mid x \in A\}$, where $x/F = \{y \in A \mid x \sim_F y\}$. Then the binary relation $\leqslant$ on A/F which is defined by:

$$x/F \leqslant y/F \text{ if and only if } x \to y \in F.$$

is a partial order relation on A/F. Thus $(A/F, \otimes, \rightsquigarrow, \sqcap, 1_{A/F})$ is a semi-hoop, where for any $x, y \in A$:

$$1_{A/F} = 1/F,\ x/F \otimes y/F = (x \odot y)/F,\ x/F \rightsquigarrow y/F = (x \to y)/F$$
$$\text{and} \quad x/F \sqcap y/F = (x \wedge y)/F.$$

Recall that a set X with a family $\mathcal{T}$ of it's subsets is called a *topological space*, denoted by $(X, \mathcal{T})$, if $X, \emptyset \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{T}$ are called *open sets* of X and the complement of $U \in \mathcal{T}$, that is U^c, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$. A subfamily $\{U_\alpha\}$ of $\mathcal{T}$ is said to be a *base* of U if for each $x \in U \in \mathcal{T}$, there exists an α such that $x \in U_\alpha \subseteq U$, or equivalently, each $U \in \mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subset P of topological space $(X, \mathcal{T})$ is said to be a *neighborhood* of $x \in X$ if there exists an open set U such that $x \in U \subseteq P$. Now, let $(A, \mathcal{T})$ be a topological space. We have the following separation axioms in $(A, \mathcal{T})$:

T_0: For each $x, y \in A$ and $x \neq y$, at least one of them has an open neighborhood not containing the other.

T_1: For each $x, y \in A$ and $x \neq y$, there exists two open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

T_2: For each $x, y \in A$ and $x \neq y$, both have disjoint open neighborhoods U and V such that $x \in U$ and $y \in V$.

3. Topological semi-hoops

Definition 3.1. Let $\mathcal{T}$ be a topology on semi-hoop $(A, \odot, \to, \wedge, 1)$ and let $*$ be one of the operations $\odot, \to, \wedge$. Then

- (i) $(A, *, 1)$ is called *right topological semi-hoop* if for each $a \in A$, the map $r_a : A \to A$, defined by $x \to x * a$ is continuous, or equivalently, for any $x \in A$ and each open neighborhood U of $x * a$, there exists an open neighborhood V of x such that $V * a \subseteq U$. In this case, we also call that operation $*$ is continuous in first variable.
- (ii) $(A, *, \mathcal{T})$ is called *topological semi-hoop*, if $* : A \times A \hookrightarrow A$ is continuous, or equivalently, if for any $x, y \in A$ and any open neighborhood W of $x * y$, there exist two open sets U and V such that $x \in U$, $y \in V$ and $U * V \subseteq W$.
- (iii) $(A, \mathcal{T})$ is called *(right)topological semi-hoop*, if $(A, \odot, \to, \wedge, \mathcal{T})$ is (right) topological semi-hoop.

For $U, V \subseteq A$ we define $U \odot V$, $U \to V$ and $U \wedge V$ as follows:

$$U \odot V = \{x \odot y \mid x \in U, y \in V\}, \quad U \to V = \{x \to y \mid x \in U, y \in V\}$$

$$\text{and} \quad U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$$

Example 3.1. (cf. [4]) (i) Let $A = \{0, a, b, c, 1\}$ and

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$x \wedge y = x \odot (x \to y)$. By routine calculations, A with these operations is a bounded semi-hoop. Define the topology $\mathcal{T} = \{\emptyset, \{0\}, \{a, b\}, \{1, c\}, \{a, b, c, 1\}, A\}$. Then it is easy to see that $(A, \mathcal{T})$ is a topological semi-hoop.

(ii) Let $A = \{a, b, 1\}$ be a chain. Then define, for any $x, y \in A$, $x \wedge y = \min\{x, y\}$ and the operations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	a

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	b	1

It is easy to see that A with these operations is a semi-hoop. We define the topology $\mathcal{T} = \{\emptyset, \{a\}, A\}$. Then by routine calculations, $(A, \odot, \to, \wedge, \mathcal{T})$ is a right topological semi-hoop. But $(A, \to, \mathcal{T})$ is not one topological semi-hoop. Because $1 \to a = a \in \{a\}$ such that A and $\{a\}$ are two open sets of 1 and a, respectively, such that $A \to \{a\} = A \nsubseteq \{a\}$.

Theorem 3.2. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. If $\{1\}$ is an open set, then $(A, \mathcal{T})$ is a topological semi-hoop.*

Proof. Let $\{1\}$ be an open set and $x \in A$. Since $(A, \to, \mathcal{T})$ is a topological semi-hoop and $x \to x = 1 \in \{1\}$, there is an open sets U such that $x \in U$, $x \to U = \{1\}$ and $U \to x = \{1\}$, which implies that $U = \{x\}$. Hence, $\mathcal{T}$ is a discrete topology on A and so $(A, \mathcal{T})$ is a topological semi-hoop. □

Theorem 3.3. *Let $(A, \odot, \to, \wedge, 1)$ be a semi-hoop and $\mathcal{F}$ be a family of filters which is closed under intersections. Then there exists a topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop.*

Proof. Define $\mathcal{T} = \{U \subseteq A \mid \forall x \in U, \exists F \in \mathcal{F}(\mathcal{A})$ such that $x/F \subseteq U\}$. For each $x \in A$ and $F \in \mathcal{F}$, the set $x/F \in \mathcal{T}$, because if y is an arbitrary element of x/F, then $y \in y/F = x/F$. It is easy to see that $\mathcal{T}$ is a topology on A. We prove that $* \in \{\odot, \to, \wedge\}$ is continuous. For this, suppose $x * y \subseteq U \in \mathcal{T}$ such that $* \in \{\odot, \to, \wedge\}$. Then for some $F \in \mathcal{F}$, $(x * y)/F \subseteq U$, and so $x/F * y/F \subseteq U$. Since x/F and y/F are two open neighborhoods of x and y, respectively, such that $x/F * y/F \subseteq (x * y)/F \subseteq U$. Hence, $*$ is continuous. Therefore, $(A, \mathcal{T})$ is a topological semi-hoop. $\square$

