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On ordered semigroups containing covered one-sided ideals

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Abstract. In this paper, the notion of covered left ideals of ordered semigroups will be introduced, and it is proved that the set of all covered left ideals of a given ordered semigroup is a sublattice of the lattice of all left ideals if the ordered semigroup. And then the structure of ordered semigroups containing covered left ideals will be studied. For the results of covered right ideals of ordered semigroups can be considered similarly.

1. Preliminaries

A proper left ideal M of a semigroup (without order) S is said to be a *covered left ideal* of S if

$$M \subseteq S(S \backslash M).$$

This notion was first introduced and studied by I. Fabrici [2]. Indeed, the author studied the structure of semigroups containing covered one-sided ideals. The purpose of this paper is to extend Fabrici's results to ordered semigroups. In fact, we introduce the concept of covered one-sided ideals of ordered semigroups, and study the structure of ordered semigroups containing covered one-sided ideals. For the concept of covered two-sided ideals of ordered semigroups can be found in [1].

A *partially ordered semigroup* (or simply an *ordered semigroup*) is a semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any $x, y, z \in S$,

$$x \leqslant y \text{ implies } zx \leqslant zy \text{ and } xz \leqslant yz.$$

For A, B nonempty subsets of an ordered semigroup $(S, \cdot, \leqslant)$, the set product AB is defined to be the set of all elements xy in S where $x \in A$ and $y \in B$. We write $(A]$ for the set of all elements x in S such that $x \leqslant a$ for some $a \in A$, i.e.,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and similarly for $\{x\}A$. It is observed that the following hold:

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(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $(A \bigcup B] = (A] \bigcup (B]$;

(5) $((A]] = (A]$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Analogousely to the concept in lattice ordered rings (see [4], p. 142), a non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

(ii) for any $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. A left ideal A of S is called a *proper left* ideal of S if $A \subset S$. The symbol $\subset$ stands for proper subset of sets. For proper right ideals and proper ideals of S are defined similarly. A proper left (resp. right, two-sided) ideal A of S is said to be *maximal* if for any left (resp. right, two-sided) ideal B of S, $A \subset B \subseteq S$ implies $B = S$. Finally, if S does not contain proper left (resp. right, two-sided) ideals, we call it *left* (resp. *right, two-sided*) *simple*. It is easy to see that the union or intersection of two two-sided ideals of S is a two-sided ideal of S. For any element a of S the principal left ideal generated by a of S is of the form

$$L(a) = (a \bigcup Sa].$$

Analogousely to the concept of Green's relations in semigroups, the equivalence relation $\mathcal{L}$ is defined on S by, for any a, b in S,

$$a\mathcal{L}b \Longleftrightarrow L(a) = L(b).$$

The $\mathcal{L}$-class containing a in S will be denoted by L_a. The set of all $\mathcal{L}$-classes of S forms a quasi-ordered:

$$L_a \preceq L_b \Longleftrightarrow L(a) \subseteq L(b).$$

The symbol $L_a \prec L_b$ means $L_a \preceq L_b$, but $L_a \neq L_b$.

Let a be any element of an ordered semigroup $(S, \cdot, \leqslant)$. If L_a is not maximal, then $L_a \prec L_b$ for some b in S. Then $L(a) \subset L(b)$. We now have the following lemma.

Lemma 1.1. *Let a be any element of an ordered semigroup $(S, \cdot, \leqslant)$. If $L(a)$ is not proper subset of any principal left ideal of S, then L_a is maximal with respect to the quasi-order $\preceq$.*

Lemma 1.2. *Let L be a subset of an ordered semigroup $(S, \cdot, \leqslant)$. Then L is a maximal $\mathcal{L}$-class (with respect to $\preceq$) of S if and only if $S \backslash L$ is a maximal left ideal of S.*