Theorem 3.4. *Let $(A, \odot, \to, \wedge, \mathcal{T})$ be a topological semi-hoop such that, for any $\emptyset \neq U \in \mathcal{T}$, $1 \in U$ and $a \notin A$. Suppose $A_a = A \cup \{a\}$. Then there exists a topology $\mathcal{T}_a$ on A_a such that $(A_a, \mathcal{T}_a)$ is a topological semi-hoop.*

Proof. Define the operation $\sqcap$, $\otimes$ and $\rightsquigarrow$ on A_a as follows,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A, y \in A \\ a & \text{if } x \in A_a, y = a \\ a & \text{if } x = a, y \in A_a \end{cases}, \quad x \rightsquigarrow y = \begin{cases} x \to y & \text{if } x \in A, y \in A \\ a & \text{if } x \in A, y = a \\ 1 & \text{if } x = a, y \in A_a \end{cases}$$

$$x \sqcap y = x \otimes (x \rightsquigarrow y)$$

By routine calculation, we can see that $(A_a, \otimes, \rightsquigarrow, \sqcap, 1)$ is a semi-hoop. It is easy to verify that $\mathcal{T}_a = \{U \cup \{a\} \mid U \in \mathcal{T}\} \cup \{\emptyset\}$ is a topology on A_a. Now, we prove that $(A_a, \mathcal{T}_a)$ is a topological semi-hoop. For this, we prove that $\otimes$ and $\rightsquigarrow$ are continuous.

Let $x \otimes y \in U \cup \{a\}$. In the following cases, we find two sets $V, W \in \mathcal{T}_a$ such that $x \in V$, $y \in W$ and $V \otimes W \subseteq U \cup \{a\}$.

Case 1. If $x, y \in A$, then $x \otimes y = x \odot y \in U$. Since $\odot$ is continuous, there exist $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $x \odot y \in V \odot W \subseteq U$. If $z_1 \in V \cup \{a\}$ and $z_2 \in W \cup \{a\}$, then $z_1 \otimes z_2 \in \{z_1 \odot z_2, a\} \subseteq U \cup \{a\}$. Hence, $V \cup \{a\} \otimes W \cup \{a\} \subseteq U \cup \{a\}$.

Case 2. If $x = a$ and $y \in A$, then $x = a \in \{a\} \in \mathcal{T}_a$, $y \in A_a \in \mathcal{T}_a$ and $\{a\} \otimes A_a = \{a\} \subseteq U \cup \{a\}$.

Case 3. If $x = y = a$, then $x = y = a \in \{a\} \in \mathcal{T}_a$ and $\{a\} \otimes \{a\} = \{a\} \in U \cup \{a\}$.

These Cases prove that $(A_a, \otimes, \mathcal{T}_a)$ is a topological semi-hoop.

Now, we prove that $\rightsquigarrow$ is continuous. For this, let $x \rightsquigarrow y \in U \cup \{a\}$. In the following cases, we find two sets $V, W \in \mathcal{T}_a$ such that $x \in V$, $y \in W$ and $V \rightsquigarrow W \subseteq U \cup \{a\}$.

Case 1. If $x, y \in A$, then $x \rightsquigarrow y = x \to y \in U$. Since $\to$ is continuous, there exist $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $x \to y \in V \to W \subseteq U$. If $z_1 \in V \cup \{a\}$ and $z_2 \in W \cup \{a\}$, since, for any $U \in \mathcal{T}$, $1 \in U$, then $z_1 \rightsquigarrow z_2 \in \{z_1 \to z_2, a, 1\} \subseteq U \cup \{a\}$. Hence, $V \cup \{a\} \rightsquigarrow W \cup \{a\} \subseteq U \cup \{a\}$.

Case 2. If $x = a$ and $y \in A$, then $x = a \in \{a\} \in \mathcal{T}_a$, $y \in A_a \in \mathcal{T}_a$ and $\{a\} \rightsquigarrow A_a = \{1\} \subseteq U \cup \{a\}$.

Case 3. If $x \in A$ and $y = a$, then $x \in A_a \in \mathcal{T}_a$, $y = a \in \{a\} \in \mathcal{T}_a$ and $A_a \rightsquigarrow \{a\} = \{a, 1\} \subseteq U \cup \{a\}$.

Case 4. If $x = y = a$, then $x = y = a \in \{a\} \in \mathcal{T}_a$ and $\{a\} \rightsquigarrow \{a\} = \{1\} \in U \cup \{a\}$.

These Cases prove that $(A_a, \rightsquigarrow, \mathcal{T}_a)$ is a topological semi-hoop. According to definition of $\sqcap$, since $\otimes$ and $\rightsquigarrow$ are continuous, it is clear that $\sqcap$ is continuous, too. Therefore, $(A_a, \mathcal{T}_a)$ is a topological semi-hoop. $\square$

Theorem 3.5. *For any $n \geqslant 2$ there exists a topological semi-hoop of order n.*

Proof. Let A be a semi-hoop of order $n \geqslant 1$. It is clear that, $\mathcal{T} = \{A, \emptyset\}$ is a topology on A, and so $(A, \mathcal{T})$ is a topological semi-hoop. Now, suppose $x \notin A$. Define $A_x = A \cup \{x\}$. Then by Theorem 3.4, there exist the operations $\sqcap$, $\otimes$, $\rightsquigarrow$ and topology $\mathcal{T}_x$ on A_x such that $(A_x, \mathcal{T}_x)$ is a topological semi-hoop. Since $\mathcal{T}_x = \{\emptyset, \{x\}, A_x\}$, it is clear that $\mathcal{T}_x$ is a non-trivial topology on A_x. $\square$

Theorem 3.6. *For any countable set A such that $1 \in A$, there exists a topological semi-hoop algebra on A.*

Proof. Consider $A = \{x_0 = 1, x_1, x_2, \ldots\}$ as a countable subset and define the operation $\wedge$, $\odot$ and $\to$ on A as follows,

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \qquad \text{and} \qquad x_i \to x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \to x_j = 1.$$

By routine calculation, we can see that $(A, \odot, \to, \wedge, 1)$ is a semi-hoop. The set $F_n = \{1, x_1, \ldots, x_n\}$, for any $n \geqslant 1$ is a filter of A. Let $B = \{F_n \mid n \geqslant 1\}$. By Theorem 3.3, there is a non-trivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. $\square$

Theorem 3.7. *Let $(A, \odot, \to, \wedge, 1, \mathcal{T})$ be a topological semi-hoop and α be a cardinal number. If $\mid A \mid\leqslant \alpha$, then there exists a topological semi-hoop $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U})$ such that $\mid B \mid\geqslant \alpha$, $1 \in U \in \mathcal{U}$ and A is a subalgebra of B.*

Proof. Let Γ be a collection of a topological semi-hoops $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ such that for any $A \subseteq H$ we have $\circ\mid_A = \odot$, $\dashrightarrow\mid_A = \to$ and $\cap\mid_A = \wedge$.

The following relation is a partial order on Γ:

$$(H,\circ,\dashrightarrow,\cap,1,\mathcal{U}) \leqslant (K,\star,\looparrowright,\sqcap,1,\mathcal{V}) \Leftrightarrow H \subseteq K, \star|_H = \circ, \looparrowright|_H = \dashrightarrow, \sqcap|_H = \cap,\ \mathcal{U} \subseteq \mathcal{V}.$$

Let $\sum = \{(H_i, \circ_i, \dashrightarrow_i, \cap_i, 1, \mathcal{U}_i) \mid i \in I\}$ be a chain in Γ. Put $H = \bigcup_{i\in I} H_i$ and $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i$. If x and y are two elements of H, since $\sum$ is a chain, then for some $i \in I$, $x, y \in H_i$. Define $x \circ y = x \circ_i y$, $x \dashrightarrow y = x \dashrightarrow_i y$ and $x \cap y = x \cap_i y$. We prove that $\circ$, $\dashrightarrow$ and $\cap$ are operations on H. Suppose $x, y \in H_i \cap H_j$. Since $\sum$ is a chain, $H_i \subseteq H_j$ or $H_j \subseteq H_i$. Without the lost of generality, assume that $H_i \subseteq H_j$. Let $* \in \{\circ, \dashrightarrow, \cap\}$. Then $*_j\mid_{H_i} = *_i$. So $x *_j y = x *_i y$. This proves that $*$ is an operation on H. Now, it is easy to see that $(H, \circ, \dashrightarrow, \cap, 1)$ is a semi-hoop such that $\circ\mid_A = \odot$, $\dashrightarrow\mid_A = \to$ and $\cap\mid_A = \wedge$.

On the other hand, since $\sum$ is a chain, $\mathcal{U}$ is a topology on H. We prove that $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ is a topological semi-hoop. Let $* \in \{\circ, \dashrightarrow, \cap\}$ and $x*y \in U \in \mathcal{U}$.

Then there exists an $i \in I$ such that $x * y = x *_i y \in U \in \mathcal{U}_i$. Since $*_i$ is continuous in $(H_i, \mathcal{U}_i)$, there are $V, W \in U_i$ such that $x \in V$, $y \in W$ and $V *_i W \subseteq U$. This proves that $*$ is continuous in $(H, \mathcal{U})$. Thus, $(H, \circ, \dashrightarrow, \cap, 1, \mathcal{U})$ is an upper bound for $\sum$. By Zorn's Lemma, Γ has a maximal element. Suppose $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U})$ is a maximal element of Γ. We prove that $\mid B \mid \geqslant \alpha$. If $\mid B \mid < \alpha$, then for some non-empty set C, $\mid B \cup C \mid = \alpha$. Take $a \in C - B$ and put $H = B \cup \{a\}$. Then by Theorem 3.4, it is easy to see that H with the following operations

$$x \bullet y = \begin{cases} x \otimes y & \text{if } x \in B, y \in B \\ a & \text{if } x \in H, y = a \\ a & \text{if } x = a, y \in H \end{cases} \qquad x \curvearrowright y = \begin{cases} x \rightsquigarrow y & \text{if } x \in B, y \in B \\ a & \text{if } x \in B, y = a \\ 1 & \text{if } x = a, y \in H \end{cases}$$

$$\text{and} \quad x \sqcap_1 y = x \bullet (x \curvearrowright y)$$

is a semi-hoop. The set $\mathcal{D} = \mathcal{U} \cup \{\{a\}\}$ is a subbase for a topology $\mathcal{V}$ on H. Similar to the proof of Theorem 3.4, we can see that, $(H, \mathcal{V})$ is a topological semi-hoop. But $(H, \bullet, \curvearrowright, \sqcap_1, \mathcal{V})$ is a member of Γ that $(B, \otimes, \rightsquigarrow, \sqcap, 1, \mathcal{U}) < (H, \bullet, \curvearrowright, \sqcap_1, \mathcal{V})$, which is a contradiction. Therefore, $\mid B \mid \geqslant \alpha$ and A is a subalgebra of B. $\square$

Theorem 3.8. *Let α be an infinite cardinal number. Then there is a topological semi-hoop of order α.*

Proof. Let X be a set of cardinality α, $0, 1 \in X$, $A = \{x_0 = 1, x_1, x_2, \ldots\}$ – a countable subset of X such that $0 \notin A$. Similar to Theorem 3.6, define the operation $\wedge$, $\odot$, $\rightarrow$ and $\leqslant$ on A as follows,

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \qquad \text{and} \qquad x_i \rightarrow x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \rightarrow x_j = 1.$$

By routine calculation, we can see that $(A, \odot, \rightarrow, \wedge, 1)$ is a semi-hoop. The set $F_n = \{1, x_1, \ldots, x_n\}$, for any $n \geqslant 1$ is a filter of A. Let $B = \{F_n \mid n \geqslant 1\}$. By Theorem 3.3, there is a non-trivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological semi-hoop. Now, define the binary operation $\otimes$, $\rightsquigarrow$ and $\sqcap$ on X as follows,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A, y \in A \\ x & \text{if } x \notin A, y \in A \\ y & \text{if } x \in A, y \notin A \\ 0 & \text{if } x \notin A, y \notin A \end{cases} \qquad x \rightsquigarrow y = \begin{cases} x \rightarrow y & \text{if } x \in A, y \in A \\ y & \text{if } x \in A, y \notin A \\ 1 & \text{if } x \notin A, y \in A \\ 1 & \text{if } x, y \notin A, x = y \\ 1 & \text{if } x, y \notin A \cup \{0\}, x \neq y \\ 1 & \text{if } x = 0, y \notin A \cup \{0\} \\ 0 & \text{if } x \notin A \cup \{0\}, y = 0 \end{cases}$$

$$\text{and} \quad x \sqcap y = \begin{cases} 0 & \text{if } x, y \notin A \cup \{0\}, x \neq y \\ x \otimes (x \rightsquigarrow y) & \text{otherwise.} \end{cases}$$