Proof. Assume first that L is a maximal $\mathcal{L}$-class of S. Let $L = L_a$ for some $a \in S$. Let $y \in S$ and $x \in S \backslash L_a$. If $yx \in L_a$, then, by $x \notin L_a$, we have $L(a) = L(yx) \subset L(x)$. That is $L_a \prec L_x$, which contradicts to the assumption. Hence $S(S \backslash L_a) \subseteq S \backslash L_a$. Let $x \in S \backslash L_a$ and $y \in S$ be such that $y \leqslant x$. Then $L_y \preceq L_x$. If $y \in L_a$, then L_y is a maximal $\mathcal{L}$-class of S; hence $L_a = L_x$. This is a contradiction. Thus $y \in S \backslash L_a$. This shows that $S \backslash L_a$ is a left ideal of S. To show that $S \backslash L_a$ is a maximal left ideal of S, we suppose that there is a left ideal A of S such that $(S \backslash L_a) \subset A$. Let $z \in A \backslash (S \backslash L_a)$, and thus $L(a) = L(z)$. If $b \in L_a$, then

$$L(b) = L(a) = L(z) \subseteq A.$$

Thus $L_a \subseteq A$, and $S = A$.

Conversely, assume that $S \backslash L$ is a maximal left ideal of S. Choose a in $S \backslash (S \backslash L)$. We have $L = L_a$. To see this, let $x \in L_a$. If $x \in S \backslash L$, then

$$a \in L(a) = L(x) \subseteq S \backslash L.$$

This is a contradiction. Thus $x \in L$, and $L_a \subseteq L$. Let $x \in L$. Since $S \backslash L \subset (S \backslash L) \bigcup L(x)$, we have $(S \backslash L) \bigcup L(x) = S$. Similarly, $(S \backslash L) \bigcup L(a) = S$. Since $x \in (S \backslash L) \bigcup L(a)$, we have $x \in L(a)$, and so $L(x) \subseteq L(a)$. Similarly, $L(a) \subseteq L(x)$. Thus $L(x) = L(a)$, and $x \in L_a$. Therefore, $L \subseteq L_a$. Finally, by Lemma 1.1, it suffices to show that $L(a)$ is not proper subset of $L(x)$ for all x in S. Let $x \in S$. If $x \in L$, then $L(a) = L(x)$. If $x \in S \backslash L$, then $L(x) \subseteq S \backslash L$; hence $L(a) \nsubseteq L(x)$. This completes the proof. □

2. Covered left ideals in ordered semigroups

We begin this section with the definition of covered left ideals of an ordered semigroup. For the concept of covered right ideals of an ordered semigroups can be defined dually, and the results for ordered semigroups containing covered right ideals are left-right dual.

Definition 2.1. *A proper left ideal L of an ordered semigroup $(S, \cdot, \leqslant)$ is called a covered left ideal of S if*

$$L \subseteq (S(S \backslash L)].$$

In this example we consider an ordered semigroup from [7] and [8].

Example 2.2. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that $S = \{a, b, c, d, e\}$ and

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leqslant\ = \{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,d),(d,b),(d,e),(e,e)\}.$$

The covering relation is given by:

$$<\ = \{(a,b),(a,e),(c,b),(c,e),(d,b),(d,e)\}.$$

The left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,c,d,e\}$ and S. It can be observed that the covered left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$.

In this example we consider an ordered semigroup from [9].

Example 2.3. Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that $S = \{a,b,c,d,f\}$ and

$\cdot$	a	b	c	d	e
a	b	d	a	b	e
b	d	b	b	d	e
c	d	b	c	d	e
d	b	d	d	b	e
e	e	e	e	e	e

$$\leqslant\ = \{(a,a),(b,b),(b,c),(b,e),(c,c),(d,a),(d,d),(d,e),(e,e)\}.$$

The covering relation is given by:

$$<\ = \{(b,c),(b,e),(d,a),(d,e)\}.$$

The left ideals of S are $\{b,d,e\}$, $\{a,b,d,e\}$ and S, and the covered left ideal of S is $\{b,d,e\}$.

Now, we will prove that the set of all covered left ideals of an ordered semigroup is a sublattice of the lattice of all left ideals.