By routine calculation, we can see that $(X, \otimes, \rightsquigarrow, \sqcap, 0, 1)$ is a bounded semi-hoop of order α and the set $C = \mathcal{T} \cup \{\{x\} \mid x \notin A\}$ is a subbase for a topology $\mathcal{U}$ on X. Since $\{1\} \notin \mathcal{U}$, $\mathcal{U}$ is a non-trivial topology on X. In the following cases we will show that $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ is a topological semi-hoop. For this, let $x \otimes y \in U \in C$.

Case 1. If $x, y \in A$, then $x \otimes y = x \odot y \in U \in \mathcal{T}$. Since $\odot$ is continuous in $(A, \mathcal{T})$, there are $V, W \in \mathcal{T}$ containing x, y, respectively, such that $V \odot W \subseteq U$. Hence, $V \otimes W \subseteq U$, which implies that $\otimes$ is continuous in $(X, \mathcal{U})$.

Case 2. If $x \notin A$ and $y \in A$, then $x \otimes y = \{x\} \subseteq U$. Now $\{x\}$ and A, both, belong to $\mathcal{U}$ and $x \in \{x\}$, $y \in A$ and $\{x\} \otimes A = \{x\} \subseteq U$.

Case 3. If $x \in A$ and $y \notin A$, then A and $\{y\}$ are two elements of $\mathcal{U}$ such that $x \in A$, $y \in \{y\}$ and $x \otimes y = \{y\}$, and so $A \otimes \{y\} = \{y\} \subseteq U$.

Case 4. If $x, y \notin A$, then $x \otimes y = \{0\} \subseteq U$. Then $\{x\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \otimes \{y\} = \{0\} \subseteq U$.

These Cases prove that $(X, \otimes, \mathcal{U})$ is a topological semi-hoop.

Now, we prove that $\rightsquigarrow$ is continuous. For this, let $x \rightsquigarrow y \in U$. In the following cases, we find two sets $V, W \in \mathcal{U}$ such that $x \in V$, $y \in W$ and $V \rightsquigarrow W \subseteq U$.

Case 1. If $x, y \in A$, then $x \rightsquigarrow y = x \rightarrow y \in U \in \mathcal{T}$. Since $\rightarrow$ is continuous in $(A, \mathcal{T})$, there are $V, W \in \mathcal{T}$ containing x, y, respectively, such that $V \rightarrow W \subseteq U$. Hence, $V \rightsquigarrow W \subseteq U$, which implies that $\rightsquigarrow$ is continuous in $(X, \mathcal{U})$.

Case 2. If $x \in A$ and $y \notin A$, then $x \rightsquigarrow y = \{y\} \subseteq U$. Thus, A and $\{y\}$ are two elements of $\mathcal{U}$ such that $x \in A$, $y \in \{y\}$ and $x \rightsquigarrow y = \{y\}$, and so $A \rightsquigarrow \{y\} = \{y\} \subseteq U$.

Case 3. If $x \notin A$ and $y \in A$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Now $\{x\}$ and A, both, belong to $\mathcal{U}$ and $x \in \{x\}$, $y \in A$ and $\{x\} \rightsquigarrow A = \{1\} \subseteq U$.

Case 4. If $x, y \notin A$ and $x = y$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{x\}$ is an open set in $\mathcal{U}$ which contains x and $\{x\} \rightsquigarrow \{x\} = \{1\} \subseteq U$.

Case 5. If $x, y \notin A \cup \{0\}$ and $x \neq y$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{x\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{1\} \subseteq U$.

Case 6. If $x = 0$ and $y \notin A \cup \{0\}$, then $x \rightsquigarrow y = \{1\} \subseteq U$. Then $\{0\}$ and $\{y\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{1\} \subseteq U$.

Case 7. If $x \notin A \cup \{0\}$ and $y = 0$, then $x \rightsquigarrow y = \{0\} \subseteq U$. Then $\{x\}$ and $\{0\}$ are two open sets in $\mathcal{U}$ which contains x, y, respectively, and $\{x\} \rightsquigarrow \{y\} = \{0\} \subseteq U$.

These Cases prove that $(X, \rightsquigarrow, \mathcal{U})$ is a topological semi-hoop. According to definition of $\sqcap$, since $\otimes$ and $\rightsquigarrow$ are continuous, then $\sqcap$ is continuous, too. Therefore, there is a topological semi-hoop of order α. $\square$

Theorem 3.9. *Let α be an infinite cardinal number. Then there is a right topological semi-hoop of order α, which is not a topological semi-hoop.*

Proof. Let A be a set with cardinal number α such that $0, 1 \in A$. Consider a countable subset $A_1 = \{x_0 = 1, x_1, x_2, \ldots\}$ of A and define

$$x_i \wedge x_j = x_i \odot x_j = x_{\max\{i,j\}} \qquad \text{and} \qquad x_i \rightarrow x_j = \begin{cases} 1 & \text{if } i \geqslant j \\ x_j & \text{if } i < j \end{cases}$$