Proposition 2.4. *If L_1 and L_2 are different proper left ideals of an ordered semigroup $(S,\cdot,\leqslant)$ such that $L_1 \bigcup L_2 = S$, then both L_1, L_2 are not covered left ideals of S.*

Proof. Assume that L_1 and L_2 are different proper left ideals of an ordered semigroup $(S,\cdot,\leqslant)$ such that $L_1 \bigcup L_2 = S$. Sine $L_1 \bigcup L_2 = S$, we have $S \backslash L_1 \subseteq L_2$ and $S \backslash L_2 \subseteq L_1$. If L_1 is a covered left ideal of S, then

$$L_1 \subseteq (S(S \backslash L_1)] \subseteq (SL_2] \subseteq L_2.$$

Since $L_1 \bigcup L_2 = S$, it follows that $S = L_2$. This is a contradiction. Similarly, L_2 is a covered left ideal of S implies $S = L_1$. This is a contradiction. Thus the assertion is proved. $\square$

The following corollary is a consequence of Proposition 2.4.

Corollary 2.5. *If an ordered semigroup $(S, \cdot, \leqslant)$ contains more than one maximal left ideals, then non of them is a covered left ideal of S.*

Proposition 2.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If L_1 and L_2 are covered left ideals of S, then $L_1 \bigcup L_2$ is a covered left ideal of S.*

Proof. Assume that L_1 and L_2 are covered left ideals of S; thus $L_1 \subseteq (S(S\backslash L_1)]$ and $L_2 \subseteq (S(S\backslash L_2)]$. Let $x \in L_1 \bigcup L_2$. If $x \in L_1$, then, by $L_1 \subseteq (S(S\backslash L_1)]$, we have $x \in (Sa]$ for some $a \in S\backslash L_1$. If $a \in S\backslash(L_1 \bigcup L_2)$, then $x \in (S(S\backslash(L_1 \bigcup L_2))]$. If $a \in L_1 \bigcup L_2$, then $a \in L_2$; hence $a \in (Sb]$ for some b in $S\backslash L_2$. We have

$$x \in (Sa] \subseteq ((S](Sb]] = (SSb] \subseteq (Sb].$$

If $b \in L_1$, then $a \in L_1$. This is a contradiction. Thus $b \in S\backslash(L_1 \bigcup L_2)$, and so $x \in (S(S\backslash(L_1 \bigcup L_2))]$. Similarly, $x \in L_2$ implies $x \in (S(S\backslash(L_1 \bigcup L_2))]$. This proves that $L_1 \bigcup L_2$ is a covered left ideal of S. $\square$

Proposition 2.7. *Let L be a left ideal of an ordered semigroup $(S, \cdot, \leqslant)$. If L_1 is a covered left ideal of S, then $L_1 \cap L$ is a covered left ideal of S.*

Proof. If L_1 is a covered left ideal of S, then $L_1 \subseteq (S(S\backslash L_1)]$; hence

$$L_1 \cap L \subseteq L_1 \subseteq (S(S\backslash L_1)] \subseteq (S(S\backslash(L_1 \cap L))].$$

This shows that $L_1 \cap L$ is a covered left ideal of S. $\square$

Corollary 2.8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If L_1 and L_2 are covered left ideals of S, then $L_1 \cap L_2$ is a covered left ideal of S.*

We now state the main theorem of this section followed by Proposition 2.6 and Corollary 2.8.

Theorem 2.9. *The set of all covered left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ is a sublattice of the lattice of all left ideals of $(S, \cdot, \leqslant)$.*

3. Ordered semigroups with covered left ideals

The purpose of this section is to study the structure of ordered semigroups containing covered left ideals.

Theorem 3.1. *An ordered semigroup $(S, \cdot, \leqslant)$ with the cardinal $|S| > 1$ contains no covered left ideals if and only if S is a union of disjoint minimal left ideals.*

Proof. Assume first that S contains no covered left ideals. Let $a \in S$. If $a \notin (Sa]$, then $(Sa] \subseteq (S(S\backslash(Sa])]$. Thus $(Sa]$ is a covered left ideal of S. This is a contradiction. Hence $a \in (Sa]$. Let L be a proper left ideal of S. If $L \subset (Sa]$, then L is a covered left ideal of S. This is a contradiction. Hence $(Sa]$ is a minimal left ideal of S. Let $a, b \in S$ such that $a \neq b$ and $(Sa] \neq (Sb]$. If $L = (Sa] \cap (Sb]$, then L is a proper subset of $(Sa]$ or $(Sb]$. Thus L is a covered left ideal of S. This is a contradiction. Hence $(Sa] \cap (Sb] = \emptyset$. Therefore, $S = \bigcup_{i \in I}(Sa_i]$.

Conversely, assume that $S = \bigcup_{i \in I} L_i$ where, for each $i \in I$, L_i is a minimal left ideal of S. We set

$$A = \bigcup_{i \in J} L_i, \quad B = \bigcup_{i \in I-J} L_i.$$

Then $S = A \bigcup B$. By Proposition 2.4, neither A nor B is a covered left ideal of S. This completes the proof. □

Theorem 3.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If S is not left simple, then S contains a covered left ideal.*

Proof. Assume that S is not left simple. Then S contains a proper left ideal L. Since $(S(S\backslash L)]$ is a left ideal of S, we have $L \cap (S(S\backslash L)]$ is a proper left ideal of S. By

$$L \cap (S(S\backslash L)] \subseteq (S(S\backslash L)] \subseteq (S(S\backslash(L \cap (S(S\backslash L)]))],$$

it follows that $L \cap (S(S\backslash L)]$ is a covered left ideal of S. □

The concept of right bases of an ordered semigroup was defined in [3] as follows:

Definition 3.3. *A subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a right base of S if it satisfies the following conditions:*

(i) $S = (A \bigcup SA]$;

(ii) *if B is a subset of A such that $S = (B \bigcup SB]$, then $B = A$.*

Here, we provide some more examples: In this example we consider an ordered semigroup from [6].

Example 3.4. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

$$\leqslant = \{(a,a), (a,d), (b,b), (c,c), (c,e), (d,d), (e,e)\}.$$

The covering relation is given by:

$$< = \{(a,d),(c,e)\}.$$

We have $\{b,d\}$ is the right base of S.

In this example we consider an ordered semigroup from [5].

Example 3.5. Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\leqslant = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),\\ (d,d),(d,e),(e,e)\}.$$

The covering relation is given by:

$$< = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}.$$

The right bases of S are $\{e\}$ and $\{c\}$.

A covered left ideal L of an ordered semigroup $(S,\cdot,\leqslant)$ is called the *greatest covered left ideal* of S if every covered left ideal of S contained in L. If an ordered semigroup contains the greatest covered left ideal, we shall denote it by L^g.

To give a necessary condition so that an ordered semigroup contains one-sided bases we need the following lemma.

Lemma 3.6. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup containing the greatest covered left ideal L^g. If $L^g\subset(S^2]$, then the following conditions hold:*

(1) *for every $\mathcal{L}$-class in $(S^2]\backslash L^g$ is maximal;*

(2) *$L(a)=(Sa]$ for all a in $(S^2]\backslash L^g$.*

Proof. Assume that $L^g\subset(S^2]$. Then $(S^2]\backslash L^g$ is non-empty. Frits we prove the second assertion. Let a be an element of $(S^2]\backslash L^g$. Since L^g is an left ideal of S, it follows that $L_a\subseteq(S^2]\backslash L^g$. Then $a\in(Sb]$ for some b in S. Sine $(Sb]\subseteq L(b)$, we have $L(a)\subseteq L(b)$. Suppose that $b\notin L_a$; thus $L_a\neq L_b$. If $b\in L(a)$, then $L(a)=L(b)$; hence $L_a=L_b$. This is a contradiction. Then $b\in S\backslash L(a)$. This implies $L(a)\subseteq(S(S\backslash L(a))]$. Thus $L(a)$ is a covered left ideal of S. By Proposition 2.6, $L^g\bigcup L(a)$ is a covered left ideal of S. Since $a\notin L^g$, $L^g\subset L^g\bigcup L(a)$. This is a contradiction. This shows that $b\in L_a$. Hence $L(a)\subseteq(Sb]\subseteq L(b)=L(a)$. Then $L(a)=(Sb]=L(b)$. Clearly, $(Sa]\subseteq L(a)$. If $b\leqslant a$, then $L(a)=(Sb]\subseteq(Sa]$;

hence $L(a) \subseteq (Sa]$. If $b \in (Sa]$, then $Sb \subseteq S(Sa] \subseteq (S(Sa]] \subseteq (SSa] \subseteq (Sa]$. Therefore, $L(a) = L(b) = (Sb] \subseteq (Sa]$.