$$x_i \leqslant x_j \text{ if and only if } x_i \to x_j = 1$$

By routine calculations, we can see that $(A_1, \odot, \to, \wedge, 1)$ is a semi-hoop. If $U_i = \{x_i, x_{i+1}, x_{i+2}, \ldots\}$, then $B = \{U_i \mid i = 1, 2, 3, \ldots\}$ is a base for a topology $\mathcal{T}_{A_1}$ on A_1. We prove that $(A_1, \odot, \to, \wedge, 1, \mathcal{T}_{A_1})$ is a right topological semi-hoop. For this, let $x_i \odot x_j \in U \in \mathcal{T}_{A_1}$. If $i \leqslant j$, then $x_i \odot x_j = x_j \in U$. Since $x_j \in U_j$, we have $x_j \in U_j \cap U$, then $x_i \odot x_j = x_i \odot (U_j \cap U) = U_j \cap U \subseteq U$. By the similar way, if $i > j$, then $x_i \odot x_j = x_i \in U$. Since $\odot$ is commutative, $x_j \odot x_i \in x_j \odot (U_i \cap U) = U_i \cap U \subseteq U$. Hence, $(A_1, \odot, \mathcal{T}_{A_1})$, and so $(A_1, \odot, \wedge, \mathcal{T}_{A_1})$ is a topological semi-hoop. Now, suppose $x_i \to x_j \in U \in \mathcal{T}_{A_1}$. If $i \geqslant j$, then $x_i \to x_j = 1 \in U$. Since A_1 is only open neighborhood of $\{1\}$, $U = A_1$. Clearly, $x_j \in A_1$ and $x_i \to A_1 \subseteq A_1 = U$. If $i < j$, then $x_i \to x_j = x_j \in U$. Since B is a base for $\mathcal{T}_{A_1}$, $x_j \in U_j \subseteq U$. Since $i < j$, $x_i \to U_j = U_j \subseteq U$. Therefore, $(A_1, \to, \mathcal{T}_{A_1})$ is a right topological semi-hoop. But this space is not a topological semi-hoop, because $x_1 \in U_1$, $x_2 \in U_2$ and $x_1 \to x_2 = x_2 \in U_2$, but $1 = x_2 \to x_2 \in U_1 \to U_2 \notin U_2$. Similar to Theorem 3.8, let A with the following operations,

$$x \otimes y = \begin{cases} x \odot y & \text{if } x \in A_1, y \in A_1 \\ x & \text{if } x \notin A_1, y \in A_1 \\ y & \text{if } x \in A_1, y \notin A_1 \\ 0 & \text{if } x \notin A_1, y \notin A_1 \end{cases} \qquad x \rightsquigarrow y = \begin{cases} x \to y & \text{if } x \in A_1, y \in A_1 \\ y & \text{if } x \in A_1, y \notin A_1 \\ 1 & \text{if } x \notin A_1, y \in A_1 \\ 1 & \text{if } x, y \notin A_1, x = y \\ 1 & \text{if } x, y \notin A_1 \cup \{0\}, x \neq y \\ 1 & \text{if } x = 0, y \notin A_1 \cup \{0\} \\ 0 & \text{if } x \notin A_1 \cup \{0\}, y = 0 \end{cases}$$

$$\text{and} \quad x \sqcap y = \begin{cases} 0 & \text{if } \ x, y \notin A \cup \{0\}, x \neq y \\ x \otimes (x \rightsquigarrow y) & \text{otherwise.} \end{cases}$$

Then $(A, \otimes, \rightsquigarrow, \sqcap, 0, 1)$ is a bounded semi-hoop. As the proof of Theorem 3.8, we can prove that $\mathcal{B} = \mathcal{T}_{A_1} \cup \{\{x\} \mid x \notin A_1\}$ is a subbase for a topology $\mathcal{U}$ on A such that $(A, \otimes, \rightsquigarrow, \sqcap, 0, 1, \mathcal{U})$ is a right topological bounded semi-hoop. But $\rightsquigarrow$ is not continuous in $(A, \mathcal{U})$, because $\to$ is not continuous in $(A_1, \mathcal{T}_{A_1})$. □

4. Hausdorff topological semi-hoops

Theorem 4.1. *Let $(A, \mathcal{T})$ be a topological semi-hoop and $\mathcal{F}(A)$ a basis of $\mathcal{T}$. Then, for all $x \in A$, $x^2 = x$ if and only if $(A, \mathcal{T})$ is a T_0-space.*

Proof. $(\Rightarrow)$ Let $x^2 = x$, for all $x \in A$. Then $x^n = x$, for all $n \in \mathbb{N}$. Suppose $x, y \in A$ and $x \neq y$. Since $\mathcal{F}(A)$ is a basis of $\mathcal{T}$, the filters $\langle x \rangle$ and $\langle y \rangle$ are two open neighborhoods of x and y, respectively. If $y \in \langle x \rangle$ and $x \in \langle y \rangle$, then there exist $n, m \in \mathbb{N}$ such that $x^n \leqslant y$ and $y^m \leqslant x$. Hence, $x \leqslant y$ and $y \leqslant x$, and so $x = y$, which is a contradiction.
$(\Leftarrow)$ Let $(A, \mathcal{T})$ be a T_0-space and $x \in A$. If $x^2 \neq x$, then there exists $U \in \mathcal{F}(A)$ such that $x \in U$ and $x^2 \notin U$ or there exists $V \in \mathcal{F}(A)$ such that $x \notin V$ and $x^2 \in V$. But both statements are not correct, because U and $V \in \mathcal{F}(A)$. □

Theorem 4.2. *Let $(A, \mathcal{T})$ be a topological semi-hoop and U be an open neighborhood of* 1*. Then,*

(*i*) *if for each $x \in A$, $U \to x$ is an open neighborhood of x, then $(A, \mathcal{T})$ is T_0-space,*

(*ii*) *if for each $x \in A$, $U \odot x$ is an open neighborhood of x, then $(A, \mathcal{T})$ is T_0-space.*

Proof. (i). Let $x, y \in A$ and $x \neq y$. Then $U \to x \in \mathcal{T}$ and $U \to y \in \mathcal{T}$. If $x \in U \to y$ and $y \in U \to x$, then by Proposition 2.2(v), $y \leqslant x$ and $x \leqslant y$. Hence, $x = y$, which is a contradiction. Therefore, $(A, \mathcal{T})$ is T_0-space.