We now prove the rest of the assertion. Let L_a be a $\mathcal{L}$-class in $(S^2]\backslash L^g$. Suppose that $L(a) \subseteq L(c)$ for some c in S. Then $a \in (c \bigcup Sc]$; thus $a \in (c]$ or $a \in (Sc]$. Each of the cases implies $(Sa] \subseteq (Sc]$, so $L(a) \subseteq (Sc]$. Sine $c \in S\backslash L(a)$, it follows that $L(a)$ is a covered left ideal of S; hence $L^g \subset L^g \bigcup L(a)$. This is a contradiction. This proves that any $\mathcal{L}$-class in $(S^2]\backslash L^g$ is maximal. □

Theorem 3.7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup containing the greatest covered left ideal L^g. Then S contains a right base if satisfies the following two condition:*

(1) $L^g \subset (S^2]$;

(2) *any two element a, b in $S\backslash(S^2]$, neither $L_a \preceq L_b$ nor $L_b \preceq L_a$*

Proof. Assume that $L^g \subset (S^2]$ and for any two elements a, b in $S\backslash(S^2]$ are incompairable. By

$$L^g \subseteq (S(S\backslash L^g)] \subseteq (S^2] \subseteq S$$

there are two families of $\mathcal{L}$-class to consider: $C_1 = \{L_a | a \in S\backslash(S^2]\}$, $C_2 = \{L_a | a \in (S^2]\backslash L^g\}$. We take one element from each $\mathcal{L}$-class in C_1 and C_2, and let A be the set of all elements we take, we claim that A is a right base of S. For convenience we let $L(A) = (A \bigcup SA]$. To show that $S = L(A)$, it suffices to show that L^g, $(S^2]\backslash L^g$ and $S\backslash(S^2]$ are subset of $L(A)$.

a) Let $x \in L^g$. Then $x \in (S(S\backslash L^g)]$, or equivalent $x \in (Sb]$ for some b in $S\backslash L^g$ We have $b \in L_a$ for some $a \in S\backslash(S^2]$ or $a \in (S^2]\backslash L^g$. Then by the constructing A we have $b \in L(A)$. Thus $x \in L(A)$.

b) If $x \in (S^2]\backslash L^g$, then there exists $a_1 \in A$ such that $x \in L(a_1)$; hence $x \in L(A)$.

c) If $x \in S\backslash(S^2]$, then there exists $a_2 \in A$ such that $x \in L(a_2) \subseteq L(A)$.

Finally, we show that A is the minimal subset of S such that $S = L(A)$. By Lemma 1.2, it follow that for every L_a in C_2 is maximal. Moreover, every L_a in C_1 is also maximal since for any elements a, b in $S\backslash(S^2]$, neither $L_a \preceq L_b$ nor $L_b \preceq L_a$. Now, let B be a proper subset of A such that $S = (B \bigcup SB]$. Let $x \in A\backslash B$. Then $x \leqslant y$ for some $y \in B \bigcup SB$. Since $y \in L(b)$ for some b in B, it follows that $L(x) \subset L(b)$. This contradicts to the constructing of A. The proof is completed. □

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An left ideal L of S is called the *greatest left ideal* of S if every proper left ideal of S contained in L. If an ordered semigroup $(S, \cdot, \leqslant)$ contains the greatest left ideal, we denote The left ideal by L^*.

Theorem 3.8. *Assume an ordered semigroup $(S, \cdot, \leqslant)$ contains only one maximal ideal L. If L is a covered left ideal of S, then L is the greatest left ideal of S.*

Proof. This is easy to see because if T is a proper left ideal of S, then $T \subseteq L$. Hence $L = L^*$. □

Theorem 3.9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with the property every proper ideal of S is a covered left ideal of S. Then either*

(1) *S contains L^*, or*

(2) *$S = (S^2]$ and for any proper left ideal L and for every left ideal $L(a) \subseteq L$, there is b in $S \backslash L$ such that $L(a) \subset L(b) \subset S$.*

Proof. If L_x and L_y are maximal $\mathcal{L}$-classes of S such that $L_x \neq L_y$, then by Lemma 1.2, we have $L_x^c = S \backslash L_x$ and $L_y^c = S \backslash L_y$ are maximal proper left ideals of S such that non of them is a covered left ideal of S. This is a contradiction. Then S contain no different maximal $\mathcal{L}$-classes; hence S contains one maximal $\mathcal{L}$-class or S does not contain maximal $\mathcal{L}$-class.