(ii). The proof is similar (i). □

Theorem 4.3. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then $(A, \to, \mathcal{T})$ is T_0-space if and only if for any $1 \neq x \in A$, there exist $U \in \mathcal{T}$ such that $x \in U$ and $1 \notin U$.*

Proof. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Then there exist a $U \in \mathcal{T}$ such that $x \to y \in U$ and $1 \notin U$. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. So, $x \notin Q$. Hence, $(A, \to, \mathcal{T})$ is T_0-space. The proof of converse is clear. □

Theorem 4.4. *If α is an infinite cardinal number, then there is a T_0 topological semi-hoop of order α, which it's topology is non-trivial.*

Proof. Let $(A, \odot, \to, \wedge, \mathcal{T})$ and $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ be topological semi-hoops in Theorem 3.8. It is clear that $\mathcal{U}$ is non-trivial. Let $x \in X - \{1\}$. If $x \in A$, then for some $n \geqslant 1$, $x \notin F_n$. Hence, $x \in x/F_n \in \mathcal{U}$ and $1 \notin x/F_n$. If $x \notin A$, then $x \in \{x\} \in \mathcal{U}$ and $1 \notin \{x\}$. Now, by Theorem 4.3, $(X, \otimes, \rightsquigarrow, \sqcap, \mathcal{U})$ is a topological semi-hoop of order α. □

In the next example, we have a topological semi-hoop that is T_1-space.

Example 4.2. Let A be a $\sqcup$-semi-hoop such that $x^2 = x$, for all $x \in A$. Suppose $A_a = \{x \in A \mid x \geqslant a\}$ and $B = \{A_a \mid a \in A\}$. We claim that B is a basis of a topology on A. For this, it is clear that $x \in A_x$, for all $x \in A$. Suppose $x \in A_a \cap A_b$, for $a, b \in A$. Then $a \leqslant x$ and $b \leqslant x$. Since A is a $\sqcup$-semi-hoop, we have $a \sqcup b \leqslant x$, and so $x \in A_{a \sqcup b}$. Also, if $x \in A_{a \sqcup b}$, then $a, b \leqslant a \sqcup b \leqslant x$. Hence, $x \in A_a \cap A_b$. Thus, $A_a \cap A_b = A_{a \sqcup b}$. Therefore, B is a basis of a topology $\mathcal{T}$ on A. Now, we prove that $(A, \odot, \mathcal{T})$ is a T_1 topological semi-hoop. For this, let $x, y, c \in A$ such that $x \odot y \in A_c$. Then $c \leqslant x \odot y$. By Proposition 2.2(ii), $c \leqslant x \odot y \leqslant x, y$, then $x, y \in A_c$. Thus, $x \odot y \in A_c \odot A_c$. Let $z \in A_c \odot A_c$. Then there exist $a, b \in A_c$ such that $z = a \odot b$. Since $a, b \geqslant c$, we have $a \odot b \geqslant c \odot c$. Then by assumption, $a \odot b \geqslant c$. Hence, $z = a \odot b \in A_c$, and so $A_c \odot A_c \subseteq A_c$. Then $(A, \odot, \mathcal{T})$ is a topological semi-hoop. Now, suppose $x, y \in A$ such that $x \neq y$. Then $x \in A_x$ and $y \in A_y$. If $y \in A_x$ and $x \in A_y$, then $x \leqslant y$ and $y \leqslant x$. This implies that $x = y$, which is a contradiction. Therefore, $(A, \odot, \mathcal{T})$ is a T_1 topological semi-hoop.

Theorem 4.5. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then $(A, \mathcal{T})$ is a T_1-space if and only if it is a T_0-space.*

Proof. Let $(A, \mathcal{T})$ be a T_0-space and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Then there exist a $U \in \mathcal{T}$ such that $x \to y \in U$ and $1 \notin U$ or $x \to y \notin U$ and $1 \in U$. First assume that $x \to y \in U$ and $1 \notin U$. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. Similarly, $y \notin P$. Now, if $1 \in U$ and $x \to y \notin U$, then since $1 = x \to x = y \to y \in U$, there are $V, W \in \mathcal{T}$ such that $x \in V$ and $y \in W$ such that $V \to V \subseteq U$ and $W \to W \subseteq U$. If $y \in V$, then $x \to y \in V \to V \subseteq U$, which is a contradiction. Similarly, $x \notin W$. Therefore, $(A, \mathcal{T})$ is a T_1-space. The proof of converse is clear. □

Corollary 4.6. *If α is an infinite cardinal number, then there is a T_1 topological semi-hoop of order α which it's topology is non-trivial.*

Proof. By Theorems 4.4 and 4.5, the proof is clear. □

Theorem 4.7. *Let $(A, \to, \mathcal{T})$ be a topological semi-hoop. Then the following statements are equivalent:*

- (i) $(A, \to, \mathcal{T})$ *is Hausdorff.*
- (ii) $\{1\}$ *is closed.*
- (iii) *for any $1 \neq x \in A$, there exist two open sets U and V of 1 and x, respectively, such that $U \cap V = \emptyset$.*
- (iv) $(A, \to, \mathcal{T})$ *is T_1-space.*

Proof. $(i) \Rightarrow (ii)$. Since A is Hausdorff, it is clear that $\{1\}$ is closed.

$(ii) \Rightarrow (iii)$. Let $\{1\}$ be closed and $x \neq 1$. Then $1 \to x = x \in A - \{1\} \in \mathcal{T}$. Since $\to$ is continuous, there exist two open neighborhoods U and V of 1 and x, respectively, such that $U \to V \subseteq A - \{1\}$. If $z \in U \cap V$, then $1 = z \to z \in U \to V \subseteq A - \{1\}$, which is a contradiction. Therefore, $U \cap V = \emptyset$.

$(iii) \Rightarrow (iv)$. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. By (iii), there exist two disjoint open sets U and V which contain $x \to y$ and 1, respectively. Since $\to$ is continuous, there are $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $x \in Q$, then $1 = x \to x \in P \to Q \subseteq U$, which is a contradiction. So, $x \notin Q$. Similarly, $y \notin P$. Hence, $(A, \to, \mathcal{T})$ is T_1-space.