If S contains one maximal $\mathcal{L}$-class L_x. Then $L_x^c = S \backslash L_x$ is a maximal proper left ideal of S. By assumption, L_x^c is a covered left ideal of S. By Theorem 3.8, $L_x^c = L^*$.

Assume that S does not contain maximal $\mathcal{L}$-classes. We will show that $S = (S^2]$.Suppose $(S^2] \subset S$. Then there exists y in $S \backslash (S^2]$. If $L(y) = S$, then S contains a maximal $\mathcal{L}$-class. This is impossible. Then $L(y) \subset S$, and thus $L(y) \subseteq (S(S \backslash L(y))]$. Then $y \in (S^2]$. This is a contradiction.

Let L be a proper left ideal of S and let $L(a) \subseteq L$. Since $L \subseteq (S(S \backslash L)]$, there exists b in $S \backslash L$ such that $a \in (Sb]$, and hence $L(a) \subseteq L(b) \subseteq S$. Since $b \in S \backslash M$, so $L(a) \subset L(b)$. By assumption, $L(b) \subset S$. □

Theorem 3.10. *Assume an ordered semigroup $(S, \cdot, \leqslant)$ satisfies one the following two condition:*

(1) *S contains L^* which is a covered left ideal of S.*

(2) *$S = (S^2]$ and for any proper left ideal L and for every left ideal $L(a) \subseteq L$, there is b in $S \backslash L$ such that $L(a) \subseteq L(b)$.*

Then every proper left ideal of S is a covered left ideal of S.

Proof. Let L be a proper left ideal of S. First we assume that S satisfies (1). Then $L \subseteq L^*$. Since $S \backslash L^* \subseteq S \backslash L$, it follows that $L \subseteq L^* \subseteq (S(S \backslash L^*)] \subseteq (S(S \backslash L)]$. This shows that L is a covered left ideal of S.

Secondary, we assume that S satisfies (2). Let $x \in L$; thus $L(x) \subseteq L$. Then there is b in $S \backslash L$ such that $L(x) \subseteq L(b)$. We have $S = (S^2]$, so $b \in (Sd]$ for some d in S. Since $b \in S \backslash L$, $d \in S \backslash L$. Hence $x \in (Sd] \subseteq (S(S \backslash L)]$. This shows that $L \subseteq (S(S \backslash L)]$. □

Definition 3.11. *We say that the principal left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ are updirected if for every $a, b \in S$ there is $c \in S$ such that $\{a, b\} \subseteq L(c)$.*

Theorem 3.12. *If all proper left ideals of an ordered semigroup $(S, \cdot, \leqslant)$ are covered, then the principal left ideals of S are updirected.*

Proof. Suppose that there exist two elements $a, b \in S$ such that there is no $c \in S$ with $\{a, b\} \subseteq L(c)$. It is sufficient to show that there exists a left ideal of S which is not covered. Let

$$L = \{x \in S \mid a \notin L(x)\}.$$

We have $L \neq \emptyset$ because $a \in L$. If $x \in S$ and $y \in L$, then $xy \in L$. Otherwise, we have $a \in L(xy) \subseteq L(y)$. This is a contradiction. If $x \in S$ and $y \in L$ such that $x \leqslant y$, then $L(x) \subseteq L(y)$. Since $a \notin L(y)$, $a \notin L(x)$. Thus $x \in L$. Hence L is a left ideal of S. Moreover, $b \in L$. Indeed, if $b \notin L$, then $a \in L(b)$; hence $\{a, b\} \subseteq L(b)$. This is a contradiction. Finally, we have to show that $b \notin (S(S\backslash L)]$. Suppose that $b \in (S(S\backslash L)]$ such that $b \leqslant xy$ for some $x \in S$ and $y \in S\backslash L$. This implies that $\{a, b\} \subseteq L(y)$ which is a contradiction. □

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