$(iv) \Rightarrow (i)$. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Without the lost of generality, suppose $x \to y \neq 1$. Since $\mathcal{T}$ is a T_1-space, there exist two open neighborhoods U and V of $x \to y$ and 1, respectively, such that $1 \notin U$ and $x \to y \notin V$. Since $\to$ is continuous, there exist $P, Q \in \mathcal{T}$ such that $x \in P$, $y \in Q$ and $P \to Q \subseteq U$. If $z \in P \cap Q$, then $1 = z \to z \in U$, which is a contradiction, and so $P \cap Q = \emptyset$. By the similar way, other case is clear. Therefore, $(A, \to, \mathcal{T})$ is Hausdorff. □

Corollary 4.8. *If α is an infinite cardinal number, then there is a Hausdorff topological semi-hoop of order α, which it's topology is non-trivial.*

Proof. By Corollary 4.6 and Theorem 4.7, the proof is clear. $\square$

Suppose A is a semi-hoop algebra and F is a proper filter of A. Define $\sum = \{U \in \mathcal{F}(A) \mid \exists\ F \in \mathcal{F}(A)$ such that $F \subseteq U\}$ and $f : \sum \hookrightarrow \mathcal{F}(A/F)$ is a map such that $f(U) = U/F$, for all $U \in \sum$. Then it is easy to prove that f is a one to one corresponding among $\sum$ and $\mathcal{F}(A/F)$.

Let $\mathcal{T}$ be a topology on semi-hoop algebra A, $F \in \mathcal{F}(A)$ and $\pi : A \hookrightarrow A/F$ be canonical epimorphism. Then the set $\widetilde{\mathcal{T}} = \{U \subseteq A/F \mid \pi^{-1}(U) \in \mathcal{T}\}$ is a topology on A/F. $\widetilde{\mathcal{T}}$ is called *quotient topology.*

It is easy to see that $\pi_F : (A, \mathcal{T}) \hookrightarrow (A/F, \widetilde{\mathcal{T}})$ is continuous. Also, it is easy to prove that if $* \in \{\odot, \rightarrow, \wedge\}$ and $(A, *, \mathcal{T})$ is a topological semi-hoop algebra, then $(A/F, *, \widetilde{\mathcal{T}})$ is a topological quotient semi-hoop algebra provided $\pi_F : A \hookrightarrow A/F$ is an open map.

Proposition 4.9. *Let $(A, \rightarrow, \mathcal{T})$ be a topological semi-hoop, $F \in \mathcal{F}(A)$ and $\widetilde{\mathcal{T}}$ be quotient topology on A/F. If $\pi_F : A \hookrightarrow A/F$ is an open map, then*

(i) *F is open if and only if $(A/F, \widetilde{\mathcal{T}})$ is discrete.*

(ii) *F is closed if and only if $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is Hausdorff.*

Proof. (i). Let F be open. Since $\pi_F : A \hookrightarrow A/F$ is an open map, the set $\pi_F(F) = 1/F$ belongs to $\widetilde{\mathcal{T}}$. Since $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is a topological semi-hoop, by Theorem 3.2, $(A/F, \widetilde{\mathcal{T}})$ is discrete. Conversely, suppose $(A/F, \widetilde{\mathcal{T}})$ is discrete. Then $1/F$ is an open set. Since $\pi_F : A \hookrightarrow A/F$ is continuous, $F = {\pi_F}^{-1}(1/F) \in \mathcal{T}$.

(ii). ($\Rightarrow$) By assumption, F is closed, then F^c is open. Thus, for any $x, y \in A$, if $x \rightarrow y \in F^c$, then there are two open neighborhood U and V of x and y, respectively, such that $U \rightarrow V \subseteq F^c$, because $\rightarrow$ is continuous. Also, since π is open, so $\pi(U)$ and $\pi(V)$ are two open neighborhoods of x/F and y/F, respectively, such that $\pi(U) \rightarrow \pi(V) \subseteq \pi(U \rightarrow V) \subseteq \pi(F^c)$. If $z/F \in \pi(U) \cap \pi(V)$, then $1/F = z/F \rightarrow z/F \in \pi(U) \rightarrow \pi(V) \subseteq \pi(F^c)$, which is a contradiction. Therefore, $(A/F, \rightarrow, \widetilde{\mathcal{T}})$ is Hausdorff.

($\Leftarrow$) Since A/F is Hausdorff, the set $\{1/F\}$ is closed in A/F, and so $F = \pi^{-1}(1/F)$ is closed in A. $\square$

5. Connected topological semi-hoop

A topological space A is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise, A is said to be *connected.* Also, $(A, \mathcal{T})$ is called *locally connected* at $x \in A$, if for every open subset V containing x, there exists a connected, open subset U with $x \in U \subseteq V$. *Connected component*, a maximal subset of a topological space that can not be covered by the union of two disjoint open sets. The components of any topological space X form a partition of X, they are disjoint, non-empty, and their union is the whole space. A topological space X is *totally disconnected* if the connected components in X are the one-point sets. Also, we know that the image of a connected space under a continuous map is connected and a finite cartesian product of connected spaces is connected (cf. [10]).

Proposition 5.1. *Let $(A, \mathcal{T})$ be a topological semi-hoop. If C is connected components of* 1*, then C is a closed filter of A.*

Proof. Let C be connected component of 1 and $x \in C$. Since $\odot$ is continuous, $x \odot C$ is a connected set which contains x. Since $x \in (x \odot C) \cap C$, the set $(x \odot C) \cup C$ is a connected set containing 1. Hence, $(x \odot C) \cup C \subseteq C$. This implies that $x \odot C \subseteq C$, so $C \odot C \subseteq C$. Now, suppose $x \leqslant y$ and $x \in C$, then $1 = x \to y \in C \to y$. Since $\to$ is continuous, $C \to y$ is a connected set. Hence, $C \to y \subseteq C$. So, $y = 1 \to y \in C \to y \subseteq C$. Therefore, C is a filter of A. Since C is component, clearly it is closed. $\square$

Recall a semi-hoop A is *locally finite* if only filters of A are $\{1\}$ and A.

Theorem 5.2. *Let $(A, \mathcal{T})$ be a topological locally finite semi-hoop. Then $(A, \mathcal{T})$ is connected or totally disconnected.*

Proof. Suppose $(A, \mathcal{T})$ is not connected. Let C be connected component of 1. Then by Proposition 5.1, C is a closed filter of A. Since $(A, \mathcal{T})$ is not connected, $C = \{1\}$. Let C_x be connected component of $x \in A$. Since $\to$ is continuous, $x \to C_x$ is a connected set containing $1 = x \to x$. Hence, $x \to C_x \subseteq C = \{1\}$. Thus, $x \to C_x = 1$. By the similar way, $C_x \to x = 1$. This implies that $C_x = \{x\}$, and so $(A, \mathcal{T})$ is totally disconnected. $\square$

Lemma 5.3. *Let A be a semi-hoop and $F \in \mathcal{F}(A)$. Then $x \odot a = y \odot b$, for some $a, b \in F$ if and only if $x/F = y/F$ in A/F.*

Proof. ($\Rightarrow$) Let $x \odot a = y \odot b$, for some $a, b \in F$. Since $x \odot a \leqslant y \odot b$, by Proposition 2.2(i), $a \leqslant x \to (y \odot b)$. Since $F \in \mathcal{F}(A)$ and $a \in F$, by (F2), $x \to (y \odot b) \in F$. Moreover, by Proposition 2.2(ii) and (v), $y \odot b \leqslant y$, and so $x \to (y \odot b) \leqslant x \to y$. Since $F \in \mathcal{F}(A)$ and $x \to (y \odot b) \in F$, by (F2), $x \to y \in F$. By the similar way, $y \to x \in F$. Therefore, $x/F = y/F$.
($\Leftarrow$) Let $x/F = y/F$, for $x, y \in A$. Then $x \to y \in F$ and $y \to x \in F$. Thus, there exists $a \in F$ such that $y \to x = a$. Since $a \to (y \to x) = 1$, by (SH3),

$(a \odot y) \to x = 1$, and so $a \odot y \leqslant x$. Then

$$\begin{aligned}
& [a \odot (x \to y)] \to [x \to (a \odot y)] && \text{by (SH3)} \\
= \;& (x \to y) \to [a \to (x \to (a \odot y))] && \text{by (SH3)} \\
= \;& (x \to y) \to [(x \odot a) \to (a \odot y))] && \text{by (SH3)} \\
= \;& (x \to y) \to [x \to (a \to (a \odot y))] && \text{by (SH3)} \\
= \;& (x \odot (x \to y)) \to (a \to (a \odot y)) && \text{by Proposition 2.2(iv) and (vi)} \\
\geqslant \;& y \to (a \to (a \odot y)) && \text{by (SH3)} \\
= \;& (a \odot y) \to (a \odot y) \\
= \;& 1
\end{aligned}$$

Then $a \odot (x \to y) \leqslant x \to (a \odot y)$. Since $F \in \mathcal{F}(A)$, $a, x \to y \in F$, by (F1), $a \odot (x \to y) \in F$ and by (F2), $x \to (a \odot y) \in F$. Then there exists $b \in F$ such that $x \to (a \odot y) = b$. Thus, $b \to [x \to (a \odot y)] = 1$. By (SH3), $(b \odot x) \to (a \odot y) = 1$, and so $b \odot x \leqslant a \odot y$. By the similar way, $a \odot y \leqslant b \odot x$. Therefore, $a \odot y = b \odot x$. $\square$

Proposition 5.4. *Let $(A, \mathcal{T})$ be a topological semi-hoop and C be a connected component of 1 in A. Then the following statements hold,*

(*i*) *if D is a closed subset of A/C such that $\pi^{-1}(D)$ is disconnected, then D is disconnected,*

(*ii*) *if $(A, \mathcal{T})$ is disconnected, then $(A/C, \widetilde{\mathcal{T}})$ is disconnected.*

Proof. (*i*). $\pi^{-1}(D) = X \cup Y$, where X, Y are two non-empty disjoint closed subsets of $\pi^{-1}(D)$ and hence, A. It is clear that $X \subseteq \pi^{-1}(\pi(X))$. Let $z \in \pi^{-1}(\pi(X))$. Then there exists $x \in X$ such that $x/F = z/F$. By Lemma 5.3, $x \odot a = z \odot b$, for some $a, b \in C$. Given C_x and C_z, two connected component of x and z, respectively. Then $x \odot a \in x \odot C \subset C_x$ and $z \odot b \in z \odot C \subset C_z$. Since $z \odot b = x \odot a$, $C_x \cap C_z \neq \emptyset$. Hence, $C_x \cup C_z$ is connected. This means that $C_x = C_z$, and so $z \in X$. Therefore, $X = \pi^{-1}(\pi(X))$. Since X is closed in A, $\pi(X)$ is closed in A/C. By the similar way, $Y = \pi^{-1}(\pi(Y))$ and $\pi(Y)$ is a closed subset of A/C. On the other hand, $\pi^{-1}(\pi(X) \cap \pi(Y)) = X \cap Y = \emptyset$ implies that $\pi(X) \cap \pi(Y) = \emptyset$. So, $D = \pi(X) \cup \pi(Y)$, where $\pi(X)$ and $\pi(Y)$ are two disjoint closed subsets of A/C. Hence, D is disconnected.

(*ii*). Let $(A, \mathcal{T})$ be disconnected. Since $\pi^{-1}(A/C) = A$, by (i), $(A/C, \widetilde{\mathcal{T}})$ is disconnected. $\square$

Theorem 5.5. *Let $(A, \mathcal{T})$ be a topological semi-hoop, C be a connected component of 1 in A and $\pi : A \to A/C$ be open canonical epimorphism. Then A/C is totally disconnected.*

Proof. Let C be the connected component of 1 in A. Then by Proposition 5.1, C is a closed filter of A. Let K be a connected component of $1/C$ in A/C. If $1/C \neq x/C$, for some $x/C \in A/C$, then C is a proper subset of $\pi^{-1}(K)$. Hence, $\pi^{-1}(K)$ is not connected. Since K is closed in A/C, by Proposition 5.4,

K is disconnected, which is contradiction. Therefore, $K = \{1\}$. Suppose K_x is connected component of x/C in A/C. Since $\to$ is continuous in A/C, $K_x \to x/C$ is connected and $1/C \in K_x \to x/C$. Then $K_x \to x/C \subseteq K = \{1/C\}$. Similarly, $x/C \to K_x \subseteq K = \{1/C\}$. Hence, for each $y/C \in K_x$, $y/C \to x/C = 1/C$ and $x/C \to y/C = 1/C$. So, $x/C = y/C$. This implies that $K_x = \{x/C\}$. Therefore, A/C is totally disconnected. □

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